MATH447

Fall 2020

Practice Midterm 1 (Edition 1)

1. Three cards are withdrawn from a standard 52 card deck without replacement.

(a) What is the probability of getting two distinct suits?

$$\begin{aligned} P(E) &= 12P(\text{2 Diamonds and 1 Heart}) \\ &= 12\frac{\binom{13}{2}\binom{13}{1}\binom{26}{0}}{\binom{52}{3}} \\ &= \frac{234}{425} \end{aligned}$$

(b) What is the probability of getting all three cards of the same rank?

$$\begin{aligned} P(E) &= 13\frac{\binom{4}{3}\binom{48}{0}}{\binom{52}{3}} \\ &= \frac{1}{425} \end{aligned}$$

2. The Lakers and Heat are playing in the NBA Finals. The series is a best-of-seven (first team to win four games clinches the series). The Lakers will win each game with probability 3/4.

(a) Given that the Heat won game one, what is the probability the Lakers go on to win the series?

$$\begin{aligned} P(F|E) &= \frac{P(E\cap F)}{P(E)} \\ &= \frac{P(BAAAA) + 4P(BAAABA) + 10P(BAAABBA)}{\frac{1}{4}} \\ &= \frac{\frac{1}{4}\left(\frac{3}{4}\right)^4 + 4\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^4 + 10\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^4}{\frac{1}{4}} \\ &= \frac{1701}{2048} \end{aligned}$$

(b) Given that the Heat win at least two games in the series, what is the probability the Lakers go on to win the series?

$$
\begin{aligned}
P(F|E) &= \frac{10P(AABBAA) + 20P(AAABBBA)}{1 - P(AAAA) - 4P(AABAA)} \\
&= \frac{10\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^2 + 20\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^3}{1 - \left(\frac{3}{4}\right)^4 - 4\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)} \\
&= \frac{1215}{1504}
\end{aligned}
$$

3. 10 people consist of five men and five women. Four people are selected from this group of 10 to form a committee.

(a) What is the expected number of men in the committee?

$$E(X) = 4 \times \frac{5}{10} = 2$$

(b) What is the probability that the number of men in the committee is unequal to the number of women in the committee?

$$
\begin{aligned}
&P(X \neq 4 - X) \\
&= 1 - P(X = 2) \\
&= 1 - \frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}} \\
&= \frac{11}{21}
\end{aligned}
$$

(c) Given that the number of men and women are unequal in the committee, what is the probability that the committee either consists of all men or all women?

$$
\begin{aligned}
&P\left(\{X=0\}\cup\{X=4\}\,|X\neq 4-X\right)\\
&=P\left(X=0|X\neq 4-X\right)+P\left(X=4|X\neq 4-X\right)\\
&=\frac{P\left(X=0\right)+P\left(X=4\right)}{1-P\left(X=2\right)}\\
&=\frac{2\times\frac{\binom{5}{0}\binom{5}{4}}{\binom{10}{4}}}{\frac{11}{21}}\\
&=\frac{1}{11}
\end{aligned}
$$

4. A multiple choice test consists of 25 questions. For each question, there are five choices, but only one correct answer. A student has not studied for this test and will guess each question randomly.

(a) Use Chebyshev's Inequality to give a lower bound on the probability that the number of questions answered correctly is between 2 and 8, inclusive.

$$X \stackrel{d}{=} binomial\left(25, \frac{1}{5}\right)$$

$$\mu = 5$$

$$\sigma^2 = 25 \times \frac{1}{5} \times \frac{4}{5} = 4$$

$$P\left(2 \leq X \leq 8\right) = P\left(1 < X < 9\right) = P\left(|X-5| < 4\right) \geq 1 - \frac{1}{2^2} = \frac{3}{4}$$

(b) Compute the probability that the student answers no more than one question correctly.

$$
\begin{aligned}
&P\left(X\leq 1\right)\\
&=P\left(X=0\right)+P\left(X=1\right)\\
&=\left(\frac{4}{5}\right)^{25}+25\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^{24}\\
&\approx 0.027\,389\,725\,6
\end{aligned}
$$

5. A coin is tossed until a head appears. The probability that the coin shows heads is 1/1000, and let X be the number of coin tosses. Let $\mu = E\left(X\right)$ and $\sigma^2 = Var\left(X\right)$.

(a) Compute μ and σ^2.

$$\mu = \frac{1}{\frac{1}{1000}} = 1000$$

$$\sigma^2 = \frac{\frac{999}{1000}}{\left(\frac{1}{1000}\right)^2} = 999000$$

(b) Compute $P\left(X \geq 1098\right)$.

$$P\left(X \geq 1098\right) = P\left(FFFF.....FF\right) = \left(\frac{999}{1000}\right)^{1097} \approx 0.333\,687\,996\,1$$

6. A random variable X has the moment generating function (mgf)

$$M_X\left(t\right) = c\sum_{k=0}^{4} e^{k^2 t}$$

where $c > 0$.

(a) Determine c.

Use the fact that $M_X\left(0\right) = 1$ to obtain

$$5c = 1$$

so that

$$c = \frac{1}{5}$$

and therefore X has the pmf

$$P\left(X = k^2\right) = \frac{1}{5} \text{ for } k = 0, 1, 2, 3, 4$$

(b) Compute $E\left(\sqrt{X}\right)$.

$$E\left(\sqrt{X}\right) = \frac{1}{5}\left(0 + 1 + 2 + 3 + 4\right) = 2$$

(c) Compute $Var\left(\sin\left(\frac{\pi}{2}X\right)\right)$.

$$\begin{aligned} & E\left(\sin\left(\frac{\pi}{2}X\right)\right) \\ &= \frac{1}{5}\sum_{k=0}^{4}\sin\left(\frac{\pi}{2}k^2\right) \\ &= \frac{1}{5}\left(0 + 1 + 0 + 1 + 0\right) \\ &= \frac{2}{5} \end{aligned}$$

$$
\begin{aligned}
&E\left(\sin^2\left(\frac{\pi}{2}X\right)\right) \\
&= \frac{1}{5}\sum_{k=0}^{4}\sin^2\left(\frac{\pi}{2}k^2\right) \\
&= \frac{1}{5}\left(0^2+1^2+0^2+1^2+0^2\right) \\
&= \frac{2}{5}
\end{aligned}
$$

$$
Var\left(\sin\left(\frac{\pi}{2}X\right)\right) = \frac{2}{5} - \frac{4}{25} = \frac{6}{25}
$$

You could have also seen this if you observe that $\sin\left(\frac{\pi}{2}X\right) \stackrel{d}{=} Bernoulli\left(\frac{2}{5}\right)$.