Math 330 Section 5 - Spring 2022 - Homework 08

Published: Thursday, March 3, 2022 *Running total:* 35 *points*
Last submission: Friday, March 25(!), 2022

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

MF lecture notes:
ch.2.1 - 2.4, ch.3, ch.4 (skim), ch.5 - 7.2 (skip 6.5)

B/G (Beck/Geoghegan) Textbook:
ch.1 - 7.1

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

New reading assignments:

Reading assignment 1 - due Monday, March 7:

- Prepare for the midterm.

Reading assignment 2 - due: Wednesday, March 9:

- Read carefully the remainder of MF ch.7.

Reading assignment 3 - due Friday, March 11:

a. Read carefully MF ch.8.1. Be sure you understand the connection between arbitrary intersections and the $\forall$ quantifier and the connection between arbitrary unions and the $\exists$ quantifier.
b. Skim MF ch.8.2.
c. Read carefully MF ch.8.3.

General note on written assignments: Unless expressly stated otherwise, to prove a proposition or theorem you are allowed to make use of everything in the book up to but NOT including the specific item you are asked to prove.

Written assignment 1:
Prove prop.6.10**(c)**: Let $\beta \in (R, \oplus, \odot, P)$ and $k, m \in [0, \infty[_{\mathbb{Z}}$. Then $(\beta^m)^k \ = \ \beta^{mk}$.

Hint: Use induction on k.

#2 and #3 are about proving MF thm.6.9 (Division Algorithm for Integers): Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. There exists a unique combination of two integers q ("quotient") and r ("remainder") such that

$$m \ = \ n \cdot q \ + \ r \quad \text{and } 0 \leq r < n.$$

Do not use induction for assignments 2 and 3. It would only make your task more difficult!

Written assignment 2:

Prove uniqueness of the "decomposition" $m = qn + r$ such tbat $0 \leq r < n$: If you have a second such decomposition $m = \tilde{q}n + \tilde{r}$ then show that this implies $q = \tilde{q}$ and $r = \tilde{r}$. Start by assuming that $r \neq \tilde{r}$ which means that one of them is smaller than the other and take it from there.

Written assignment 3:

Much harder than #2: Prove the existence of q and r.

Hints for #3: Review the Extended Well–Ordering principle MF thm.6.8. Its use will give the easiest way to prove this assignment: Let

$$A \ := \ A(m,n) \ := \ \{r' \in [0, \infty[_{\mathbb{Z}} \colon r' = m - q'n \text{ for some } q' \in \mathbb{Z}\}\,.$$

Show that $A \neq \emptyset$ by separately examining the cases

- $m \geqq 0$ (easy)
- $m < 0$ (probably the hardest part of the proof!)

Now you can apply the Extended Well–Ordering principle to the set A. What can you do with $\min(A)$?

Hint for both #2 and #3: MF prop. 3.61 and cor.3.5 at the end of ch.3.5 will come in handy in connection with using or proving $0 \leqq r < n$. They assert for the ordered integral domain $(\mathbb{Z}, +, \cdot, \mathbb{N})$ the following.

If $a, b \in [0, n[_{\mathbb{Z}}$ then

$$|a - b| \ \leqq \ \max(a,b), \quad \text{i.e.,} \tag{3.46}$$
$$-\max(a,b) \ \leqq \ a - b \ \leqq \ \max(a,b), \tag{3.47}$$
$$-n \ < \ a - b \ < \ n. \tag{3.48}$$