

Formula Collection for Math 447 Final Exam – Not all items are relevant!

- (1) (a)**
- power set $2^\Omega = \{ \text{all subsets of } \Omega \}$
 - $\forall x \dots$: For all $x \dots$ $\exists x$ s.t. \dots : There is an x such that \dots
 - $\exists! x$ s.t. \dots : There is a unique x s.t. \dots $p \Rightarrow q$ If p is true then q is true $p \Leftrightarrow q$ iff q , i.e., p true if and only if q true
 - Intervals: $]a, b[= \{x \in \mathbb{R} : a < x < b\}$, $]a, b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \leq b\}$, $[a, b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \leq x \leq b\}$, etc.
 - countable set A : can be sequenced: $\square A = \{a_1, a_2, \dots, a_n\}$ (finite set) $\square A = \{1, a_2, \dots\}$ (“countably infinite” set)
 - $\square \mathbb{Z}$ and \mathbb{Q} are countable, but \mathbb{R} is uncountable
 - family $(x_i)_{i \in I}$: index set I may be uncountable
 - $\bigcup_{i \in J} A_i = \{x : \exists i_0 \in J \text{ s.t. } x \in A_{i_0}\}$
 - $\bigcap_{i \in J} A_i = \{x : \forall i \in J \ x \in A_i\}$.
 - Can use $A \uplus B$ for $A \cup B$ if disjoint sets
 - **De Morgan**: $\square (\bigcup_k A_k)^c = \bigcap_k A_k^c$ $\square (\bigcap_k A_k)^c = \bigcup_k A_k^c$
 - **Distributivity**: $\square \bigcup_j (B \cap A_j) = B \cap \bigcup_j A_j$ $\square \bigcap_{j \in I} (B \cup A_j) = B \cup \bigcap_{j \in I} A_j$
 - Cartesian products: $|X_1 \times \dots \times X_n| = |X_1| \dots |X_n|$
 - Formulas f. preimages of $f : X \rightarrow Y$:
Arbitrary index set J and $B, B_j \subseteq Y$: $\square f^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$ $\square f^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} f^{-1}(B_j)$
 - $\square f^{-1}(B^c) = (f^{-1}(B))^c$ $\square B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$
 - $A \subseteq \Omega \Rightarrow 1_A(\omega) = 1$ if $\omega \in A$ and 0 else
 - partition B_j ($j \in \mathbb{N}$) of Ω , $A \in \Omega \Rightarrow A = \bigsqcup_j (A \cap B_j) \Rightarrow P(A) = \dots$; Used to compute $P(B_{j_0} | A)$ from $P(A | B_j)$ and $P(B_j)$ (must know for all j)
 - Indicator function: $1_A(y) =$
- (b) Sums and Riemann integrals (Riem- f) and Lebesgue integrals (Leb- f):**
- $x_n \geq 0$ or $\sum_n x_n$ abs conv $\Rightarrow \sum_n x_n$ satisfies WHAT?
 - Leb- f : positive, monotone, linear, mon. + domin. conv.
 - step function $h: \int h(\vec{y}) d\vec{y} = ?$
 - simple function $g: \int g d\lambda^d = ?$
 - If both $\int_A f(\vec{y}) d\vec{y}$, $\int_A f d\lambda^d$ exist, they are equal
 - Use Fubini for both $\int_A f(\vec{y}) d\vec{y}$ and $\int f d\lambda^d$ to compute multidim \int .
 - 1_A Riem- f -ble $\Rightarrow \lambda^d(A)$ defined how?
 - $\mathfrak{B}^d = \sigma\{d\text{-dim rectangles}\}$
 - $[f \geq 0 \text{ or } \int |f| d\lambda^d < \infty] \Rightarrow [A \mapsto \int_A f d\lambda^d \text{ is } \sigma\text{-additive}]$
