Math 330 Section 4 - Fall 2021 - Homework 16

Published: Tuesday, November 26, 2021 *Running total:* 61 *points*
Last submission: Friday, December 10, 2021

Status - previously assigned reading Assignments:

B/G (Beck/Geoghegan) Textbook:
ch.1-7 (until Theorem 7.17), ch.8-13

MF lecture notes:
ch.2-3, ch.4 (skim), ch.5-ch.12.3.2 until before Proposition 12.48

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

New reading assignments:

Reading assignment 1 - due: Monday, November 29:

- **a.** Carefully read the remainder of ch.12.3. Note that this assignment is relatively brief since you can skip lemma 12.2 and the very lengthy proof of Riemann's Rearrangement Theorem.
- **b.** Carefully read MF ch.13.1 - 13.3. This also is a brief assignment since you can skip Proposition 13.2 (ε–nets in $\mathbb{R}^n$). Draw pictures as you do this, especially when reading the proofs of thm.13.1 and thm.13.2.

Reading assignment 2 - due: Wednesday, December 1:

- **a.** Carefully read MF ch.13.4 - 13.5. Some of this material is not easy to grasp. Do your best to understand the proofs at least to some extent.

Reading assignment 3 - due Friday, December 3:

- **a.** Carefully read the remainder of MF ch.13.
- **b.** Carefully read MF ch.14.1 - 14.4. That's less than 4 1/2 pages and contains some very important stuff.

Written assignments on next page.

The first two written assignments of this homework set require you to work with sequence compact metric spaces. You can solve those problems with your knowledge of convergence in metric spaces but you also need to work with the following definition from ch.13.4 (Sequence Compactness):

We say that the metric space (X, d) is **sequence compact** if it has the following property: Given any sequence (x_n) of elements of X, there exists $x \in X$ and a subsequence $(x_{n_j})_j$ of $(x_n)_n$ such that $\lim\limits_{j\to\infty} x_{n_j} = x$.

Written assignment 1:
Let X be the open unit interval $]0, 1[$, equipped with the Euclidean metric $d(x, x') = |x' - x|$. Prove that X is **not** sequence compact by finding a sequence $x_n \in]0, 1[$ for which no subsequence possesses a limit in]0,1[.

Written assignment 2:
Let X be a (abstract) finite and nonempty set, equipped with the discrete metric. Prove that X is sequence compact.

Written assignment 3 (2 points):
Let $X := \mathbb{R}$ equipped with the standard Euclidean metric $d(x, x') = |x - x'|$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the following sequence of functions:

$$f_n(x) \;:=\; \begin{cases} 0 & \text{if } |x| > \frac{1}{n}, \\ nx + 1 & \text{if } \frac{-1}{n} \leqq x \leqq 0, \\ -nx + 1 & \text{if } 0 \leqq x \leqq \frac{1}{n}, \end{cases}$$

i.e., the point $\big(x, f_n(x)\big)$ is on the straight line between $(-\frac{1}{n}, 0)$ and $(0, 1)$ for $\frac{-1}{n} \leqq x \leqq 0$, it is on the straight line between $(0, 1)$ and $(\frac{1}{n}, 0)$ for $0 \leqq x \leqq \frac{-1}{n}$, and it is on the x–axis for all other x. Draw a picture! Let $f(x) := 0$ for $x \neq 0$ and $f(0) := 1$.

- **a.** Prove that f_n converges pointwise to f on $\mathbb{R}$.
- **b.** Prove that f_n does not converge uniformly to f on $\mathbb{R}$. $\square$

You may use without proof that each of the functions f_n is continuous on $\mathbb{R}$.

One point each for (a) and (b)!

Hint for part **(a)**: For fixed $x \neq 0$, what happens eventually, i.e., for big enough n? Example (NOT legit as a proof): If x = 0.01, what happens if $n > 1000$? Thus $\lim\limits_{n\to\infty} f_n(0.01)$ = WHAT?