Math 330 Section 5 - Fall 2023 - Homework 15

Published: Saturday, November 25, 2023 Running total: 50 points
Last submission: Thursday evening, December 7, 2023

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete before the first one of this HW.

MF lecture notes:
ch.2.1 – 2.7, ch.3, skim ch.4 (optional), ch.5 - ch.12.5; skip ch.12.6 (optional)

B/G (Beck/Geoghegan) Textbook (optional, EXCEPT for ch.3 on logic):
ch.1 – 3, ch.5 – 12

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

New reading assignments:

Reading assignment 1 - due Monday, November 27:

- Carefully read MF ch.12.7 and 12.8. You might have problems with subspaces, since this is a very subtle concept!

Reading assignment 2 - due Wednesday, November 29:

a. Carefully read MF ch.12.9 and 12.10 until Theorem 12.10.

Reading assignment 3 - due Friday, December 1:

a. Carefully read the remainder of MF ch.12.10.
b. Carefully read MF ch.13.1.1 until Example 13.1.

Be sure to read pages 2 and 3!

Supplementary instructions for reading MF ch.12:

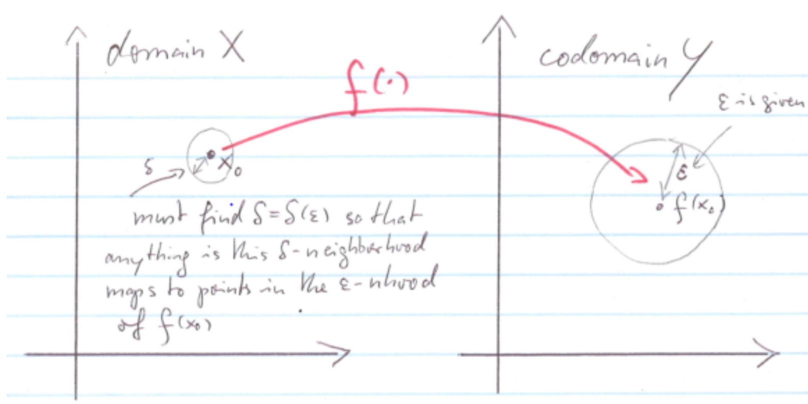
When you read or reread any topics in those chapters then the following is good advice:

a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for

- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \biguplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^\complement$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^\complement$ and A_1. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
  Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^\complement$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.

b. MF ch.12.2 (Continuity): Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 0.1: ε-δ continuity



Written assignments on pages 3 and 4

Written assignment 1: (3 points!) Prove MF prop.11.13 (Properties of the sup norm): $h \mapsto \|h\|_\infty = \sup\{|h(x)| : x \in X\}$ defines a norm on $\mathscr{B}(X, \mathbb{R})$

This assignment is worth three points: **One point each** for pos.definite, absolutely homogeneous, triangle inequality!

Hint: Go for a treasure hunt in ch.9.2 (Minima, Maxima, Infima and Suprema), and look at the properties of $\sup(A)(A \subseteq \mathbb{R})$ to prove absolute homogeneity and the triangle inequality.

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove that $\|\cdot\|_\infty$ satisfies the triangle inequality (11.27c) of a norm you will have to write something along the following lines:

c. Triangle inequality.
NTS: $\|f+g\|_\infty \leqq \|f\|_\infty + \|g\|_\infty$ for all $f, g \in \mathscr{B}(X, \mathbb{R})$.

Proof:

$\|f+g\|_\infty = \sup\{|f(x)| + |g(x)| : x \in X\}$ (definition of $\|\cdot\|_\infty$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= \|f\|_\infty + \|g\|_\infty$ (.....)

(There may be fewer or more steps in your proof. The above only serves as an illustration!)

No need to justify properties of the absolute value $|\alpha|$ of a real number α, but you will need to justify why

$$\sup\{|\alpha f(x)| : x \in X\} = |\alpha| \sup\{|f(x)| : x \in X\},$$

and why

$$\sup\{|f(x) + g(x)| : x \in X\} \leqq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\}.$$

An aside: **DO NOT** write $\|f(x)\|_\infty$ when you deal with the real number $f(x)$ (and **you probably mean** the absolute value $|f(x)|$).

$\|\cdot\|_\infty$ is defined for functions f, NOT for numbers $f(x)$!

Written assignment 2 (3 points):
Prove MF thm.12.1 (Norms define metric spaces): Let $(V, \|\cdot\|)$ be a normed vector space. Then the function

$$d_{\|\cdot\|}(\cdot,\cdot) : V \times V \to \mathbb{R}_{\geqq 0}; \qquad (x,y) \mapsto d_{\|\cdot\|}(x,y) := \|y - x\|$$

defines a metric space $(V, d_{\|\cdot\|})$.

This assignment is worth three points: **One point each** for pos.definite, symmetry, triangle inequality!

Hint: You will have to show for each one of (12.1a), (12.1b), (12.1c) how it follows from def. 11.15: Which one of (11.31a), (11.31b), (11.31c) do you use at which spot?

Careful with symmetry: What is the reason that $\|a - b\| = \|b - a\|$?

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove, e.g., that $d_{\|\cdot\|}(\cdot,\cdot)$ satisfies the triangle inequality (12.1c) of a metric you will have to write something along the following lines:

c. Triangle inequality.
NTS: $d_{\|\cdot\|}(x,z) \leqq d_{\|\cdot\|}(x,y) + d_{\|\cdot\|}(y,z)$ for all $x, y, z \in X$.

Proof:
$d_{\|\cdot\|}(x,z) = \|z - x\|$ (definition of the metric $d_{\|\cdot\|}$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= d_{\|\cdot\|}(x,y) + d_{\|\cdot\|}(y,z)$ (.....)

(There may be fewer or more steps in your proof. The above only serves as an illustration!)