 - For what functions φ, ψ is $A \mapsto \sum_{\omega \in A} \varphi(\omega)$, $A \mapsto \int_A \psi(\vec{y}) d\vec{y} (= \int_A \psi d\lambda^d)$ a probab measure?
- (c)**
- Probability space = sample space (Ω, P)
 - σ -algebra $\mathfrak{F} \subseteq 2^\Omega$: $\square A \in \mathfrak{F} \Rightarrow A^c \in \mathfrak{F}$ $\square A_n \in \mathfrak{F} \Rightarrow \bigcup_{j=1}^\infty A_j \in \mathfrak{F}$ $\square \emptyset \in \mathfrak{F}$
 - distribution of random element (rand elem) $X : (\Omega, P) \rightarrow \Omega'$: $P_X(B) = P\{X \in B\} = P(X^{-1}(B))$ on codomain.
 - Conveniences: $P_X(\{x\}) = P\{X = x\}$; $P_X(]a, b]) = P\{a < X \leq b\}$ (if X is random var. (rv), i.e., $\Omega' \subseteq \mathbb{R}$); ...
 - discrete probab spaces and random elements and rvs defined how?
 - independence for 2, n , arbitr. many events
 - $P(A | B)$
 - general addition & multiplication rules, complement rule, total probability, Bayes formula
- (d) Combinatorial Analysis**
- Think: Does order matter in your probability space or doesn't it?
 - multiplication rule for several factors
 - # of permutations P_r^n vs # of combinations $\binom{n}{r}$ vs $\binom{n}{r_1, \dots, r_k}$
 - $\square 0! = 1, n! = 1 \cdot 2 \cdot \dots \cdot n; (n \in \mathbb{N})$ \square several interpretations of $\binom{n}{r_1, \dots, r_k}$
 - deck of 52 cards: \square 4 suits (clubs, spades, hearts, diamonds) of 13 each: Ace, 2, 3, ..., 10, Jack, Queen, King \square so: 4 2's, 4 3's, 4 Aces, 4 Jacks, ...
 - Roulette: \square slots 0, 00, 1, 2, ..., 36 \square 18 black, 18 red; numbers 1 – 36 in 12 rows \times 3 cols
- (e) Random variables (rvs) and random elements**
- Discrete rand elem $X : (\Omega, P) \rightarrow \Omega', p(x) = p_X(x) = P_X\{x\}$: PMF = probab. mass func (WMS: probab. func.) for X .
 - Continuous rand vars $Y : (\Omega, P) \rightarrow \mathbb{R}, p(y) = p_Y(y)$: PDF for Y .
 - discrete & cont. rvs: CDF $F_Y(y)$; p th quantile ϕ_p
 - $E[Y], Var[Y], \sigma_Y$ of rv Y : \square Remember all formulas! $\square E[g(Y)] = \dots$
 - m'_k and m_k ; MGF $m_Y(t)$ compute how?
 - Each distribution: \square application context? $\square m_Y(t) = ?$ \square Given $m_Y(t)$: $Y \sim$ WHAT?
 - iid sequences of random elements \square Bernoulli trials and sequences \square 0–1 encoded Bernoulli trials
 - Discrete rvs: \square Bernoulli(p) \square binom(n, p) \square geom(p) \square neg. binom(p, r): $p(y) = \binom{y-1}{r-1} p^r q^{y-r}, \mu = \frac{r}{p}, \sigma^2 = \frac{r(1-p)}{p^2}$
 - \square hypergeom(N, R, n) \square Poisson(λ)
 - Contin rvs: \square uniform(θ_1, θ_2): $\sigma^2 = \frac{(\theta_1 - \theta_2)^2}{12}$
 - $\square \mathcal{N}(\mu, \sigma^2)$: empirical rule =? \square gamma(α, β) vs χ^2 (df = ν) vs expon(β)
 - $2 \times$ Tchebysheff – know them both!
- Continued on p.2!**

