Math 330 Section 6 - Fall 2020 - Homework 11

Published: Thursday, October 22, 2020 *Running total:* 38 *points*
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Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

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B/G (Beck/Geoghegan) Textbook:
ch.1 - 7.1 (skip the remainder of ch.7), ch.8 - 10, ch.13

MF lecture notes:
ch.2 - 3, ch.5 - 11

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

Other:
Stewart Calculus 7ed - ch.1.7: "The Precise Definition of a Limit". If you have a newer or older edition then you may have to search through the table of contents and/or consult the index.

New reading assignments:

Reading assignment 1 - due Monday, October 26:

a. Read B/G ch.11 and cross–reference with MF ch.9.
b. Read B/G ch.12 and cross–reference with MF ch.9.

Reading assignment 2 - due: Wednesday, October 28:

a. Read B/G ch.13 and cross–reference ch.13.1 - 13.4 with MF ch.7 and 10.
b. Carefully read MF ch.12 until the end of ch.12.1.2. Draw plenty of pictures!

Reading assignment 3 - due Friday, October 30:

a. Carefully read MF ch.12.1.3 - 12.1.5.
a. Carefully read MF ch.12.1.7. The better students also should read ch.12.1.6 without worrying about the proofs except that of Thm.12.5 ($\mathbb{R}^n$ is second countable).

Supplementary instructions for reading MF ch.12:

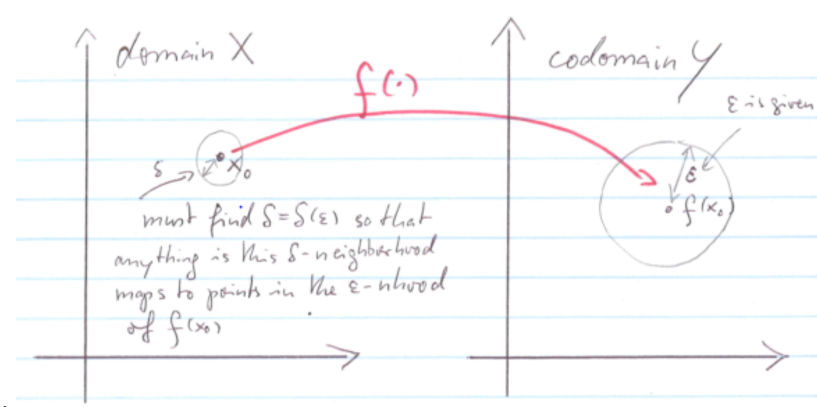
When you read or reread any topics in those chapters then the following is good advice:

a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for

- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \uplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^\complement$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^\complement$ and A_1. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^\complement$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.

b. MF ch.12.2: Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 1: ε-δ continuity



Written assignments on next page.

Written assignment 1:
Prove MF prop.9.17.**b**:

If y_n is a sequence of real numbers that is nonincreasing, i.e., $y_n \geqq y_{n+1}$ for all n, and bounded below, the analogous result is that $\lim_{n\to\infty} y_n$ exists and coincides with $\inf\{y_n : n \in \mathbb{N}\}$.

Do the proof by modifying the proof of prop.9.17.**a**. You are **NOT ALLOWED** to apply prop.9.17.**a** to the sequence $x_n := -y_n$!

Written assignment 2:

Let $f(x) = x^2$. Prove that f is continous at $x_0 = 1$. Do so by use of "ε-δ continuity" which was defined and proven in thm.9.7 to be equivalent to our definition of continuity ("sequence continuity"). Work with formula (9.38) of that theorem since use of formula (9.37) will not get you anywhere!

You are **NOT ALLOWED to work with sequence continuity**, and you cannot use any "advanced" knowledge such as the product of continuous functions being continuous, etc.

Special instructions for assignment 2: Turn in your scratchpaper where you solve for δ (see the hints below). It can be hand-written.

Hints:

a. What does $|x - x_0| < \delta$ and $|f(x) - f(x_0)| < \varepsilon$ translate to in our specific case?

b. This will help a lot: $x^2 - 1 = (x+1)(x-1)$.

c. Do the following on scratch paper: Work your way backward by establishing a relationship between $\varepsilon > 0$ and δ and then "solving for δ" That part should not be in your official proof.

c1. Only small neighborhoods matter: Given $\varepsilon > 0$ try to find δ that works for $0 < \varepsilon < 1$. The justification is given in the last proposition (no tag) at the very end of ch.9.3. zRestrict your search to $\delta < 1$. You can do that since you only need to find ONE $\delta > 0$ that satisfies the inequality

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \text{ for all } x \in \mathbb{R}.$$

c2. Given that $\varepsilon < 1$ and $\delta < 1$, what bounds do you get for $|x^2 - 1|$, $|x+1|$, $|x-1|$?

c3. Put all the above together. Show that you obtain $|f(x) - f(x_0)| \leq 3\delta$?. How then do you choose δ when you consider ε as given? You'll get the answer by "solving for δ".

c4. All of the above was done under the assumption that $\delta < 1$. Satisfy it by replacing δ with $\delta' := \min(\delta, 1)$

d. Only now you are ready to construct an acceptable proof: Let $\varepsilon > 0, \delta := \dots$, and $\delta' := \min(\delta, 1)$. Then .... Thus, if ..... $< \delta'$ then ..... $< \varepsilon$.