Formula Collection for Math 447 exams – Not all items are relevant!

(1) **(a)** • power set $2^\Omega = \{$ all subsets of $\Omega\}$ • $\forall\, x \dots$: For all $x \dots$ ⊡ $\exists\, x$ s.t. $\dots$ There is an x such that $\dots$ ⊡ $\exists!\, x$ s.t. $\dots$ There is a unique x s.t. $\dots$ ⊡ $p \Rightarrow q$ If p is true then q is true ⊡ $p \Leftrightarrow q$ p iff q, i.e., p is true if and only if q is true • Intervals: $]a,b[= \{x \in \mathbb{R} : a < x < b$, $]a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \leq b$, $[a,b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \leq x \leq b$, etc. • countable set A: can be sequenced: ⊡ $A = \{a_1, a_2, \dots, a_n\}$ (finite set) ⊡ $A = \{_1, a_2, \dots\}$ ("countably infinite" set) ⊡ $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is uncountable • family $\big(x_i\big)_{i \in I}$: index set I may be uncountable • $\bigcup_{i \in J} A_i = \{x : \exists\, i_0 \in J$ s.t. $x \in A_{i_0}\}$ and $\bigcap_{i \in J} A_i = \{x : \forall\, i \in J\; x \in A_i\}$. • Can use $A \biguplus B$ for $A \cup B$ if disjoint sets • **De Morgan:** ⊡ $\big(\bigcup_k A_k\big)^{\complement} = \bigcap_k A_k^{\complement}$ ⊡ $\big(\bigcap_k A_k\big)^{\complement} = \bigcup_k A_k^{\complement}$ • **Distributivity:** ⊡ $\bigcup_j (B \cap A_j) = B \cap \bigcup_j A_j$ ⊡ $\bigcap_{j \in I} (B \cup A_j) = B \cup \bigcap_j A_j$

• Cartesian products: $\big|X_1 \times \cdots \times X_n\big| = \big|X_1\big| \cdots \big|X_n\big|$ (the mn–rule) • Preimages: $f : X \to Y$, arbitrary index set J $B, B_j \subseteq Y$: ⊡ $f^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$ ⊡ $f^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} f^{-1}(B_j)$ ⊡ $f^{-1}(B^{\complement}) = \big(f^{-1}(B)\big)^{\complement}$ ⊡ $B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$ • $A \subseteq \Omega \Rightarrow 1_A(\omega) = 1$ if $\omega \in A$ and 0 else

(b) • Probability space (Ω, P) same as WMS sample space (S, P) • σ–algebra $\mathfrak{F} \subseteq 2^\Omega$: ⊡ $A \in \mathfrak{F} \Rightarrow A^{\complement} \in \mathfrak{F}$ ⊡ $A_n \in \mathfrak{F} \Rightarrow \bigcup_{j=1}^\infty A_j \in \mathfrak{F}$ ⊡ $\emptyset \in \mathfrak{F}$ • random var. (rv) $X : (\Omega, P) \to \mathbb{R}$ produces distribution
$P_X(B) = P\{X \in B\} = P$
$big(X^{-1}(B)\big)$ on codomain. ⊡ Conveniences: $P_Y(\{y\}) = P\{Y = y\}$; $P_Y(]a,b]) = P\{a < Y \leq b\}$; ...

• Discrete probability space and rv: STUDY IT!

(b) Combinatorial Analysis

• Think: Does order matter in your probability space or doesn't it?

• # of permutations P_r^n vs # of combinations $\binom{n}{r}$ vs $\binom{n}{r_1,\dots,r_k}$ ⊡ $0! = 1$, $n! = 1 \cdot 2 \cdots n$; $(n \in \mathbb{N})$

(c) partition B_j $(j \in \mathbb{N})$ of Ω, $A \in \Omega \Rightarrow A = \biguplus_j (A \cap B_j) \Rightarrow P(A) = \dots$; Used to compute $P(B_{j_0} \mid A)$ from $P(A \mid B_j)$ and $P(B_j)$ (must know for all j)

(d) Random variables (rvs)

• Any rv Y has cumulative distribution function = CDF $F(y) = F_Y(y)$ (WMS: distribution function).

• Any rv Y has pth quantile = $100p$th percentile ϕ_p. $\phi_{0.25}, \phi_{0.50}, \phi_{0.75}$ are named ....

• Discrete rv Y on (Ω, P), $p(y) = p_Y(y) = P_Y\{y\}$: probability mass function (WMS: probability func) for Y.

• continuous rv Y has probability density function (PDF) $f(y) = f_Y(y) = P_Y\{y\}$ (WMS: probability func).

• $g : \mathbb{R} \to \mathbb{R} \Rightarrow E[g(Y)] = \dots$ • rvs $Y_1, Y_2, \dots, Y_n$ on the same $(\Omega, P) \Rightarrow$ ⊡ always: $E\left[\sum_{j=1}^n Y_j\right] = \sum_{j=1}^n E[Y_j]$.

⊡ if the Y_j are INDEPENDENT: $Var\left[\sum_{j=1}^n Y_j\right] = \sum_{j=1}^n Var[Y_j]$. • Tchebysheff inequalities are ....

• Y is uniform w. $\theta_1 < \theta_2 \Rightarrow Var[Y] = (\theta_2 - \theta_1)^2/12$ • Y is $\mathcal{N}(\mu, \sigma^2) \Rightarrow m_Y(t) = e^{\mu t + (\sigma^2 t^2)/2}$

• Y is gamma$(\alpha, \beta) \Rightarrow m_Y(t) = 1/(1 - \beta t)^\alpha$ • Y is beta$(\alpha, \beta) \Rightarrow E[Y] = \alpha/(\alpha + \beta)$; $Var[Y] = \frac{\alpha\beta}{(\alpha+\beta)^2\,(\alpha+\beta+1)}$