

Math 330 Section 1 - Fall 2025 - Homework 12

Published: Tuesday, October 28, 2025

Running total: 43 points

Last submission: Friday, November 14, 2025

Status - Reading Assignments:

The reading assignments you were asked to complete before the first one of this HW are:

MF lecture notes:

ch.1 - 3, skim ch.4, ch.5 - ch.12.9 (skip ch.8.2 and 10.3)

B/G (Beck/Geoghegan) Textbook:

ch.2.1 – 7, ch.8-13.4

Other material (optional):

B/K lecture notes ch.1.1 and ch.1.2, except ch.1.2.4

Stewart Calculus 7ed - ch.1.7: "The Precise Definition of a Limit"

New reading assignments:

Reading assignment 1 - due Monday, November 3:

- a. Read carefully MF ch.12.10. Understand the connection AND DIFFERENCE between the completeness axiom and completeness of $(X, d) = (\mathbb{R}, d_{\|\cdot\|_2})$!
- b. Start reviewing convergence and continuity in $(X, d) = (\mathbb{R}, d_{\|\cdot\|_2})$ (Ch.9.3).

Reading assignment 2 - due: Wednesday, November 5:

- Study for the midterm!

Reading assignment 3 - due Friday, November 7:

- a. Finish reviewing convergence and continuity in $(X, d) = (\mathbb{R}, d_{\|\cdot\|_2})$ (Ch.9.3).
- b. Extra carefully read MF ch.13.1.1.
- c. Carefully read MF ch.13.1.2. (Very brief.)
- d. The stronger students are encouraged to read MF ch.13.1.3.

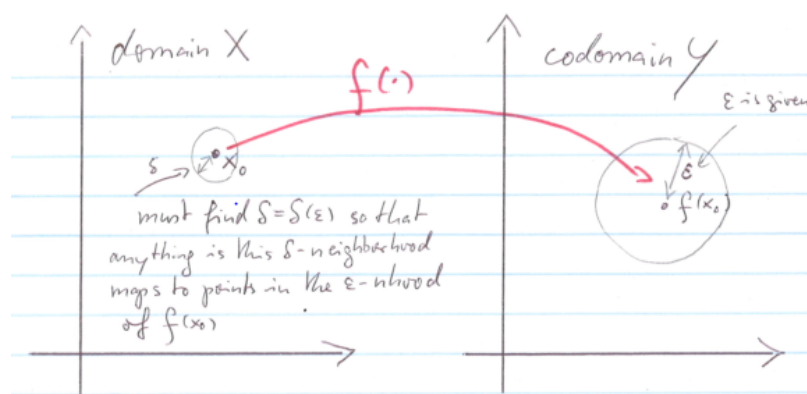
Be sure to read pages 2 and 3!

Supplementary instructions for reading MF ch.12:

When you read or reread any topics in those chapters then the following is good advice:

- a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d|_{\|\cdot\|_2})$ and $(\mathcal{B}(X, \mathbb{R}), d|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for
 - open sets and neighborhoods (ch.12.1.3)
 - convergence, expressed with neighborhoods (the end of def.12.10 in ch.12.1.4)
 - metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \uplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε -nhoods (circles with radius ε about x_j) so that some circles are entirely in A , one with $x_j \in A_1$ which reaches into A° but not into A_2 , and one with $x_j \in A_2$ which reaches both into A° and A_1 . What is $N_\varepsilon^A(x_j)$?
 - Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
 Draw points “completely inside” B , others “completely outside” B , and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B , i.e., which ones are interior points of B ? Which ones can you surround by circles that entirely stay outside the closure of B , i.e., which ones are entirely within $\bar{B}^c$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
 - Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.
- b. MF ch.12.2 (Continuity): Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 0.1: ε - δ continuity



Written assignments on page 3

Written assignments:

Written assignment 1: Partial credit up to 3 points! Can be turned in repeatedly.

Let $f(x) = x^2$. Prove by use of “ ε - δ continuity” that f is continuous at $x_0 = 1$. You **MUST** work with ε - δ continuity (thm.9.7) **NOT WITH SEQUENCE CONTINUITY**, and you cannot use any “advanced” knowledge such as the product of continuous functions being continuous, etc.

Special instructions for assignment 1: Turn in your scratchpaper where you solve for δ (see the hints below).

Hints:

- a. What does $d(x, x_0) < \delta$ and $d(f(x), f(x_0)) < \varepsilon$ translate to?
- b. $x^2 - 1 = (x + 1)(x - 1)$.
- c. Do the following on scratch paper: Work your way backward by establishing a relationship between $\varepsilon > 0$ and δ and then “solving for δ ” That part should not be in your official proof.
- c1. Only small neighborhoods matter (see Proposition 9.24 at the end of the chapter on convergence and continuity.): Given $0 < \varepsilon < 1$ try to find δ that works for such ε . Restrict your search to $\delta < 1$. What kind of bounds do you get for $|x^2 - 1|$, $|x + 1|$, $|x - 1|$ if $0 < \delta < 1$? In particular what kind of bounds do you get for $|x + 1|$?
- c2. Put all the above together. Show that you obtain $|f(x) - f(x_0)| \leq 3\delta$?. How then do you choose δ when you consider ε as given? You’ll get the answer by “solving $|f(x) - f(x_0)| \leq 3\delta$ for δ ”.
- c3. All of the above was done under the assumption that $\delta < 1$. Satisfy it by replacing δ with $\delta' := \min(\delta, 1)$
- d. Only now you are ready to construct an acceptable proof: Let $\varepsilon > 0$, $\delta := \dots$, and $\delta' := \min(\delta, 1)$. Then

Written assignment 2: Prove MF Thm. 9.9 (B/G thm.11.12):

If $m \in [0, \infty[\setminus \mathbb{Z}$ is not a perfect square, then $\sqrt{m}$ is irrational.

You MUST work with lowest term representations: Modify the proof that $\sqrt{2}$ is irrational. Key is Proposition 9.28 (B/G prop.11.5)! No partial credit for this one!