Math 330 Section 5 - Fall 2022 - Homework 15

Published: Monday, November 14, 2022 *Running total: 55 points*
Last submission: Monday, December 5, 2022

Update December 3, 2022

Last submission date was moved from Fri Dec 2 to Mon Dec 5

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

MF lecture notes:
ch.2 – ch.3; skim ch.4; ch.5.1 – ch.5.2; ch.6 - 12; ch.13.1.1 – 13.1.3

B/G (Beck/Geoghegan) Textbook:
ch.1 – ch.6; skim ch.7; ch.8, ch.10.1 – 11.2; skip ch.11.3; ch.13.1 – 13.4

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

New reading assignments:

Reading assignment 1 - due Monday, November 21:

a. Carefully read MF ch.13.2.1 and 13.2.2 through Theorem 13.7. Skim the remainder of ch.13.2.2.

Reading assignment 2 - due: Wednesday, November 23:

- Carefully read MF ch.15.1 and 15.3 and skim ch.15.2.

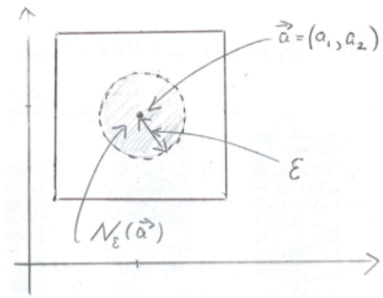
Reading assignment 3 - due Friday, November 25:

a. Carefully read MF ch.14.1 through Proposition 14.1 (but skip its optional proof).

Written assignments on the next page.

Written assignment 1: Let $A := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ be the first quadrant in the plane (the points on the coordinate axes are excluded). Prove that each element of A is an inner point, i.e., A is open in $\mathbb{R}^2$.

Hint: Find for $\vec{a} = (a_1, a_2)$ small enough ε such that $N_\varepsilon(\vec{a}) \subseteq A$ The drawing shows that $\varepsilon = \min(a_1, a_2)$ works, but I want you to be precise with inequalities to prove this.



Written assignment 2 (2 points):
Let $X := \mathbb{R}$ equipped with the standard Euclidean metric $d(x, x') = |x - x'|$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the following sequence of functions:

$$f_n(x) \;:=\; \begin{cases} 0 & \text{if } |x| > \frac{1}{n}, \\ nx + 1 & \text{if } \frac{-1}{n} \leqq x \leqq 0, \\ -nx + 1 & \text{if } 0 \leqq x \leqq \frac{1}{n}, \end{cases}$$

i.e., the point $\big(x, f_n(x)\big)$ is on the straight line between $(-\frac{1}{n}, 0)$ and $(0, 1)$ for $\frac{-1}{n} \leqq x \leqq 0$, it is on the straight line between $(0, 1)$ and $(\frac{1}{n}, 0)$ for $0 \leqq x \leqq \frac{-1}{n}$, and it is on the x–axis for all other x. Draw a picture! Let $f(x) := 0$ for $x \neq 0$ and $f(0) := 1$.

- **a.** Prove that f_n converges pointwise to f on $\mathbb{R}$. In other words, prove that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$.
- **b.** Prove that f_n does not converge uniformly to f on $\mathbb{R}$. You may use without proof that each of the functions f_n is continuous on $\mathbb{R}$.

One point each for (a) and (b)!

Hint for part **(a)**: For fixed $x \neq 0$, what happens eventually, i.e., for big enough n? Example (NOT legit as a proof): If x = 0.01, what happens if $n > 1000$? Thus $\lim\limits_{n\to\infty} f_n(0.01)$ = WHAT?