Math 330 Section 5 - Spring 2022 - Homework 14

Published: Tuesday, April 19, 2022 *Running total: 56 points*
Last submission: Friday, May 6, 2022

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

MF lecture notes:
ch.2.1 - 2.4, ch.3, ch.4 (skim), ch.5 - 8 (skip 6.5, skim 8.2, 8.4 after Prop.8.11, ch.8.5), ch.9 skim/skip after Prop.9.44), ch.10.1-10.2, ch.11, ch.14.1, ch.14.3

B/G (Beck/Geoghegan) Textbook:
ch.1 - 7.1, ch.8-11, ch.13

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

Other:
Stewart Calculus 9ed - ch.1.7: "The Precise Definition of a Limit". If you have a newer or older edition then you may have to search through the table of contents and/or consult the index.

New reading assignments:

Reading assignment 1 - due Monday, April 25:

- **a.** Carefully read MF ch.12.1.1-12.1.4. That's a lot of pages but there are very few (but all of them very important!) definitions and theorems in ch.12.1.1 and ch.12.1.2. Understanding all of the material about metric spaces is **very relevant for the final exam!**

Reading assignment 2 - due Wednesday, April 27:

- **a.** Carefully read MF ch.12.1.5.
- **b.** Skim MF ch.12.1.6. None of this will be on any quiz or the final exam. The better students should read the material more carefully and try to truly understand Theorem 12.5.
- **c.** Carefully read MF ch.12.1.7–12.1.8. Be sure to do Example 12.5
with paper and pen and redo it closed book after a day or two! Definition 12.22 (topological subspaces) is marked optional, but note that I will use that concept in lecture, so you should remember the core of it even though you will not be asked to write it down.

Reading assignment 3 - due Friday, April 29:

- **a.** Carefully read the remainder of MF ch.12.1.
- **b.** Carefully read MF ch.12.2.1. Understand how this is consistent with the material about continuity that was given in ch.9 and with what you may bave learned about continuous functions in multivariable calculus.

Be sure to read pages 2 and 3!

Supplementary instructions for reading MF ch.12:

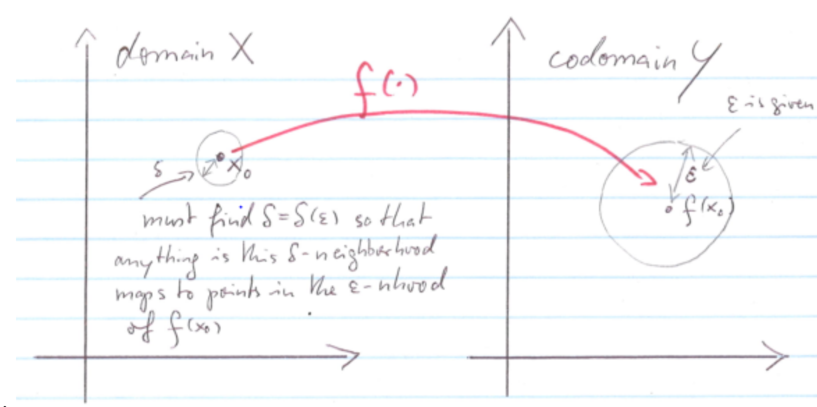
When you read or reread any topics in those chapters then the following is good advice:

a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for

- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \uplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^\complement$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^\complement$ and A_1. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
  Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^\complement$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.

b. MF ch.12.2 (Continuity): Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 1: ε-δ continuity



Written assignments on pages 3 and 4!

Written assignment 1: (3 points!) Prove MF prop.11.13 (Properties of the sup norm): $h \mapsto \|h\|_\infty = \sup\{|h(x)| : x \in X\}$ defines a norm on $\mathscr{B}(X, \mathbb{R})$

This assignment is worth three points: **One point each** for pos.definite, absolutely homogeneous, triangle inequality!

Hint: Go for a treasure hunt in ch.9.2 (Minima, Maxima, Infima and Suprema), and look at the properties of $\sup(A)(A \subseteq \mathbb{R})$ to prove absolute homogeneity and the triangle inequality.

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove that $\|\cdot\|_\infty$ satisfies the triangle inequality (11.27c) of a norm you will have to write something along the following lines:

c. Triangle inequality.
NTS: $\|f + g\|_\infty \leqq \|f\|_\infty + \|g\|_\infty$ for all $f, g \in \mathscr{B}(X, \mathbb{R})$.

Proof:

$\|f + g\|_\infty = \sup\{|f(x)| + |g(x)| : x \in X\}$ (definition of $\|\cdot\|_\infty$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= \|f\|_\infty + \|g\|_\infty$ (.....)

(There may be fewer or more steps in your proof. The above only serves as an illustration!)

No need to justify properties of the absolute value $|\alpha|$ of a real number α, but you will need to justify why

$$\sup\{|\alpha f(x)| : x \in X\} = |\alpha| \sup\{|f(x)| : x \in X\},$$

and why

$$\sup\{|f(x) + g(x)| : x \in X\} \leqq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\}.$$

> An aside: **DO NOT** write $\|f(x)\|_\infty$ when you deal with the real number $f(x)$ (and **you probably mean** the absolute value $|f(x)|$).
>
> $\|\cdot\|_\infty$ is defined for functions f, NOT for numbers $f(x)$!

Written assignment 2 (3 points):
Prove MF thm.12.1 (Norms define metric spaces): Let $(V, \|\cdot\|)$ be a normed vector space. Then the function

$$d_{\|\cdot\|}(\cdot,\cdot) : V \times V \to \mathbb{R}_{\geqq 0}; \qquad (x,y) \mapsto d_{\|\cdot\|}(x,y) := \|y - x\|$$

defines a metric space $(V, d_{\|\cdot\|})$.

This assignment is worth three points: **One point each** for pos.definite, symmetry, triangle inequality!

Hint: You will have to show for each one of (12.1a), (12.1b), (12.1c) how it follows from def. 11.15: Which one of (11.31a), (11.31b), (11.31c) do you use at which spot?

Careful with symmetry: What is the reason that $\|a - b\| = \|b - a\|$?

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove, e.g., that $d_{\|\cdot\|}(\cdot,\cdot)$ satisfies the triangle inequality (12.1c) of a metric you will have to write something along the following lines:

c. Triangle inequality.
NTS: $d_{\|\cdot\|}(x,z) \leqq d_{\|\cdot\|}(x,y) + d_{\|\cdot\|}(y,z)$ for all $x, y, z \in X$.

Proof:

$d_{\|\cdot\|}(x,z) = \|z - x\|$ (definition of the metric $d_{\|\cdot\|}$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= d_{\|\cdot\|}(x,y) + d_{\|\cdot\|}(y,z)$ (.....)

(There may be fewer or more steps in your proof. The above only serves as an illustration!)