Math 330 Section 5 - Fall 2022 - Homework 09

Published: Saturday, October 1, 2022 *Running total: 34 points*
Last submission: Wednesday, October 19, 2022 *(Extended from Friday, Oct 14)*

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

MF lecture notes:
ch.2 – ch.3; skim ch.4; ch.5.1 – ch.5.2; ch.6 - 8.4

B/G (Beck/Geoghegan) Textbook:
ch.1 – ch.6; skim ch.7

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

New reading assignments:

Reading assignment 1 - due Monday, October 10:

- **a.** Skim MF ch.8.5 but visualize the definition of an indicator function and the bijectivity between 2^Ω and $\{0,1\}^\Omega$.
- **b.** Read carefully MF ch.9.1.

Reading assignment 2 - due: Wednesday, October 12:

- **Prepare for midterm 1!**

Reading assignment 3 - due Friday, October 14:

- **a.** Read carefully MF ch.9.2.
- **b.** Read extra carefully MF ch.9.3 through Proposition 9.17 (Rules of arithmetic for limits).

The written assignments are about proving MF thm.6.9 (Division Algorithm for Integers): Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. There exists a unique combination of two integers q ("quotient") and r ("remainder") such that

$$m = n \cdot q + r \quad \text{and } 0 \leq r < n.$$

Do not use induction for assignments 2 and 3. It would only make your task more difficult!

Written assignment 1:

Prove uniqueness of the "decomposition" $m = qn + r$ such tbat $0 \leq r < n$: If you have a second such decomposition $m = \tilde{q}n + \tilde{r}$ then show that this implies $q = \tilde{q}$ and $r = \tilde{r}$. Start by assuming that $r \neq \tilde{r}$ which means that $|r - \tilde{r}| \geqq \ldots$ and find a contradiction. More hints further down!

Written assignment 2:

Much harder than #1: Prove the existence of q and r.

Hints for #2: Review the Extended Well–Ordering principle MF thm.6.8. Its use will give the easiest way to prove this assignment: Let

$$A \ := \ A(m,n) \ := \ \{r' \in [0,\infty[_{\mathbb{Z}} : \ r' = m - q'n \text{ for some } q' \in \mathbb{Z}\}\,.$$

Show that $A \neq \emptyset$ by separately examining the cases

- $m \geqq 0$ (easy)
- $m < 0$ (probably the hardest part of the proof!)

Now you can apply the Extended Well–Ordering principle to the set A. What can you do with $\min(A)$?

Hint for both #1 and #2: MF prop. 3.61 and cor.3.5 at the end of ch.3.5 will come in handy in connection with using or proving $0 \leqq r < n$. They assert for the ordered integral domain $(\mathbb{Z},+,\cdot,\mathbb{N})$ the following.

If $a,b \in [0,n[_{\mathbb{Z}}$, then $|a-b| \ \leqq \ \max(a,b)$, i.e., $-n \ < \ a-b \ < \ n$.