Formula Collection for Math 447 Final Exam – Not all items are relevant!

(1) **(a)** • power set $2^\Omega = \{$ all subsets of $\Omega\}$ • $\forall\, x \dots$: For all $x \dots$ $\boxdot$ $\exists\, x$ s.t. $\dots$ There is an x such that $\dots$
$\boxdot$ $\exists!\, x$ s.t. $\dots$ There is a unique x s.t. $\dots$ $\boxdot$ $p \Rightarrow q$ If p is true then q is true $\boxdot$ $p \Leftrightarrow q$ p iff q, i.e., p true if and only if q true
• Intervals: $]a,b[= \{x \in \mathbb{R} : a < x < b$, $]a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \leq b$, $[a,b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \leq x \leq b$, etc.
• countable set A: can be sequenced: $\boxdot$ $A = \{a_1, a_2, \dots, a_n\}$ (finite set) $\boxdot$ $A = \{_1, a_2, \dots\}$ ("countably infinite" set)
$\boxdot$ $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is uncountable • family $(x_i)_{i \in I}$: index set I may be uncountable • $\bigcup_{i \in J} A_i = \{x : \exists\, i_0 \in J$ s.t. $x \in A_{i_0}\}$ • $\bigcap_{i \in J} A_i = \{x : \forall\, i \in J\, x \in A_i\}$. • Can use $A \uplus B$ for $A \cup B$ if disjoint sets • **De Morgan:** $\boxdot$ $\left(\bigcup_k A_k\right)^{\complement} = \bigcap_k A_k^{\complement}$ $\boxdot$ $\left(\bigcap_k A_k\right)^{\complement} = \bigcup_k A_k^{\complement}$ • **Distributivity:** $\boxdot$ $\bigcup_j (B \cap A_j) = B \cap \bigcup_j A_j$ $\boxdot$ $\bigcap_{j \in I} (B \cup A_j) = B \cup \bigcap_j A_j$
• Cartesian products: $\left| X_1 \times \dots \times X_n \right| = \left|X_1\right| \cdots \left|X_n\right|$ • Formulas f. preimages of $f : X \to Y$:
Arbitrary index set J and $B, B_j \subseteq Y$: $\boxdot$ $f^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$ $\boxdot$ $f^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} f^{-1}(B_j)$
$\boxdot$ $f^{-1}(B^{\complement}) = \left(f^{-1}(B)\right)^{\complement}$ $\boxdot$ $B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$ • $A \subseteq \Omega \Rightarrow \mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 else
• partition B_j ($j \in \mathbb{N}$) of Ω, $A \in \Omega \Rightarrow A = \biguplus_j (A \cap B_j) \Rightarrow P(A) = \dots$; Used to compute $P(B_{j_0} \mid A)$ from $P(A \mid B_j)$ and $P(B_j)$ (must know for all j) • Indicator function: $\mathbf{1}_A(y) =$

(b) Sums and Riemann integrals (Riem-$\int$) and Lebesgue integrals (Leb-$\int$):
• $x_n \geq 0$ or $\sum_n x_n$ abs conv $\Rightarrow \sum_n x_n$ satisfies WHAT? • Leb-$\int$: positive, monotone, linear, mon. + domin. conv.
• step function h: $\int h(\vec{y})d\vec{y} =$? • simple function g: $\int g d\lambda^d =$? • If both $\int_A f(\vec{y})d\vec{y}$, $\int_A f d\lambda^d$ exist, they are equal
• Use Fubini for both $\int_A f(\vec{y})d\vec{y}$ and $\int f d\lambda^d$ to compute multidim $\int$. • $\mathbf{1}_A$ Riem-$\int$-ble $\Rightarrow \lambda^d(A)$ defined how?
• $\mathfrak{B}^d = \sigma\{$ d–dim rectangles $\}$ • $\left[f \geq 0 \text{ or } \int |f| d\lambda^d < \infty\right] \Rightarrow \left[A \mapsto \int_A f\, d\lambda^d \text{ is } \sigma\text{–additive}\right]$
• For what functions φ, ψ is $A \mapsto \sum_{\omega \in A} \varphi(\omega)$, $A \mapsto \int_A \psi(\vec{y})d\vec{y}$ $\left(= \int_A \psi d\lambda^d\right)$ a probab measure?

(c) • Probability space = sample space (Ω, P) • σ–algebra $\mathfrak{F} \subseteq 2^\Omega$: $\boxdot$ $A \in \mathfrak{F} \Rightarrow A^{\complement} \in \mathfrak{F}$ $\boxdot$ $A_n \in \mathfrak{F} \Rightarrow \bigcup_{j=1}^\infty A_j \in \mathfrak{F}$ $\boxdot$ $\emptyset \in \mathfrak{F}$ • distribution of random element (rand elem) $X : (\Omega, P) \to \Omega'$: $P_X(B) = P\{X \in B\} = P\left(X^{-1}(B)\right)$ on codomain.
• Conveniences: $P_X(\{x\}) = P\{X = x\}$; $P_X(]a,b]) = P\{a < X \leq b\}$ (if X is random var. (rv), i.e., $\Omega' \subseteq \mathbb{R}$); ...
• discrete probab spaces and random elements and rvs defined how?

