Math 330 Section 5 - Fall 2023 - Homework 10

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Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete before the first one of this HW.

MF lecture notes:
ch.2.1 – 2.7, ch.3, skim ch.4 (optional), ch.5 - 8.4, ch.9.1

B/G (Beck/Geoghegan) Textbook (optional, EXCEPT for ch.3 on logic):
ch.1 – 3, ch.5 – 7

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

New reading assignments:

Reading assignment 1 - due Monday, October 16:

- **a.** Skim or skip the remainder of MF ch.8 (that's ch.8.5)
- **b.** Carefully read MF ch.9.2.
- **b.** Carefully read MF ch.9.3 through Proposition 9.17 (Rules of arithmetic for limits).

Reading assignment for the break (optional):

- **a.** Catch up on the topics you were not able to study thoroughly enough.
- **b.** If you have not taken or are not currently taking linear algebra, this would be a good time to look at MF ch.11 through Example 11.11. That's nine pages of very easy material.

Written assignments are on the next page.

The written assignments are about proving MF thm.6.9 (Division Algorithm for Integers): Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. There exists a unique combination of two integers q ("quotient") and r ("remainder") such that

$$m \;=\; n \cdot q \;+\; r \quad \text{and } 0 \leq r < n.$$

Do not use induction for assignments 2 and 3. It would only make your task more difficult!

Written assignment 1:

Prove uniqueness of the "decomposition" $m = qn + r$ such that $0 \leq r < n$: If you have a second such decomposition $m = \tilde{q}n + \tilde{r}$ then show that this implies $q = \tilde{q}$ and $r = \tilde{r}$. Start by equating $qn+r \;=\; m \;=\; \tilde{q}n+\tilde{r}$ and see what can be said about $|r - \tilde{r}|$ if $q \neq \tilde{q}$. More hints further down!

Written assignment 2:

Much harder than #1: Prove the existence of q and r.

Hints for #2: Review the Extended Well–Ordering principle MF thm.6.8. Its use will give the easiest way to prove this assignment: Let

$$A \;:=\; A(m,n) \;:=\; \{r' \in [0,\infty[_{\mathbb{Z}} : \; \exists\, q' \in \mathbb{Z} \text{ such that } r' = m - q'n\}\,.$$

Show that $A \neq \emptyset$ by separately examining the cases

- $m \geqq 0$ (easy)
- $m < 0$ (probably the hardest part of the proof!)

Now you can apply the Extended Well–Ordering principle to the set A. How is $\min(A)$ related to your problem? (It is!) To better see what is going on, this may help:

- What is $m = nq + r$ for $n = 10$ and $m = 43,\; m = -43,\; m = -3$?
- What is $\min(A(m,n))$ in those three cases? Draw a picture!

Hint for both #1 and #2: MF prop. 3.61 and cor.3.5 at the end of ch.3.5 will come in handy in connection with using or proving $0 \leqq r < n$. They assert for the ordered integral domain $(\mathbb{Z}, +, \cdot, \mathbb{N})$ the following.

If $a, b \in [0, n[_{\mathbb{Z}}$, then $|a - b| \;\leqq\; \max(a,b)$, i.e., $\;-n \;<\; a - b \;<\; n$.