Math 330 Section 4 - Fall 2021 - Homework 14

Published: Sunday, November 7, 2021 *Running total:* 49 + __ *points*
Last submission: Wednesday, December 1, 2021

Status - previously assigned reading Assignments:

B/G (Beck/Geoghegan) Textbook:
ch.1-7 (until Theorem 7.17), ch.8-11, ch.13.1

MF lecture notes:
ch.2-3, ch.4 (skim), ch.5-11

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

New reading assignments:

Reading assignment 1 - due: Monday, November 15:

a. Read B/G ch.12. Only skim the proofs.
b. Carefully read B/G ch.13.2.

Reading assignment 2 - due: Wednesday, November 17:

a. Carefully read B/G ch.13.3-13.4.
b. Skim or skip G/G ch.13.5.
a. Carefully read MF ch.12.1.1-12.1.2. That's a lot of pages but there are very few (but all of them very important!) definitions and theorems. Understanding all of the material about metric spaces is **very relevant for the final exam!**

Reading assignment 3 - due Friday, November 19:

a. Carefully read MF ch.12.1.3-12.1.5

Be sure to read pages 2 and 3!

Supplementary instructions for reading MF ch.12:

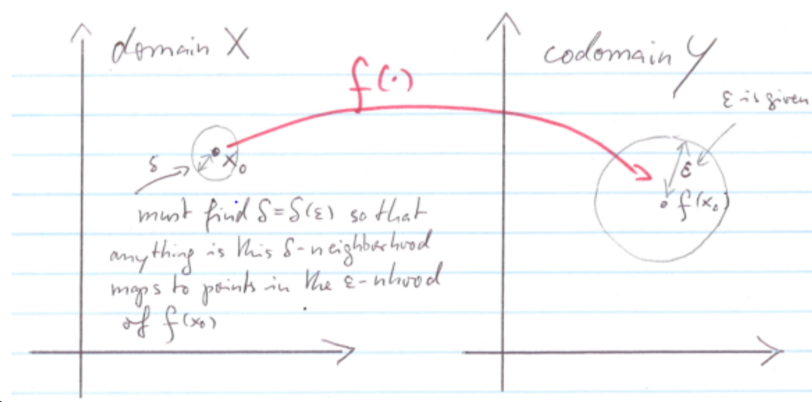
When you read or reread any topics in those chapters then the following is good advice:

a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for

- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \biguplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^{\complement}$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^{\complement}$ and A_1. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
  Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^{\complement}$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.

b. MF ch.12.2: Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 1: ε-δ continuity



Written assignments on next page.

Written assignment 1:
Prove MF thm. 9.8: If $m \in [0, \infty[_{\mathbb{Z}}$ is not a perfect square then $\sqrt{m}$ is irrational.

Written assignment 2: (3 points!) Prove MF prop.11.13 (Properties of the sup norm): $h \mapsto \|h\|_\infty = \sup\{|h(x)| : x \in X\}$ defines a norm on $\mathscr{B}(X, \mathbb{R})$

This assignment is worth three points: **One point each** for pos.definite, absolutely homogeneous, triangle inequality!

Hint: Go for a treasure hunt in ch.9.2 (Minima, Maxima, Infima and Suprema), and look at the properties of $\sup(A)(A \subseteq \mathbb{R})$ to prove absolute homogeneity and the triangle inequality.

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove that $\|\cdot\|_\infty$ satisfies the triangle inequality (11.27c) of a norm you will have to write something along the following lines:

c. Triangle inequality.
NTS: $\|f+g\|_\infty \leqq \|f\|_\infty + \|g\|_\infty$ for all $f, g \in \mathscr{B}(X, \mathbb{R})$.

Proof:

$\|f+g\|_\infty = \sup\{|f(x)| + |g(x)| : x \in X\}$ (definition of $\|\cdot\|_\infty$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= \|f\|_\infty + \|g\|_\infty$ (.....)

(There may be fewer or more steps in your proof. The above only serves as an illustration!)

No need to justify properties of the absolute value $|\alpha|$ of a real number α, but you will need to justify why

$$\sup\{|\alpha f(x)| : x \in X\} = |\alpha| \sup\{|f(x)| : x \in X\},$$

and why

$$\sup\{|f(x) + g(x)| : x \in X\} \leqq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\}.$$

An aside: **DO NOT** write $\|f(x)\|_\infty$ when you deal with the real number $f(x)$ (and **you probably mean** the absolute value $|f(x)|$).

$\|\cdot\|_\infty$ is defined for functions f, NOT for numbers $f(x)$!