# Math 330 - Number Systems - Sample Exam Problems

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The following list of problems is typical for what you might find on my first Math 330 exam. I plan to add to this list in the future and also include sample problems for exam 2 and the final exam.

*Problem* 0.1. (**Induction**). Let $x_1 = 1, x_2 = 1 + \frac{1}{2}, \dots, x_k = \sum_{j=1}^{k} \frac{1}{j}$ $(k \in \mathbb{N})$.
Prove by induction that $\sum_{k=1}^{n} x_k \; = \; (n+1)x_n - n$ $(n \in \mathbb{N})$.

*Problem* 0.2. (**Induction**). Prove by induction that $\sum_{j=1}^{n} j(j!) \; = \; (n+1)! - 1$ $(n \in \mathbb{N})$.

**Solution to #0.2:**

Base case $n = 1$: LS = $1 \cdot (1!) = \cdot 1 = 1 = 2 - 1 = (2!) - 1$ = RS.

Induction assumption $(\star)$: $\sum_{j=1}^{n} j(j!) \; = \; (n+1)! - 1$.

Need to show $(\star\star)$: $\sum_{j=1}^{n+1} j(j!) \; = \; (n+2)! - 1$.

$$\begin{aligned} \text{LS} \; &= \sum_{j=1}^{n} j(j!) + (n+1)(n+1)! \overset{(\star)}{=} (n+1)! - 1 + (n+1)(n+1)! \\ &= (1)(n+1)! + (n+1)(n+1)! - 1 \; = \; (n+2)(n+1)! - 1 \; = \; \text{RS}. \; \blacksquare \end{aligned}$$

*Problem* 0.3. (**Strong Induction**). Let $x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ \dots,\ x_n = x_{n-1} + x_{n-2} + x_{n-3}$ $(n \in \mathbb{N}, n \geqq 3)$.
Prove by strong induction that $x_n \leqq 3^n$ for all $n \in \mathbb{Z}_{\geqq 0}$.

*Problem* 0.4. (**Strong Induction**).

Let $x_0 = 2,\ x_1 = 4,\ x_{n+1} = 3x_n - 2x_{n-1}$ for $n \in \mathbb{N}$. Prove by strong induction that $x_n = 2^{n+1}$ for every integer $n \geq 0$. Hint: Is one number enough for the base case?

**Solution to #0.4:**

Base cases: $n = 0, 1$: $x_0 = 2 = 2^{0+1}$. Further, $x_1 = 4 = 2^{1+1}$. This proves the base cases.

Induction assumption $(\star)$: Let $n \in \mathbb{N}$. Assume that $x_j \; = \; 2^{j+1}$ for **all** $0 \leq j \leq n$.

Need to show $(\star\star)$: $x_{n+1} \; = \; 2^{n+2}$.

$$\begin{aligned} \text{LS} \; &= x_{n+1} \; = \; 3x_n - 2x_{n-1} \quad \text{(the recursive definition)} \\ &= \; 3(2^{n+1}) - 2(2^n) \quad ((\star) \text{ was applied both to } j = n \text{ and } j = n-1) \\ &= \; 6 \cdot 2^n - 2 \cdot 2^n \; = \; 4 \cdot 2^n \; = \; 2^{n+2} \; = \; \text{RS}. \; \blacksquare \end{aligned}$$

*Problem* 0.5. (**Strong Induction**).

Let $x_0 = 1,\ x_1 = 3,\ x_{n+1} = 2x_n + 3x_{n-1}$ for $n \in \mathbb{N}$. Prove by strong induction that $x_n = 3^n$ for every integer $n \geq 0$. Hint: Is one number enough for the base case?

**Solution to #0.5:**

Base cases: $n = 0, 1$: $x_0 = 1 = 3^0$. Further, $x_1 = 3 = 3^1$. This proves the base cases.

Induction assumption $(\star)$: Let $n \in \mathbb{N}$. Assume that $x_j = 3^j$ for **all** $0 \leq j \leq n$.

Need to show $(\star\star)$: $x_{n+1} = 3^{n+1}$..

$$\begin{aligned} \text{LS } &= x_{n+1} = 2x_n + 3x_{n-1} \quad \text{(the recursive definition)} \\ &= 2(3^n) + 3(3^{n-1}) \quad ((\star) \text{ was applied both to } j = n \text{ and } j = n-1\text{ )} \\ &= 2 \cdot 3^n + 1 \cdot 3^n = 3 \cdot 3^n = 3^{n+1} = \text{RS. } \blacksquare \end{aligned}$$

*Problem* 0.6. (**Recursion**). Let $x_1 = 3,\ x_{n+1} = x_n + 2n + 3$ $(n \in \mathbb{N}$. Prove by induction that $x_n = n(n+2)$ $(n \in \mathbb{N})$.

