Math 330 Section 1 - Fall 2024 - Homework 16

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Status - Reading Assignments:

The reading assignments you were asked to complete before the first one of this HW are:

MF lecture notes:
ch.1; ch.2.1 - 2.6, ch.3; skim ch.4; ch.5 - 15.3

B/G (Beck/Geoghegan) Textbook:
ch.2 – 13.4

B/K lecture notes:
ch.1.1 (Introduction to sets) (optional)
ch.1.2 (Introduction to Functions) but skip ch.1.2.4: Floor and Ceiling Functions (optional)

New reading assignments:

Reading assignment 1 - due Monday, November 25:

Read carefully MF ch.15.4 and ch.15.5.1.

Reading assignment 2 - due: Tuesday, November 26:

Read carefully MF ch.15.5.2, but skip the proofs of Lemma 15.3 and Lemma 15.4 if you want.

Written assignments are on the next page

Written assignment 1: Let $X := \mathbb{R}$ equipped with the standard Euclidean metric $d(x, x') = |x - x'|$. Let $f_n : \mathbb{R} \to \mathbb{R}$ be the following sequence of functions:

$$f_n(x) \ := \ \begin{cases} 0 & \text{if } |x| > \frac{1}{n}, \\ nx + 1 & \text{if } \frac{-1}{n} \leqq x \leqq 0, \\ -nx + 1 & \text{if } 0 \leqq x \leqq \frac{1}{n}, \end{cases}$$

i.e., the point $\big(x, f_n(x)\big)$ is on the straight line between $(-\frac{1}{n}, 0)$ and $(0, 1)$ for $\frac{-1}{n} \leqq x \leqq 0$, it is on the straight line between $(0, 1)$ and $(\frac{1}{n}, 0)$ for $0 \leqq x \leqq \frac{-1}{n}$, and it is on the x–axis for all other x. Draw a picture! Let $f(x) := 0$ for $x \neq 0$ and $f(0) := 1$.

a. Prove that f_n converges pointwise to f on $\mathbb{R}$. In other words, prove that $\lim\limits_{n \to \infty} f_n(x) = f(x)$ for all $x \in \mathbb{R}$.

b. Prove that f_n does not converge uniformly to f on $\mathbb{R}$. You may use without proof that each of the functions f_n is continuous on $\mathbb{R}$.

One point each for (a) and (b)!

Hint for part **(a)**: For $\boxed{\text{fixed } x \neq 0}$, what happens eventually, i.e., for big enough n? Transform the inequalities $\dots \leq x \leq \dots$ into inequalities for n and you should see what happens.

Example (NOT legit as a proof): If $x = 0.01$, what happens if $n > 1000$? Thus $\lim\limits_{n \to \infty} f_n(0.01)$ = WHAT?

No need to submit those hints as part of your HW. Just use them!

Hint for part **(b)**: Look at the (very few propositions and theorems of Ch.13.2.1).