# Math 330 Section 1 - Fall 2024 - Homework 14

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**Status - Reading Assignments:**

The reading assignments you were asked to complete before the first one of this HW are:

MF lecture notes:
ch.1; ch.2.1 - 2.6, ch.3; skim ch.4; ch.5 - 13.1

B/G (Beck/Geoghegan) Textbook:
ch.2 – 13.4

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

**New reading assignments:**

**Reading assignment 1 - due Monday, November 11:**

- **a.** Read carefully MF ch.13.2.1-13.2.2. Understand the DIFFERENCE between convergence and uniform convergence, continuity and uniform continuity, convergence and uniform continuity,
- **b.** Skim the contents of ch.13.2.3. The better students are ecouraged to take a closer look at this easily disgestible material.
- **c.** Review your Calc II knowledge about series, in particular the concept of absolute convergence.

**Reading assignment 2 - due: Wednesday, November 13:**

- **a.** Read carefully MF ch.13.2.1. Definitely on quizzes and/or the final exam and one of the homework sets!
- **b.** Skim the contents of ch.13.2.2, but look closely at the Cauchy criteria (Prop.13.10), absolute convergence, Theorem 13.7., and Riemann's Rearrangement Theorem and its consequences. It is not important that you understand or even look at the proofs, but you must understand what they assert.

**Reading assignment 3 - due Friday, November 15:**

- **a.** Read carefully MF ch.14.1. Draw plenty of pictures to understand the assertions of Proposition 14.1.

**Be sure to read pages 2 – 4!**

**Supplementary instructions for reading MF ch.12 - 14:**

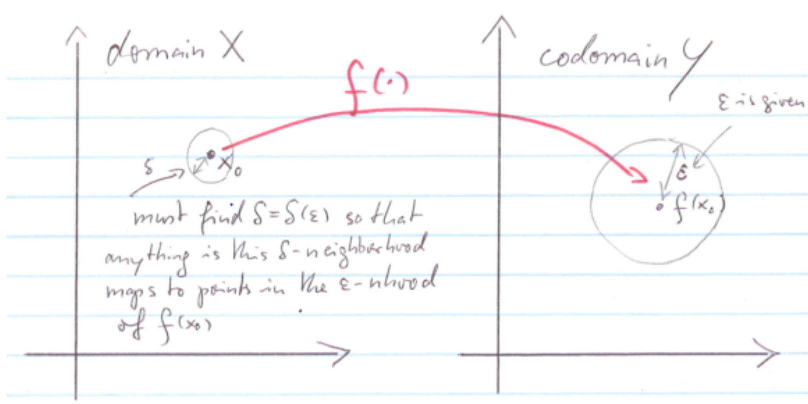
When you read or reread any topics in those chapters then the following is good advice:

**a.** MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for

- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \uplus A_2$ which do not overlap. Draw some points $x_j \in A$ with $\varepsilon$–nhoods (circles with radius $\varepsilon$ about $x_j$) so that some circles are entirely in $A$, one with $x_j \in A_1$ which reaches into $A^{\complement}$ but not into $A_2$, and one with $x_j \in A_2$ which reaches both into $A^{\complement}$ and $A_1$. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to $B$ and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
  Draw points "completetely inside" $B$, others "completetely outside" $B$, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within $B$ by sequences? Which ones can you surround by circles that entirely stay within $B$, i.e., which ones are interior points of $B$? Which ones can you surround by circles that entirely stay outside the closure of $B$, i.e., which ones are entirely within $\bar{B}^{\complement}$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set $A$ which is meant to be a metric subspace of $\mathbb{R}^2$.

**b.** MF ch.12.2 (Continuity): Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 0.1: $\varepsilon$-$\delta$ continuity




**Written assignments on pages 3 and 4**

**Written assignment 1:** (3 points!) Prove MF prop.11.13 (Properties of the sup norm): $h \mapsto \|h\|_\infty = \sup\{|h(x)| : x \in X\}$ defines a norm on $\mathscr{B}(X, \mathbb{R})$

This assignment is worth three points: **One point each** for pos.definite, absolutely homogeneous, triangle inequality!

**Hint:** Go for a treasure hunt in ch.9.2 (Minima, Maxima, Infima and Suprema), and look at the properties of $\sup(A)(A \subseteq \mathbb{R})$ to prove absolute homogeneity and the triangle inequality.

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove that $\|\cdot\|_\infty$ satisfies the triangle inequality (11.29c) of a norm you will have to write something along the following lines:

**c.** Triangle inequality.
NTS: $\|f+g\|_\infty \leqq \|f\|_\infty + \|g\|_\infty$ for all $f, g \in \mathscr{B}(X, \mathbb{R})$.

Proof:

$\|f+g\|_\infty = \sup\{|f(x)| + |g(x)| : x \in X\}$ (definition of $\|\cdot\|_\infty$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= \|f\|_\infty + \|g\|_\infty$ (.....)

(There may be fewer or more steps in your proof. The above only serves as an illustration!)

No need to justify properties of the absolute value $|\alpha|$ of a real number $\alpha$, but you will need to justify why

$$\sup\{|\alpha f(x)| : x \in X\} = |\alpha| \sup\{|f(x)| : x \in X\},$$

and why

$$\sup\{|f(x) + g(x)| : x \in X\} \leqq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\}.$$

An aside: **DO NOT** write $\|f(x)\|_\infty$ when you deal with the real number $f(x)$ (and **you probably mean** the absolute value $|f(x)|$).

$\|\cdot\|_\infty$ is defined for functions $f$, NOT for numbers $f(x)$!

**Written assignment 2 (3 points):**
Prove MF thm.12.1 (Norms define metric spaces): Let $(V, \|\cdot\|)$ be a normed vector space. Then the function

$$d_{\|\cdot\|}(\cdot,\cdot) : V \times V \to \mathbb{R}_{\geqq 0}; \qquad (x,y) \mapsto d_{\|\cdot\|}(x,y) := \|y - x\|$$

defines a metric space $(V, d_{\|\cdot\|})$.

This assignment is worth three points: **One point each** for pos.definite, symmetry, triangle inequality!

**Hint**: You will have to show for each one of (12.1a), (12.1b), (12.1c) how it follows from def. 11.17: Which one of (11.31a), (11.31b), (11.31c) do you use at which spot?

Careful with symmetry: What is the reason that $\|a - b\| = \|b - a\|$?

This assignment is worth three points, and you will have to earn them! The following exemplifies the level of detail I expect you to provide.

To prove, e.g., that $d_{\|\cdot\|}(\cdot,\cdot)$ satisfies the triangle inequality (12.1c) of a metric you will have to write something along the following lines:

**c.** Triangle inequality.
NTS: $d_{\|\cdot\|}(x,z) \leqq d_{\|\cdot\|}(x,y) + d_{\|\cdot\|}(y,z)$ for all $x,y,z \in X$.

Proof:

$d_{\|\cdot\|}(x,z) = \|z - x\|$ (definition of the metric $d_{\|\cdot\|}$)
$= \dots$ (.....)
$\leqq \dots$ (.....)
$= d_{\|\cdot\|}(x,y) + d_{\|\cdot\|}(y,z)$ (.....)

(There may be fewer or more steps in your proof. The above only serves as an illustration!)