Math 330 Section 4 - Fall 2021 - Homework 15

Published: Tuesday, November 16, 2021 *Running total: 57 points*
Last submission: Friday, December 3, 2021

Status - previously assigned reading Assignments:

B/G (Beck/Geoghegan) Textbook:
ch.1-7 (until Theorem 7.17), ch.8-13

MF lecture notes:
ch.2-3, ch.4 (skim), ch.5-12.1.8

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

New reading assignments:

Reading assignment 1 - due: Monday, November 22:

a. Carefully read the remainder of MF ch.12.1.
b. Carefully read MF ch.12.2.1. Understand how this is consistent with the material about continuity that was given in ch.9. and with what you may bave learned about continuous functions in multivariable calculus.

Reading assignment 2 - due: Wednesday, November 24:

a. Carefully read the remainder of MF ch.12.2.

Reading assignment 3 - due Friday, November 26:

a. Carefully read MF ch.12.3.1.
b. Carefully read MF ch.12.3.2 until before Proposition 12.48.

Be sure to read pages 2 (unchanged) and 3 (written assignments)!

Supplementary instructions for reading MF ch.12:

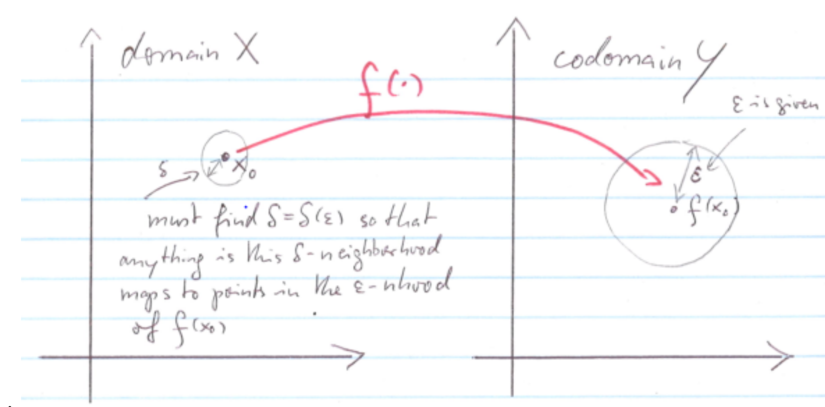
When you read or reread any topics in those chapters then the following is good advice:

a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for

- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \uplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^{\complement}$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^{\complement}$ and A_1. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
  Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^{\complement}$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.

b. MF ch.12.2: Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 1: ε-δ continuity



Written assignments on next page.

Written assignment 1 (3 points):
Prove MF thm.12.1 (Norms define metric spaces): Let $(V, \|\cdot\|)$ be a normed vector space. Then the function

$$d_{\|\cdot\|}(\cdot,\cdot) : V \times V \to \mathbb{R}_{\geqq 0}; \qquad (x,y) \mapsto d_{\|\cdot\|}(x,y) := \|y - x\|$$

defines a metric space $(V, d_{\|\cdot\|})$.

This assignment is worth three points: **One point each** for pos.definite, symmetry, triangle inequality!

Hint: You will have to show for each one of (12.1a), (12.1b), (12.1c) how it follows from def. 11.15: Which one of (11.31a), (11.31b), (11.31c) do you use at which spot?

Careful with symmetry: What is the reason that $\|a - b\| = \|b - a\|$?

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove, e.g., that $d_{\|\cdot\|}(\cdot,\cdot)$ satisfies the triangle inequality (12.1c) of a metric you will have to write something along the following lines:

c. Triangle inequality.
NTS: $d_{\|\cdot\|}(x,z) \leqq d_{\|\cdot\|}(x,y) + d_{\|\cdot\|}(y,z)$ for all $x, y, z \in X$.

Proof:

$d_{\|\cdot\|}(x,z) = \|z - x\|$ (definition of the metric $d_{\|\cdot\|}$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= d_{\|\cdot\|}(x,y) + d_{\|\cdot\|}(y,z)$ (.....)

(There may be fewer or more steps in your proof. The above only serves as an illustration!)

Written assignment 2: Let $A := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ be the first quadrant in the plane (the points on the coordinate axes are excluded). Prove that each element of A is an inner point, i.e., A is open in $\mathbb{R}^2$.

Hint: Find for $\vec{a} = (a_1, a_2)$ small enough ε such that $N_\varepsilon(\vec{a}) \subseteq A$