Math 330 Section 6 - Fall 2019 - Homework 15

Published: Thursday, November 14, 2019 *Running total: 53 points*
Last submission: Wednesday, December 4, 2019

Update November 29, 2019

Date for last submission has been pushed back from Monday, 12/2, to Wednesday, 12/4 (campus shutdown)

Status - Reading Assignments: You were asked to complete the following reading so far:

B/G (Beck/Geoghegan) Textbook:
ch.1 – ch.13 (ch.7 only until thm.7.17)

MF lecture notes:
ch.2, ch.3, ch.5 – ch.10, ch.11, ch.12 through ch.12.1.3

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

Stewart Calculus 7ed - ch.1.7: "The Precise Definition of a Limit".

New reading assignments:

Reading assignment 1 - due Monday, November 18:

a. Carefully read MF ch.12.1.4 and ch.12.1.5. Draw plenty of pictures!
b. Skim the contents of the optional MF ch.12.1.6.

Reading assignment 2 - due: Wednesday, November 20:

a. Carefully read the remainder of MF ch.12.1. Draw plenty of pictures!

Reading assignment 3 - due Friday, November 22:

a. Carefully read MF ch.12.2.1 and ch.12.2.2.
b. Skim the contents of MF ch.12.2.3.

Supplementary instructions for reading MF ch.12:

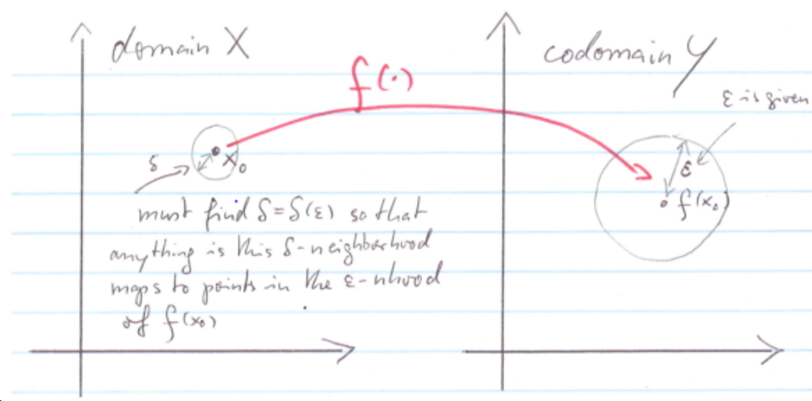
When you read or reread any topics in those chapters then the following is good advice:

a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for
- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)

- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \uplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^{\complement}$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^{\complement}$ and A_1. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
  Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^{\complement}$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.

b. MF ch.12.2: Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 1: ε-δ continuity



The written assignments are on the next page!

Written assignment 1: (3 points!) Prove MF prop.11.13 (Properties of the sup norm): $h \mapsto \|h\|_\infty = \sup\{|h(x)| : x \in X\}$ defines a norm on $\mathscr{B}(X, \mathbb{R})$

This assignment is worth three points: **One point each** for pos.definite, absolutely homogeneous, triangle inequality!

Hint: Go for a treasure hunt in ch.9.2 (Minima, Maxima, Infima and Suprema) and look at the properties of $\sup(A) (A \subseteq \mathbb{R})$ to prove absolute homogeneity and the triangle inequality.

Written assignment 2: (3 points!) Prove MF thm.12.1 (Norms define metric spaces): Let $(V, \|\cdot\|)$ be a normed vector space. Then the function

$$d_{\|\cdot\|}(\cdot,\cdot) : V \times V \to \mathbb{R}_{\geqq 0}; \qquad (x, y) \mapsto d_{\|\cdot\|}(x, y) := \|y - x\|$$

defines a metric space $(V, d_{\|\cdot\|})$.

This assignmemt is worth three points: **One point each** for pos.definite, symmetry, triangle inequality!

Hint for assignment 2: You will have to show for each one of (12.1a), (12.1b), (12.1c) how it follows from def. 11.15: Which one of (11.30a), (11.30b), (11.30c) do you use at which spot?

Each one of the two assignments above is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove that $\|\cdot\|_\infty$ satisfies the triangle inequality (11.27c) of a norm you will have to write something along the following lines:

c. Triangle inequality.
NTS: $d(x, y) \leqq d(x, z) + d(z, y)$ for all $x, y, z \in V$.
Proof: $d(x, y) \overset{(\text{def } d_{\|\cdot\|})}{=} \|y - x\| \overset{(\dots)}{=} \cdots \overset{(\dots)}{\leqq} \cdots \overset{(\dots)}{=} d(x, z) + d(z, y)$

Of course you can also write some or all of your proof over several lines and use lengthier explanations or write the numeric references. For example, def. $d_{\|\cdot\|}$ would be (12.3). Thus the last line of the above can also be written as

Proof: $d(x, y) = \|y - x\|$ (definition of $d_{\|\cdot\|}$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= d(x, z) + d(z, y)$ (.....)

No need to justify properties of the absolute value $|\alpha|$ of a real number α, but you will need to justify why $\sup\{|\alpha f(x)| : x \in X\} = |\alpha| \sup\{|f(x)| : x \in X\}$ and why $\sup\{|f(x) + g(x)| : x \in X\} \leqq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\}$.

An aside: **DO NOT** write $\|f(x)\|$ or even $\|f(x)\|_\infty$ when you deal with the real number $f(x)$!