Formula Collection for Math 447 exams – Not all items are relevant!

(0) REMOVED: ⊡ $E\left[\sum\limits_{j=1}^{n} Y_j\right] = \dots$ ⊡ $Var\left[\sum\limits_{j=1}^{n} Y_j\right] = \dots$ ⊡ MGFs for $\mathcal{N}(\mu,\sigma)$, gamma(α,β)

(1) **(a)** • power set $2^\Omega = \{$ all subsets of $\Omega\}$ • $\forall\, x \dots$: For all $x \dots$ ⊡ $\exists\, x$ s.t. … There is an x such that … ⊡ $\exists!\, x$ s.t. … There is a unique x s.t. … ⊡ $p \Rightarrow q$ If p is true then q is true ⊡ $p \Leftrightarrow q$ p iff q, i.e., p is true if and only if q is true • Intervals: $]a,b[= \{x \in \mathbb{R} : a < x < b$, $]a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \leq b$, $[a,b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \leq x \leq b$, etc. • countable set A: can be sequenced: ⊡ $A = \{a_1, a_2, \dots, a_n\}$ (finite set) ⊡ $A = \{_1, a_2, \dots\}$ ("countably infinite" set) ⊡ $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is uncountable • family $(x_i)_{i\in I}$: index set I may be uncountable • $\bigcup_{i\in J} A_i = \{x : \exists\, i_0 \in J \text{ s.t. } x \in A_{i_0}\}$ and $\bigcap_{i\in J} A_i = \{x : \forall\, i \in J\, x \in A_i\}$. • Can use $A \uplus B$ for $A \cup B$ if disjoint sets • **De Morgan:** ⊡ $(\bigcup_k A_k)^{\complement} = \bigcap_k A_k^{\complement}$ ⊡ $(\bigcap_k A_k)^{\complement} = \bigcup_k A_k^{\complement}$ • **Distributivity:** ⊡ $\bigcup_j (B \cap A_j) = B \cap \bigcup_j A_j$ ⊡ $\bigcap_{j\in I} (B \cup A_j) = B \cup \bigcap_j A_j$

• Cartesian products: $\left|X_1 \times \dots \times X_n\right| = \left|X_1\right| \cdots \left|X_n\right|$ (the mn–rule) • Preimages: $f : X \to Y$, arbitrary index set J $B, B_j \subseteq Y$: ⊡ $f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j)$ ⊡ $f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$ ⊡ $f^{-1}(B^{\complement}) = \left(f^{-1}(B)\right)^{\complement}$ ⊡ $B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$ • $A \subseteq \Omega \Rightarrow 1_A(\omega) = 1$ if $\omega \in A$ and 0 else

• partition B_j $(j \in \mathbb{N})$ of Ω, $A \in \Omega \Rightarrow A = \biguplus_j (A \cap B_j) \Rightarrow P(A) = \dots$; Used to compute $P(B_{j_0} \mid A)$ from $P(A \mid B_j)$ and $P(B_j)$ (must know for all j)

(b) • Probability space (Ω, P) same as WMS sample space (S, P) • σ–algebra $\mathfrak{F} \subseteq 2^\Omega$: ⊡ $A \in \mathfrak{F} \Rightarrow A^{\complement} \in \mathfrak{F}$ ⊡ $A_n \in \mathfrak{F} \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathfrak{F}$ ⊡ $\emptyset \in \mathfrak{F}$ • random var. (rv) $X : (\Omega, P) \to \mathbb{R}$ produces distribution

$P_X(B) = P\{X \in B\} = P\left(X^{-1}(B)\right)$ on codomain. ⊡ Conveniences: $P_Y(\{y\}) = P\{Y = y\}$; $P_Y(]a,b]) = P\{a < Y \leq b\}$; ...

(c) Combinatorial Analysis

• Think: Does order matter in your probability space or doesn't it?