(f) Multivariate $\vec{Y} = (Y_1, \dots, Y_k)$:

- joint CDF $F_{\vec{Y}}(\vec{y})$, joint PMF $p_{\vec{Y}}(\vec{y})$, joint PDF $f_{\vec{Y}}(\vec{y})$ allow you to see whether the rvs are independent. HOW? What condition on $\{(y_1, y_2) : \text{the PDF or PMF is } > 0\}$? Must determine! • $E[g(Y_1, \dots, Y_n)] = \dots$ • $Cov[X, Y] = 0$ vs. X, Y independent. Relationship? • $\vec{Y} = (Y_1, Y_2)$: marginal CDFs F_{Y_j} , PMFs p_{Y_j} and PDFs f_{Y_j} conditional PMF $p_{Y_1|Y_2}(y_1|y_2)$ and PDF $f_{Y_1|Y_2}(y_1|y_2)$ define $E[Y_1 | Y_2]$, $Var[Y_1 | Y_2]$ how? • $E[E[Y_1 | Y_2]] = (8.50)$ • $Var[Y_1] = (8.53)$

- connection indep rvs and MGFs of ___

- Given a small 2-dim table (say, 3×4 entries) for a joint PMF, be able to compute marginal and conditional distributions and conditional expectations and variances.

- multinomial sequence X_1, X_2, \dots vs multinomial random vector (!) $\vec{Y} = (Y_1, \dots, Y_k)$: How does Y_j relate to $(X_n)_n$?

- Order stats: We only do them for continuous, iid rvs. Find CDFs for $Y(1)$ and $Y(n)$ directly; differentiate to get PDFs For $1 < j < n$: maybe find a corresponding multinomial sequence $f_{(\bullet)}(\vec{y}) = (9.40)$; proof done how?

(g) Functions (transforms) of rvs $U = h \circ Y$: • Method of transformations needs injectivity Formulas (9.24) and (9.26) • Method of distrib functions always works • MGF method good for sums of indep rvs. WHY?

(h) Sampling: • “sample” sometimes = the random vector (sampling action) \vec{Y} and sometimes = “the” realization

$\vec{y} = \vec{Y}(\omega)$ Random sample vs SRS: which is iid? Neither need be on a normal rv. • random sample with sample picks $Y_j (j = 1, \dots, n)$: $E[Y_j] = \mu$, $\sigma_{Y_j} = \sigma \Rightarrow E[\tilde{Y}] = ??$, $\sigma_{\tilde{Y}} = ??$

(i) Convergence of random variables and limit theorems:

- 4 modes of convergence $Y_n \rightarrow Y$; Imply what for $P\{a < Y_n \leq b\}$ vs. $P\{a < Y \leq b\}$? What mode for CLTs?

- Two laws of large numbers and two CLTs - What do they state? a CLT that uses σ^2 + normal distribution a CLT that uses S^2 + t -distribution • approximate binom(n, p) w. poisson rv (CLT not used) w. normal rv (need CLT)

- t -rvs defined with help of χ^2 -rvs. HOW? Both use df = degs of freedom. • $U_n \sim t(\text{df} = n) \Rightarrow D\text{-}\lim_n U_n = ??$

- Sample SDs S_n of random samples with $E[Y_j] = \mu$, $Var[Y_j] = \sigma^2 \Rightarrow D\text{-}\lim_n S_n = ??$

- CLT lets us work random samples with non-normal sample picks. HOW?

(j) NOT ON EXAM - MIGHT HELP TO REMEMBER STUFF:

- abstract integrals $\int \dots dP$: their properties determine those of $E[Y] = \int_{\Omega} Y dP = \int_{\mathbb{R}} y dP_Y$ $E[g \circ \vec{Y}] = \int_{\Omega} g \circ \vec{Y} dP = \int_{\mathbb{R}^n} g(\vec{y}) dP_{\vec{Y}}$ $Var[Y] = \int_{\Omega} g(Y) dP$ (with $g(y) = ??$) \bar{Y} and S^2 are statistics for a sample \vec{Y} . Means what?