# Math 330 Section 5 - Spring 2022 - Homework 15

*Published: Tuesday, April 26, 2022* *Running total:* 60 *points*
*Last submission: Tuesday, May 10 (finals week!), 2022*

***Update May 3, 2022***

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| *Assignment #3(b) was changed since I cannot quickly enough get to uniform convergence in lecture.* |

**Status - Reading Assignments:**

Here is the status of the reading assignments you were asked to complete by this date.

MF lecture notes:
ch.2.1 - 2.4, ch.3, ch.4 (skim), ch.5 - 8 (skip 6.5, skim 8.2, 8.4 after Prop.8.11, ch.8.5), ch.9 skim/skip after Prop.9.44), ch.10.1-10.2, ch.11, ch.12.1.1-12.2.1, ch.14.1, ch.14.3

B/G (Beck/Geoghegan) Textbook:
ch.1 - 7.1, ch.8-11, ch.13

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

Other:
Stewart Calculus 9ed - ch.1.7: "The Precise Definition of a Limit". If you have a newer or older edition then you may have to search through the table of contents and/or consult the index.

**New reading assignments:**

**Reading assignment 1 - due Monday, April 25:**

- **a.** Carefully read MF ch.12.2.2 and skim MF ch.12.2.3.
- **b.** Carefully read MF ch.12.3.1.

**Reading assignment 2 - due Wednesday, April 27:**

- **a.** Carefully read MF ch.12.3.2. through Theorem 12.19.
- **b.** Skim the remainder of MF ch.12.3.2 but skip Lemma 12.2 and the proof of the Riemann Rearrangement Theorem. Note that this assignment is relatively brief since you can skip Lemma 12.2 and the very lengthy proof of Riemann's Rearrangement Theorem.
- **c.** Carefully read MF ch.13.1.

**Reading assignment 3 - due Friday, April 29:**

- **a.** Carefully read MF ch.13.2 - ch.13.4 but skip the proof of Proposition 13.2. Draw pictures as you do this, especially when reading the proofs of thm.13.1 and thm.13.2. and you try to visualize the assertions of ch.13.4 in 2–dimensional space.

**Written assignments on page 2.**

The first two written assignments of this homework set require you to work with sequence compact metric spaces. You can solve those problems with your knowledge of convergence in metric spaces but you also need to work with the following definition from ch.13.4 (Sequence Compactness):

We say that the metric space $(X, d)$ is **sequence compact** if it has the following property: Given any sequence $(x_n)$ of elements of $X$, there exists $x \in X$ and a subsequence $(x_{n_j})_j$ of $(x_n)_n$ such that $\lim\limits_{j\to\infty} x_{n_j} = x$.

**Written assignment 1:**
Let $X$ be the open unit interval $]0,1[$, equipped with the Euclidean metric $d(x,x') = |x' - x|$. Prove that $X$ is **not** sequence compact by finding a sequence $x_n \in ]0,1[$ for which no subsequence possesses a limit in ]0,1[.

**Written assignment 2:**
Let $X$ be a (abstract) finite and nonempty set, equipped with the discrete metric. Prove that $X$ is sequence compact.

**Written assignment 3 (2 points):**
Let $X := \mathbb{R}$ equipped with the standard Euclidean metric $d(x,x') = |x - x'|$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the following sequence of functions:

$$f_n(x) \;:=\; \begin{cases} 0 & \text{if } |x| > \frac{1}{n}, \\ nx + 1 & \text{if } \frac{-1}{n} \leqq x \leqq 0, \\ -nx + 1 & \text{if } 0 \leqq x \leqq \frac{1}{n}, \end{cases}$$

i.e., the point $\big(x, f_n(x)\big)$ is on the straight line between $(-\frac{1}{n}, 0)$ and $(0,1)$ for $\frac{-1}{n} \leqq x \leqq 0$, it is on the straight line between $(0,1)$ and $(\frac{1}{n}, 0)$ for $0 \leqq x \leqq \frac{-1}{n}$, and it is on the $x$–axis for all other $x$. Draw a picture! Let $f(x) := 0$ for $x \neq 0$ and $f(0) := 1$.

- **a.** Prove that $f_n$ converges pointwise to $f$ on $\mathbb{R}$. In other words, prove that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$.
- **b.** **REPLACED on 5/3/2022:** Prove Proposition 12.28**(d)**: Let $A$ and $B$ be subsets of a topological space $(X, \mathfrak{U})$. Then $\overline{A \cup B} = \bar{A} \cup \bar{B}$. Hint: OK to use parts **(b)** and **(c)** of that proposition. $\square$

**One point each for (a) and (b)**!

**Hint** for part **(a)**: For fixed $x \neq 0$, what happens eventually, i.e., for big enough $n$? Example (NOT legit as a proof): If $x$ = $0.01$, what happens if $n > 1000$? Thus $\lim\limits_{n\to\infty} f_n(0.01)$ = WHAT?