Math 447 - Fall 2024 - Homework 04

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Status - Reading Assignments:

Here are the reading assignments to be completed before the first one of this HW.

WMS (Wackerly, et al. Textbook):

Nothing assigned yet

MF447 lecture notes:

Ch.1 - 3, ch.4 (non-optional parts), ch.5 through ch.5.4, Remark 5.18

Other:

Nothing assigned yet

New reading assignments:

> Important: Work through the examples of the WMS reading assignments!

Reading assignment 1 - due Monday, September 9:

a. If you missed this in HW2: Carefully (re-) read the non–optional part of MF ch.4.1 (not a lot).

a. Carefully read the remainder of MF ch.5.
Do not skip Example 5.16. Some of it is likely to show up on a quiz and/or exam.

Reading assignment 2 - due: Wednesday, September 11:

a. The strong students are encouraged to study MF ch.6.1. The others should review ch.5 instead. It is possible that I'll skip it in lecture (but I will refer to the material when expedient).

b. Ch.6.2 also is optional, but a lot more important than ch.6.1. The strong students definitely should study the material. For everyone else, I suggest the following for ch.6.2:
• Ignore σ–algebras throughout. • Look at its beginning through Example 6.11
• Understand to at least some extent how similar $\int f d\mu$ and $\int f d\lambda^d$ are. • Look at Definition 6.10 and Theorem 6.9 - Theorem 6.10.

Reading assignment 3 - due Friday, September 13:

a. The strong students are encouraged to study MF ch.6.4 and ch.6.5. I plan to lecture about some of that material. • The others should carefully read MF ch.9.1 instead. • I'll assign this chapter again at a later date, after the assignments for ch.7 and ch.8 have been given.

General note on written assignments: I will not collect those assignments for grading but doing them might be helpful for your quizzes and exams.

(a) The results of quiz 3 (topic: simple integration) were abysmal. Now that I have shown you how series and integrals are used to compute probabilities $P(A) = \sum\limits_{\omega \in A} p(\omega)$ and $P(B) = \int\limits_B f(\vec{y})d\vec{y}$ for suitable functions $p(\omega)$ and $f(\vec{y})$, I will keep bringing up series and integrals in future quizzes and all exams.

I advise you once more: Work closed book through the examples given in Section 3.4 (Series and Integrals as Tools to Compute Probabilities) of MF ver 2024-09-05!

(b) Even though I marked it as optional (I will not quiz you on its content), I suggest in particular to those of you who struggle with the mathyness of the material to study Remark Remark 4.5.

(c) Write rom memory the following definitions and compare them with the MF lecture notes:

- σ–algebra, probability measure, probability space, Null event, equiprobability, discrete probability space,
- Compare Theorem 5.2 with Theorem 3.6. discrete probability space,
- STUDY Remark 5.7 and put it in perspective with integration: Theorem 3.4. and ch.3.4.2 (Integrals).
- For the better students: Recite Thm.5.3 and the def. of $\sigma\{\mathscr{A}\}$, $\mathfrak{B}^d$, Borel sets, Fact 5.1
- STUDY Thm 5.5 and Rem.5.10, 5.11: Important for quizzes and exams!
- $P(A|B)$, what condition on B?; Thm.5.8 + PROOF!, Prop.5.4 and ALL definitions of independence! Cor.5.2 and 5.3.
- *** Preimages will be for sure on quizzes AND EXAMS! *** Read Introduction 5.2! The definition of the preimage $f^{-1}(V)$,where $f : X \to Y$. $V \in$ WHAT SET? Get used to different notation: If $X : (\Omega, P) \to \widetilde{\Omega}$ is a random element with outcomes in $\widetilde{\Omega}$ and $B \subseteq \widetilde{\Omega}$, write $X \in B = \{\_\_\_ \in \Omega : \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\}$; Review formula (5.33), Definition 5.13, Rem.5.15, Thm.5.11: Recite from memory!
- For the better students: direct images + Examples 5.12, 5.13 random variable, distribution
- Understand everything in ch.5.4 that leads to the defs of random element, random variable, random vector, distribution. $P_Y(W) = P\{???\}$ Here, $W \in$ WHAT?
- For the better students: Write $\sigma\{(X_i)_{i \in I}\}$ for a family of random elements X_i.
- Everyone: Understand the meaning of Cor.5.4.

(d) Work through remark 5.5 and examples 5.2, 5.3, 5.6, 5.7, 5.11.
- You MUST work closed book through examples 5.9, 5.10 until you get them right!

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(e) Work as soon as you can through WMS examples 5.3.c and 5.4 You should be able to grasp the essentials according to the following instructions:

(1) WMS: "$f(\vec{y}) = f(y_1, y_2)$ is the joint probability density function (joint PDF) of a random vector (Y_1, Y_2)" means in MF lingo:
• $\vec{Y} = (Y_1, Y_2) : (\Omega, P) \to \mathbb{R}^2$ is a random action (=random element), where Y_1, Y_2 are real–valued random actions (=random variables). See MF Def. 5.16 and 5.17.

- The WMS joint PDF f defines $P_{\vec{Y}}$ (distribution of $\vec{Y}$) on the Riemann integrable subsets of $\mathbb{R}^d$ $(d = 2)$ as in MF Thm.3.4: Asssume that $f \geq 0$ and $\int_{\mathbb{R}^d} f(\vec{y})\, d\vec{y} = 1$. Then

(⋆)
$$P_{\vec{Y}}(A) = \int_A f(\vec{y})\, d\vec{y}$$

is a Probability measure on $\mathscr{R}$ = Riemann integrable subsets of $\mathbb{R}^d$.]
• Relationship between P on the subsets of Ω and $P_{\vec{Y}}$ on the subsets of $\mathbb{R}^d$:

(⋆⋆)
$$P_{\vec{Y}}(A) = P\{\omega : \vec{Y}(\omega) \in A\}$$

= the probability that the outcome of the random action $\vec{Y}$ ends up in A.

- So, take WMS Example 5.3(c): Let $f(\vec{y}) := 1_{[0,1]\times[0,1]}(\vec{y})$. To find

$$\alpha := P\{0.1 \leq Y_1 \leq 0.3,\, 0 \leq Y_2 \leq 0.5\} = P\{\omega \in \Omega : 0.1 \leq Y_1(\omega) \leq 0.3,\, 0 \leq Y_2(\omega) \leq 0.5\}\,,$$

we proceed as follows:

$$\alpha \overset{(\star\star)}{=} P_{\vec{Y}}\{(y_1, y_2) \in \mathbb{R}^2 : 0.1 \leq y_1 \leq 0.3,\, 0 \leq y_2 \leq 0.5\} = P_{\vec{Y}}\big([0.1, 0.3] \times [0, 0.5]\big)\,.$$
$$\overset{(\star)}{=} \iint\limits_{[0.1,0.3]\times[0,0.5]} f(\vec{y}) d\vec{y} = \int_{0.1}^{0.3}\int_0^{0.5} f(\vec{y})\, dy_1\, dy_2 = \int_{0.1}^{0.3}\int_0^{0.5} 1\, dy_1\, dy_2 = \ldots$$

The last equation is true because $f = 1$ on the unit square. Thus,

$$\alpha = (0.5 - 0)\,(0.3 - 0.1) = 0.1\,. = P\{\omega \in \Omega : 0.1 \leq Y_1(\omega) \leq 0.3,\, 0 \leq Y_2(\omega) \leq 0.5\}\,,$$