• # of permutations P_r^n vs # of combinations $\binom{n}{r}$ vs $\binom{n}{r_1,\dots,r_k}$ ⊡ $0! = 1$, $n! = 1 \cdot 2 \cdots n$; $(n \in \mathbb{N})$

• Roulette game: ⊡ slots $0, 00, 1, 2, \dots, 36$ ⊡ 18 black, 18 red; numbers 1 – 36 in 12 rows $\times$ 3 cols

• deck of 52 cards: ⊡ 4 suits (clubs, spades, hearts, diamonds) of 13 each: Ace, $2, 3, \dots, 10$, Jack, Queen, King ⊡ so: 4 2's, 4 3's, 4 Aces, 4 Jacks, …

(d) Random variables (rvs)

• Any rv Y has cumulative distribution function = CDF $F(y) = F_Y(y)$ (WMS: distribution function).

• Any rv Y has pth quantile = $100p$th percentile ϕ_p. $\phi_{0.25}, \phi_{0.50}, \phi_{0.75}$ are named ....

• Discrete rv Y on (Ω, P), $p(y) = p_Y(y) = P_Y\{y\}$: probability mass function (WMS: probability func) for Y.

⊡ 0–1 encoded Bernoulli, binomial, poisson, geometric, hypergeometric, multinomial

• continuous rv Y has probability density function (PDF) $f(y) = f_Y(y) = P_Y\{y\}$ (WMS: probability func).

⊡ uniform, normal, gamma, beta, χ^2, exponential

• $g : \mathbb{R} \to \mathbb{R} \Rightarrow E[g(Y)] = \dots$ • Tchebysheff inequalities are ....

• Y is uniform w. $\theta_1 < \theta_2 \Rightarrow Var[Y] = (\theta_2 - \theta_1)^2/12$

• Y is beta$(\alpha,\beta) \Rightarrow E[Y] = \alpha/(\alpha+\beta)$; $Var[Y] = \frac{\alpha\beta}{(\alpha+\beta)^2\,(\alpha+\beta+1)}$

(e) Multivariate distributions: • distinguish joint, marginal and conditional PMFs and PDFs! Joint CDFs, PMFs, CDFs, MGFs allow you to see whether the rvs are independent. HOW? ⊡ What condition on $\{(y_1, y_2)$: the PDF or PMF is $> 0\}$? Must determine! • $E[g(Y_1, \dots, Y_n)] = \cdots$ • $Cov[X,Y] = 0$ vs. X, Y independent. Relationship? • $E[Y_1 \mid Y_2]$, $Var[Y_1 \mid Y_2]$, Relationship? • $E\big[\,E[Y_1 \mid Y_2]\,\big] = \dots$

• Given a small 2-dim table (say, 3×4 entries) for a joint PMF, be able to compute marginal and conditional distributions and conditional expectations and variances.

• do not confuse hypergeom. distrib. and multinomial distrib!

(f) Functions of (rvs): • Method of transformations needs injectivity • Method of distrib functions always works • MGF method best for sums of indep rvs

• Order stats: ⊡ We only do them for continuous, iid rvs. ⊡ Find CDFs for $Y(1)$ and $Y(n)$ directly; differentiate to get PDFs ⊡ For $1 < j < n$: maybe find a corresponding multinomial sequence

(g) Limit theorems: ⊡ $Y_n \overset{\text{a.s.}}{\to} Y$: $P\{\lim_{n\to\infty} Y_n \neq Y\} = 0$ ⊡ $Y_n \overset{\mathbf{P}}{\to} Y$: $\forall\, \varepsilon > 0: \lim_{n\to\infty} P\{|Y_n - Y| > \varepsilon\} = 0$

⊡ The two laws of large numbers and the CLT - What do they state?

• approximate binom(n, p) with poisson rv vs. with normal rv

(h) Sampling: ⊡ Random sampling actions (RSAs) are iid, SRS actions are not. ⊡ Neither need be on a normal rv. ⊡ The CLT allows to consider non–normal RSAs WHY? ⊡ sample mean & variance $\bar{Y} = \ldots; S^2 = \ldots;$ ⊡ $\bar{Y}$ and S^2 are statistics for RSA $\vec{Y}$. Means what? ⊡ When is the median a statistic?