**Solution to #0.6:**

Base case $n = 1$: LS = $3 = 1(1+2) = 1$ = RS.

Induction assumption $(\star)$: $x_n = n(n+2)$.

Need to show $(\star\star)$: $x_{n+1} = (n+1)(n+3)$

$$\text{LS } = x_{n+1} \overset{\text{def.}}{=} x_n + 2n + 3 \overset{(\star)}{=} n(n+2) + 2n + 3 = n^2 + 4n + 3 = (n+1)(n+3) = \text{RS. } \blacksquare$$

*Problem* 0.7. (**Logic**). Given a function $f : X \to Y$, negate the following statements:

**a.** There exists $x \in X$ and $y \in Y$ such that $f(x) = y$,

**b.** For all $x \in X$ there exists $y \in Y$ such that $f(x) = y$,

**c.** $\exists x \in X$ such that $\forall y \in Y$ such that $f(x) \neq y$.

**d.** $\forall x_1, x_2 \in X$ : if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

**Solution to #0.7:**

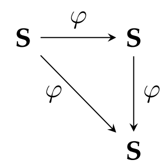
**a.** $\forall x \in X, \forall y \in Y : f(x) \neq y$,
**b.** $\exists x \in X$ such that $\forall y \in Y : f(x) \neq y$,
**c.** $\forall x \in X\ \exists y \in Y$ such that $f(x) = y$,
**d.** $\exists x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. $\blacksquare$

*Problem* 0.8. (**Functions**). Given is a function $f : A \longrightarrow B$ $(A, B \neq \emptyset)$. Give the definitions of each of the following:

**a.** $f$ is injective.
**b.** $f$ is surjective.
**c.** $f$ is bijective.

**d.** $f$ has a left-inverse $g$.
**e.** $f$ has a right-inverse $h$.

For **d** and **e**, give the "arrow diagram" which show domain and codomain for each function involved. In both cases it will like the one to the left. Each symbol **S** denotes a (possibly different) set and each symbol $\varphi$ denotes a (possibly different) function.




**Solution to #0.8:**

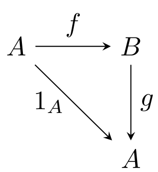
**Solution to problems a,b,c**:

Injective means one-one: If $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ then $a_1 = a_2$.

Surjective means onto: If $b \in B$ then there is $a \in A$ such that $f(a) = b$.

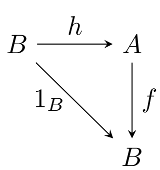
Bijective means both injective and surjective.

**Solution to problem d**: If this diagram commutes:




i.e., $g \circ f = 1_A$, then we call $g$ a **left inverse** of $f$ "to the left of the reference object $f$".

**Solution to problem e**: If this diagram commutes:




i.e., $f \circ h = 1_B$, then we call $h$ a **right inverse** of $f$ "to the right of the reference object $f$" ■

*Problem* 0.9. (**Set functions**). Given is an arbitrary collection of sets $(A_j)_{j \in J}$. Determine for each assertion below whether it is true or false. If it is true, prove it. If it is false, give a counterexample.

**a.** $f(\bigcup_{j \in J} A_j) \subseteq \bigcup_{j \in J} f(A_j)$; **b.** $\bigcup_{j \in J} f(A_j) \subseteq f(\bigcup_{j \in J} A_j)$;
**c.** $f(\bigcap_{j \in J} A_j) \subseteq \bigcap_{j \in J} f(A_j)$; **d.** $\bigcap_{j \in J} f(A_j) \subseteq f(\bigcap_{j \in J} A_j)$;

You may use the fact that the direct image is increasing with its argument: $A \subseteq B \Rightarrow f(A) \subseteq f(B)$ .

*Problem* 0.10. (**Equivalence relations and partial order relations**).

**a.** Let $a, b \in \mathbb{Z}$. State as precisely as possible the definition of $a \mid b$.

**b.** Is the relation $a|b$ **reflexive**? **symmetric**? **antisymmetric**? **transitive**? If true, prove it. If false, give a counterexample.

*Problem* 0.11. (**Functions and equivalence relations**).
Let $f : X \to Y (X, Y \neq \emptyset)$. Prove that $a \sim b \;\Leftrightarrow\; f(a) = f(b)$ is an equivalence relation on $X$.

*Problem* 0.12. (**Continuity**). Let $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. Let $f : ]a, b[ \to ]c, d[$ be bijective and strictly monotone, i.e., strictly increasing or decreasing. Prove that both $f$ and $f^{-1}$ are continuous.

Hint: Use $\varepsilon$–$\delta$ continuity.