Math 330 Section 5 - Spring 2018 - Homework 07

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Update Feb 16, 2018

The last submission for problems 3 and 4 about the Division Algorithm is due on Wednesday, February 28.

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

B/G (Beck/Geoghegan) Textbook:
- all of ch.1, ch.2 except the material on gcd(m, n), all of ch.3 – ch.5, ch.6.1 .

MF lecture notes:
- ch.1 – ch.3, ch.5, ch.6 (skip ch.6.3), ch.7
- ch.17 (Addenda to B/G): the chapters corresponding to what has been assigned so far from B/G.

B/K lecture notes (optional):
- ch.1.1 (Introduction to sets)
- ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions

New reading assignments:

It is particularly **important** that you consult the parts of MF ch.17 which cover this week's B/G reading assigments!

Reading assignment 1 - due Monday, February 12:

a. Carefully read B/G ch.9.1 (Injections and Surjections). You have encountered almost all of that material already in MF ch.5.
b. Carefully read B/G ch.6.2 and 6.3.

Reading assignment 2 - due: Wednesday, February 14:

a. Carefully read the remainder of B/G ch.6.
b. Carefully read B/G ch.7 until the formulation of thm.7.17 about the base 10 representation of the sum of two integers. Skip the proof and all that follows in ch.7.

Reading assignment 3 - due Friday, February 16:

a. Read B/G ch.8: Carefully read the items printed in normal (big) size and skip through the material in ch.8.1 and 8.2 which is printed in small size.
b. Convince yourself that B/G ax.8.1 – 8.4 imply that $\mathbb{R}$ is a commutative ring with unit (MF ch.3) and that the addition of prop.8.7 (which needs ax.8.5) implies that $\mathbb{R}$ actually is an integral domain.

General note on written assignments: Unless expressly stated otherwise, to prove a proposition or theorem you are allowed to make use of everything in the book up to but NOT including the specific item you are asked to prove.

Written assignment 1: Prove the "$\subseteq$ part" of formula b of De Morgan's Law:
Let there be a universal set Ω.
Then for any indexed family $(A_\alpha)_{\alpha\in I}$ of sets: $\quad \big(\bigcap\limits_{\alpha} A_\alpha\big)^{\complement} \subseteq \bigcup\limits_{\alpha} A_\alpha^{\complement}$.

Written assignment 2: (One point each for **a** and **b**)

Let X, Y be two nonempty sets and let $f : X \longrightarrow Y$. For $a, b \in X$ we write $a \sim b$ iff $f(a) = f(b)$.

- **a:** Prove MF prop.7.5 (Indirect image and fibers of f): $\sim$ is indeed an equivalence relation for X.
- **b:** Write $[x]_f$ for the equivalence class of $x \in X$ with respect to "$\sim$". Express $[x]_f$ in terms of the function f: $[x]_f = \{x' \in X : f(x')$.....??.......$\}$. (I do not want to see "$[x]_f = \{x' \in X : x' \sim x\}$".)

Do not use induction for any of the following assignments. It would only make your task more difficult!

#3 and #4 are about proving B/G thm.6.13 (Division algorithm for integers): Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. There exists a unique combination of two integers q ("quotient") and r ("remainder") such that

$$m = n \cdot q + r \quad \text{and } 0 \leq r < n.$$

Written assignment 3:

Prove uniqueness of the "decomposition" $m = qn + r$: If you have a second such decomposition $m = \tilde{q}n + \tilde{r}$ then show that this implies $q = \tilde{q}$ and $r = \tilde{r}$. Start by assuming that $r \neq \tilde{r}$ which means that one of them is smaller than the other and take it from there.

Written assignment 4:

Much harder than #3: Prove the existence of q and r.

Hints for #4: Review the Well-Ordering principle from B/G ch.2. Its use will give the easiest way to prove this theorem. Remember we extended the well-ordering principle as follows in MF thm.17.1 (Generalization of the Well-Ordering Principle): Let $A \subseteq \mathbb{Z}$ have lower bounds (which is especially true if $A \subseteq \mathbb{N}$ or $A \subseteq \mathbb{Z}_{\geq 0}$). If $A \neq \emptyset$ then A has a minimum.

Apply the above to the set $A := A(m,n) := \{x \in \mathbb{Z}_{\geq 0} : x = m - kn \text{ for some } k \in \mathbb{Z}\}$.

Hint for both #3 and #4: MF prop.17.8 and cor.17.3 from ch.17.6.2 (The Division Algorithm) will come in handy in connection with $0 \leqq r < n$: If $m, n \in \mathbb{Z}_{\geq 0}$ then

$$|n - m| \leqq \max(m,n), \text{ i.e.,} \tag{17.11}$$
$$-\max(x,y) \leqq x - y \leqq \max(x,y), \tag{17.12}$$
$$-n < y - x < n. \tag{17.13}$$