Math 330 Section 6 - Fall 2020 - Homework 12

Published: Saturday, October 31, 2020 *Running total:* 42 *points*
Last submission: Friday, November 13, 2020

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

B/G (Beck/Geoghegan) Textbook:
ch.1 - 7.1 (skip the remainder of ch.7), ch.8 - 13

MF lecture notes:
ch.2 - 3, ch.5 - 12.1.7

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

Other:
Stewart Calculus 7ed - ch.1.7: "The Precise Definition of a Limit". If you have a newer or older edition then you may have to search through the table of contents and/or consult the index.

New reading assignments:

Draw plenty of pictures when studying the material of ch.12 and ch.13!

Reading assignment 1 - due Monday, November 2:

- **a.** None – Prepare for the midterm!

Reading assignment 2 - due: Wednesday, November 4:

- **a.** Reread MF ch.12.1.7 for a second time, as there is much subtlety! Work example 12.5 closed book a day after you have studied it!
- **b.** Carefully read MF ch.12.1.8 and 12.1.9.
- **b.** Carefully read MF ch.12.1.10 through theorem 12.9 (Completeness of $\mathbb{R}^n$).

Reading assignment 3 - due Friday, November 6:

- **a.** Carefully read the remainder of MF ch.12.1.10.
- **b.** Carefully read MF ch.12.2.1. Convince yourself that the definitions of ε–δ continuity and sequence continuity are compatible with those of MF ch.9.3.
- **c.** Carefully read B/G appendix A (Continuity and Uniform Continuity).

Supplementary instructions for reading MF ch.12:

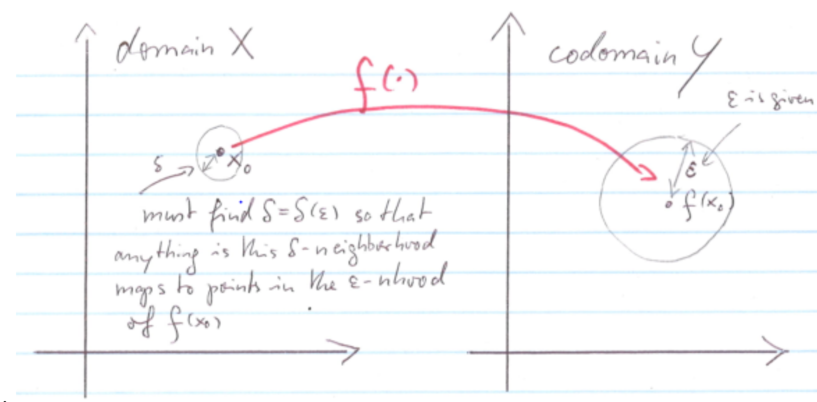
When you read or reread any topics in those chapters then the following is good advice:

a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for

- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \uplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^{\complement}$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^{\complement}$ and A_1. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^{\complement}$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.

b. MF ch.12.2: Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 1: ε-δ continuity



Written assignments on next page.

Written assignment 1:
Prove MF thm. 9.8: If $m \in [0, \infty[_{\mathbb{Z}}$ is not a perfect square then $\sqrt{m}$ is irrational.

Written assignment 2: (3 points!) Prove MF prop.11.13 (Properties of the sup norm): $h \mapsto \|h\|_\infty = \sup\{|h(x)| : x \in X\}$ defines a norm on $\mathscr{B}(X, \mathbb{R})$

This assignment is worth three points: **One point each** for pos.definite, absolutely homogeneous, triangle inequality!

> The **Latex** code for writing, e.g., $\|h\|_\infty$ is $\backslash Vert\, h \,\backslash Vert\}\backslash\_\{\backslash infty\}$. Of course that has to be enclosed in \$ ..... \$ since its math and not ordinary text. But I won't insist: I'll allow you drop the ∞ subscript and simply write $\|h\|$ for the sup–norm. Note though that I keep writing $\|h\|_\infty$ in the remainder of this assignment.
>
> It is also OK if you use $||$ (two absolute value bars) instead of the $\backslash Vert$ delimiter (BIG V!) for norms, as long as that stuff is written in math mode (enclosed in \$ .... \$)
>
> Here is the difference: Math mode: $||x + y||$. Text mode: | | x + y | |.

Hint: Go for a treasure hunt in ch.9.2 (Minima, Maxima, Infima and Suprema), and look at the properties of $\sup(A) (A \subseteq \mathbb{R})$ to prove absolute homogeneity and the triangle inequality.

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove that $\|\cdot\|_\infty$ satisfies the triangle inequality (11.27c) of a norm you will have to write something along the following lines:

c. Triangle inequality.
NTS: $\|f + g\|_\infty \leqq \|f\|_\infty + \|g\|_\infty$ for all $f, g \in \mathscr{B}(X, \mathbb{R})$.

Proof:

$\|f + g\|_\infty = \sup\{|f(x)| + |g(x)| : x \in X\}$ (definition of $\|\cdot\|_\infty$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= \|f\|_\infty + \|g\|_\infty$ (.....)

(There may be fewer or more steps in your proof. The above only serves as an illustration!)

No need to justify properties of the absolute value $|\alpha|$ of a real number α, but you will need to justify why

$$\sup\{|\alpha f(x)| : x \in X\} = |\alpha| \sup\{|f(x)| : x \in X\},$$

and why

$$\sup\{|f(x) + g(x)| : x \in X\} \leqq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\}.$$

> An aside: **DO NOT** write $\|f(x)\|_\infty$ when you deal with the real number $f(x)$ (and **you probably mean** the absolute value $|f(x)|$).
>
> $\|\cdot\|_\infty$ is defined for functions f, NOT for numbers $f(x)$!