Formula Collection for the Math 447 Final Exam – Not all items are relevant!

(1) **(a)** • power set $2^{\Omega} = \{$ all subsets of $\Omega\}$ • $\forall\, x \dots$: For all $x \dots$ $\boxdot$ $\exists\, x$ s.t. … There is an x such that …
$\boxdot$ $\exists!\, x$ s.t. … There is a unique x s.t. … $\boxdot$ $p \Rightarrow q$ If p is true then q is true $\boxdot$ $p \Leftrightarrow q$ p iff q, i.e., p true if and only if q true
• Intervals: $]a,b[= \{x \in \mathbb{R} : a < x < b$, $]a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \le b$, $[a,b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \le x \le b$, etc.
• countable set A: can be sequenced: $\boxdot$ $A = \{a_1, a_2, \dots, a_n\}$ (finite set) $\boxdot$ $A = \{_1, a_2, \dots\}$ ("countably infinite" set)
$\boxdot$ $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is uncountable • family $(x_i)_{i \in I}$: index set I may be uncountable • $\bigcup_{i \in J} A_i = \{x : \exists\, i_0 \in J$ s.t. $x \in A_{i_0}\}$ • $\bigcap_{i \in J} A_i = \{x : \forall\, i \in J\, x \in A_i\}$. • Can use $A \uplus B$ for $A \cup B$ if disjoint sets • **De Morgan:** $\boxdot$ $\left(\bigcup_k A_k\right)^{\complement} = \bigcap_k A_k^{\complement}$ $\boxdot$ $\left(\bigcap_k A_k\right)^{\complement} = \bigcup_k A_k^{\complement}$ • **Distributivity:** $\boxdot$ $\bigcup_j (B \cap A_j) = B \cap \bigcup_j A_j$ $\boxdot$ $\bigcap_{j \in I} (B \cup A_j) = B \cup \bigcap_j A_j$
• Cartesian products: $\left| X_1 \times \cdots \times X_n \right| = \left|X_1\right| \cdots \left|X_n\right|$ (the mn–rule) • Formulas f. preimages of $f : X \to Y$: Arbitrary index set J and $B, B_j \subseteq Y$: $\boxdot$ $f^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$ $\boxdot$ $f^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} f^{-1}(B_j)$
$\boxdot$ $f^{-1}(B^{\complement}) = \left(f^{-1}(B)\right)^{\complement}$ $\boxdot$ $B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$ • $A \subseteq \Omega \Rightarrow 1_A(\omega) = 1$ if $\omega \in A$ and 0 else
• partition B_j ($j \in \mathbb{N}$) of Ω, $A \in \Omega \Rightarrow A = \biguplus_j (A \cap B_j) \Rightarrow P(A) = \dots$; Used to compute $P(B_{j_0} \mid A)$ from $P(A \mid B_j)$ and $P(B_j)$ (must know for all j)

(b) • Probability space = sample space (Ω, P) • σ–algebra $\mathfrak{F} \subseteq 2^{\Omega}$: $\boxdot$ $A \in \mathfrak{F} \Rightarrow A^{\complement} \in \mathfrak{F}$ $\boxdot$ $A_n \in \mathfrak{F} \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathfrak{F}$ $\boxdot$ $\emptyset \in \mathfrak{F}$ • distribution of random element (rand elem) $X : (\Omega, P) \to \Omega'$: $P_X(B) = P\{X \in B\} = P\big(X^{-1}(B)\big)$ on codomain.
• Conveniences: $P_X(\{x\}) = P\{X = x\}$; $P_X(]a,b]) = P\{a < X \le b\}$ (if X is random var. (rv), i.e., $\Omega' \subseteq \mathbb{R}$); ...
• discrete probab spaces and random elements and rvs defined how?

• independence for $2, n$, arbitr. many events • $P(A \mid B)$ • general addition & multiplication rules, total probability, complement rule, Bayes formula

(c) Combinatorial Analysis • Think: Does order matter in your probability space or doesn't it?
• multiplication rule for several factors • # of permutations P_r^n vs # of combinations $\binom{n}{r}$ vs $\binom{n}{r_1, \dots, r_k}$
$\boxdot$ $0! = 1$, $n! = 1 \cdot 2 \cdots n$; $(n \in \mathbb{N})$ $\boxdot$ several interpretations of $\binom{n}{r_1, \dots, r_k}$

• deck of 52 cards: $\boxdot$ 4 suits (clubs, spades, hearts, diamonds) of 13 each: Ace, $2, 3, \dots, 10$, Jack, Queen, King $\boxdot$ so: 4 2's, 4 3's, 4 Aces, 4 Jacks, … • Roulette: $\boxdot$ slots $0, 00, 1, 2, \dots, 36$ $\boxdot$ 18 black, 18 red; numbers 1 – 36 in 12 rows $\times$ 3 cols

