Math 330 Section 2 - Fall 2018 - Homework 12

Published: Thursday, October 11, 2018 *Running total:* 44 *points*
Last submission: Friday, November 2, 2018

Update October 22, 2018

Added an amended proof of prop.10.14 (helpful for doing assignment #2).

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

B/G (Beck/Geoghegan) Textbook:
Preface, ch.1 – ch.6, ch.7.1 (only prop.7.9 – prop.7.12), ch.8 – ch.10,
ch.11 until before cor.11.23, ch.12

MF lecture notes:
ch.1 – ch.3, ch.5, ch.6 (skip ch.6.3), ch.7, ch.8 (skip ch.8.3), ch.9, ch.10.1 – ch.10.5
ch.19.7.2 (The Addition Algorithm for Two Nonnegative Numbers (Base 10))
Any "Addenda" subchapters: those will be added to without notice.

B/K lecture notes (optional):
ch.1.1 (Introduction to sets)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions

Other:
- Stewart Calculus 8ed - ch.1.7: "The Precise Definition of a Limit". If you have a newer or older edition then you may have to search through the table of contents and/or consult the index.

New reading assignments:

Reading assignment 1 - due Monday, October 22:

a. Read carefully MF ch.10.6. Skip the proofs of prop.10.23 – 10.25. Thm.10.7: Only read the second proof (the one that does not make use of prop.10.23 – 10.25).
b. Highly recommended for studying ch.10.6: Study the picture between def.10.15 and prop.10.23. You should be able to draw it from memory if you understand the geometry of limsup x_n as the largest possible limit and of liminf x_n as the smallest possible limit among all convergent subsequences $(x_{n_j})_j$ of $(x_n)_n$.

Reading assignment 2 - due: Wednesday, October 24:

a. Read carefully MF ch.10.7. This is very abstract: you look at liminf, limsup and limit of a sequence **of sets** rather than of numbers!
b. Read carefully MF ch.11. up to and including prop.11.5.

Reading assignment 3 - due Friday, October 26:

a. Read carefully the end of MF ch.11, but skip the proof of thm.11.4 (Cantor–Schröder–Bernstein).
b. Read carefully MF ch.10.8. We had to defer this until the proving the uncountability of $\mathbb{R}$.

Written assignment 1: Prove MF prop.10.10 for the case that $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$:

Let $(x_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} x_n \in \mathbb{R}$ exists. Let $K \in \mathbb{N}$. For $n \in \mathbb{N}$ let $y_n := x_{n+K}$. Then $(y_n)_n$ has the same limit.

Written assignment 2: Prove MF prop.10.14.**b**: If y_n is a sequence of real numbers that is nonincreasing, i.e., $y_n \geqq y_{n+1}$ for all n, and bounded below, then $\lim_{n\to\infty} y_n$ exists and coincides with $\inf\{y_n : n \in \mathbb{N}\}$.

Do the proof by modifying the proof of prop.10.14.**a**. You are **NOT ALLOWED** to apply prop.10.14.**a** to the sequence $x_n := -y_n$!

Update October 22, 2018

I found a typo in the proof of prop.10.14.**a**: I wrote $x - \varepsilon \;<\; x - \frac{\varepsilon}{2} \;\leqq\; x_N \;\leqq\; \boxed{j} \;\leqq\; x$ instead of $x - \varepsilon \;<\; x - \frac{\varepsilon}{2} \;\leqq\; x_N \;\leqq\; \boxed{x_j} \;\leqq\; x$. I decided to rewrite the entire proof. The next version of the MF doc will have that updated version and I include it here so you can use it when doing assignment #2.

Proposition 10.14.

- **a.** Let x_n be a sequence of real numbers that is nondecreasing, i.e., $x_n \leqq x_{n+1}$ for all n (see def. **??** on p.**??**), and bounded above. Then $\lim_{n\to\infty} x_n$ exists and coincides with $\sup\{x_n : n \in \mathbb{N}\}$
- **b.** If y_n is a sequence of real numbers that is nonincreasing, i.e., $y_n \geqq y_{n+1}$ for all n, and bounded below, the analogous result is that $\lim_{n\to\infty} y_n$ exists and coincides with $\inf\{y_n : n \in \mathbb{N}\}$.

PROOF of **a**: Let $x := \sup\{x_n : n \in \mathbb{N}\}$. This is an upper bound of the sequence, hence $x_j \leqq x$ for all $j \in \mathbb{N}$. Let $\varepsilon > 0$. $x - \frac{\varepsilon}{2}$ is not an upper bound, hence there exists $N \in \mathbb{N}$ such that $x - \frac{\varepsilon}{2} \leqq x_N$. Because $(x_n)_n$ is nondecreasing, it follows for all $j \geqq N$ that $x - \varepsilon \;<\; x - \frac{\varepsilon}{2} \;\leqq\; x_N \;\leqq\; x_j \;\leqq\; x$, hence

$\varepsilon - x > -x_j \geqq -x$, hence $\varepsilon > x - x_j \geqq 0$ for all $j \geqq N$, hence $|x_j - x| = x - x_j < \varepsilon$ for all $j \geqq N$.

It follows that $\lim_{j\to\infty} x_j = x$, i.e., $x = \sup_{n\in\mathbb{N}} x_n \;=\; \lim_{j\to\infty} x_j$.

The proof of **b** is similar to **a**. (Alternate proof of **b**: apply **a** to the sequence $x_n := -y_n$.) ■