(d) Random variables (rvs) and random elements
• Discrete rand elem $X : (\Omega, P) \to \Omega'$, $p(x) = p_X(x) = P_X\{x\}$: PMF = probab. mass func (WMS: probab. func.) for X.
• Continuous rand vars $Y : (\Omega, P) \to \mathbb{R}$, $p(y) = p_Y(y)$: PDF for Y. • discrete & cont. rvs: CDF $F_Y(y)$; pth quantile ϕ_p
$\boxdot$ $\phi_{0.25}, \phi_{0.50}, \phi_{0.75}$ are named ....
• $E[Y], Var[Y], \sigma_Y$ of rv Y: $\boxdot$ Remember all formulas! $\boxdot$ $E[g(Y)] = \dots$ • m_k' and m_k; MGF $m_Y(t)$ compute how?
• Each distribution: $\boxdot$ application context? $\boxdot$ $m_Y(t) =$? $\boxdot$ Given $m_Y(t)$: $Y \sim$ WHAT?
• iid sequences of random elements $\boxdot$ Bernoulli trials and sequences $\boxdot$ 0–1 encoded Bernoulli trials
• Discrete rvs: $\boxdot$ Bernoulli(p) $\boxdot$ binom(n, p) $\boxdot$ geom(p) $\boxdot$ neg. binom(p, r): $p(y) = \binom{y-1}{r-1} p^r q^{y-r}$, $\mu = \frac{r}{p}$, $\sigma^2 = \frac{r(1-p)}{p^2}$
$\boxdot$ hypergeom(N, R, n) $\boxdot$ Poisson(λ) hypergeom $\neq$ multinomial distrib! • Contin rvs: $\boxdot$ uniform(θ_1, θ_2): $\sigma^2 = \frac{(\theta_1 - \theta_2)^2}{12}$
$\boxdot$ $\mathcal{N}(\mu, \sigma^2)$: empirical rule =? $\boxdot$ gamma(α, β) vs $\chi^2(\text{df} = \nu)$ vs expon(β) • $2\times$ Tchebysheff – know them both!

(e) Multivariate $\vec{Y} = (Y_1, \dots, Y_k)$:
• joint CDF $F_{\vec{Y}}(\vec{y})$, joint PMF $p_{\vec{Y}}(\vec{y})$, joint PDF $f_{\vec{Y}}(\vec{y})$ $\boxdot$ allow you to see whether the rvs are independent. HOW? $\boxdot$ What condition on $\{(y_1, y_2)$: the PDF or PMF is $> 0\}$? Must determine! • $E[g(Y_1, \dots, Y_n)] = \cdots$ • $Cov[X, Y] = 0$ vs. X, Y independent. Relationship? • $\vec{Y} = (Y_1, Y_2)$: $\boxdot$ marginal CDFs F_{Y_j}, PMFs p_{Y_j} and PDFs f_{Y_j} $\boxdot$ conditional PMF $p_{Y_1|Y_2}(y_1|y_2)$ and PDF $f_{Y_1|Y_2}(y_1|y_2)$ define $E[Y_1 \mid Y_2]$, $Var[Y_1 \mid Y_2]$ how? • $E\big[\, E[Y_1 \mid Y_2] \,\big] =$ (8.50) • $Var[Y_1] =$ (8.53)
• Given a small 2-dim table (say, 3×4 entries) for a joint PMF, be able to compute marginal and conditional distributions and conditional expectations and variances.

Continued on p.2!

(f) Functions (transforms) of rvs $U = h \circ Y$:

• Method of transformations needs injectivity Formulas (9.24) and (9.26) $\boxdot$ • Method of distrib functions always works
• MGF method best for sums of indep rvs

• Order stats: $\boxdot$ We only do them for continuous, iid rvs. $\boxdot$ Find CDFs for $Y(1)$ and $Y(n)$ directly; differentiate to get PDFs $\boxdot$ For $1 < j < n$: maybe find a corresponding multinomial sequence $\boxdot$ $f_{(\bullet)}(\vec{y}) =$ (9.40); proof done how?

(g) Convergence of random variables and limit theorems:

• 4 modes of convergence $Y_n \to Y$; $\boxdot$ Imply what for $P\{a < Y_n \leq b\} vs.\ P\{a < Y \leq b\}$? $\boxdot$ What mode for CLT?

$\boxdot$ Two laws of large numbers and two CLTs - What do they state? $\boxdot$ a CLT that uses σ^2 + normal distribution $\boxdot$ a CLT that uses S^2 + t–distribution • approximate binom(n, p) $\boxdot$ w. poisson rv (CLT not used) $\boxdot$ w. normal rv (need CLT)

(h) Sampling: • "sample" sometimes denotes the random vector (sampling action) $\vec{Y}$ and sometimes "the" realization $\vec{y} = \vec{Y}(\omega)$ $\boxdot$ Random sampling actions (RSAs) are iid, SRS actions are not. $\boxdot$ Neither need be on a normal rv.

• CLT lets us work with non–normal RSAs HOW? • sample mean & variance $\bar{Y} = \dots$; $S^2 = \dots$; When independent?

$\boxdot$ $\bar{Y}$ and S^2 are statistics for samples $\vec{Y}$. Means what? • t–rvs and F–rvs defined with help of χ^2–rvs. HOW? All 3 use df = degs of freedom. $\boxdot$ df $\to \infty \Rightarrow$ t–distrib converges to WHAT?