• independence for $2, n$, arbitr. many events • $P(A \mid B)$ • general addition & multiplication rules, complement rule, total probability, Bayes formula

(d) Combinatorial Analysis • Think: Does order matter in your probability space or doesn't it?
• multiplication rule for several factors • # of permutations P_r^n vs # of combinations $\binom{n}{r}$ vs $\binom{n}{r_1, \dots, r_k}$
$\boxdot$ $0! = 1$, $n! = 1 \cdot 2 \cdots n$; $(n \in \mathbb{N})$ $\boxdot$ several interpretations of $\binom{n}{r_1, \dots, r_k}$

• deck of 52 cards: $\boxdot$ 4 suits (clubs, spades, hearts, diamonds) of 13 each: Ace, $2, 3, \dots, 10$, Jack, Queen, King $\boxdot$ so: 4 2's, 4 3's, 4 Aces, 4 Jacks, $\dots$ • Roulette: $\boxdot$ slots $0, 00, 1, 2, \dots, 36$ $\boxdot$ 18 black, 18 red; numbers 1 – 36 in 12 rows $\times$ 3 cols

(e) Random variables (rvs) and random elements
• Discrete rand elem $X : (\Omega, P) \to \Omega'$, $p(x) = p_X(x) = P_X\{x\}$: PMF = probab. mass func (WMS: probab. func.) for X.
• Continuous rand vars $Y : (\Omega, P) \to \mathbb{R}$, $p(y) = p_Y(y)$: PDF for Y. • discrete & cont. rvs: CDF $F_Y(y)$; pth quantile ϕ_p
• $E[Y], Var[Y], \sigma_Y$ of rv Y: $\boxdot$ Remember all formulas! $\boxdot$ $E[g(Y)] = \dots$ • m_k' and m_k; MGF $m_Y(t)$ compute how?
• Each distribution: $\boxdot$ application context? $\boxdot$ $m_Y(t) =$? $\boxdot$ Given $m_Y(t)$: $Y \sim$ WHAT?
• iid sequences of random elements $\boxdot$ Bernoulli trials and sequences $\boxdot$ 0–1 encoded Bernoulli trials
• Discrete rvs: $\boxdot$ Bernoulli(p) $\boxdot$ binom(n, p) $\boxdot$ geom(p) $\boxdot$ neg. binom(p, r): $p(y) = \binom{y-1}{r-1} p^r q^{y-r}$, $\mu = \frac{r}{p}$, $\sigma^2 = \frac{r(1-p)}{p^2}$
$\boxdot$ hypergeom(N, R, n) $\boxdot$ Poisson(λ) • Contin rvs: $\boxdot$ uniform(θ_1, θ_2): $\sigma^2 = \frac{(\theta_1 - \theta_2)^2}{12}$
$\boxdot$ $\mathscr{N}(\mu, \sigma^2)$: empirical rule =? $\boxdot$ gamma(α, β) vs $\chi^2(\text{df} = \nu)$ vs expon(β) • $2\times$ Tchebysheff – know them both!
Continued on p.2!

(f) Multivariate $\vec{Y} = (Y_1, \dots, Y_k)$:

• joint CDF $F_{\vec{Y}}(\vec{y})$, joint PMF $p_{\vec{Y}}(\vec{y})$, joint PDF $f_{\vec{Y}}(\vec{y})$ ⊡ allow you to see whether the rvs are independent. HOW? ⊡ What condition on $\{(y_1, y_2)$: the PDF or PMF is $> 0\}$? Must determine! • $E[g(Y_1, \dots, Y_n)] = \cdots$ • $Cov[X, Y] = 0$ vs. X, Y independent. Relationship? • $\vec{Y} = (Y_1, Y_2)$: ⊡ marginal CDFs F_{Y_j}, PMFs p_{Y_j} and PDFs f_{Y_j} ⊡ conditional PMF $p_{Y_1|Y_2}(y_1|y_2)$ and PDF $f_{Y_1|Y_2}(y_1|y_2)$ define $E[Y_1 \mid Y_2]$, $Var[Y_1 \mid Y_2]$ how? • $E\big[\, E[Y_1 \mid Y_2] \,\big]$ = (8.50) • $Var[Y_1]$ = (8.53)

• connection indep rvs and MGFs of ___

• Given a small 2-dim table (say, 3×4 entries) for a joint PMF, be able to compute marginal and conditional distributions and conditional expectations and variances.

• multinomial sequence $X_1, X_2, \dots$ vs multinomial random vector(!) $\vec{Y} = \big(Y_1, \dots, Y_k\big)$: How does Y_j relate to $(X_n)_n$?

• Order stats: ⊡ We only do them for continuous, iid rvs. ⊡ Find CDFs for $Y(1)$ and $Y(n)$ directly; differentiate to get PDFs ⊡ For $1 < j < n$: maybe find a corresponding multinomial sequence ⊡ $f_{(\bullet)}(\vec{y})$ = (9.40); proof done how?

(g) Functions (transforms) of rvs $U = h \circ Y$: • Method of transformations needs injectivity Formulas (9.24) and (9.26) ⊡ • Method of distrib functions always works • MGF method good for sums of indep rvs. WHY?

(h) Sampling: • "sample" sometimes = the random vector (sampling action) $\vec{Y}$ and sometimes = "the" realization $\vec{y} = \vec{Y}(\omega)$ ⊡ Random sample vs SRS: which is iid? ⊡ Neither need be on a normal rv. • random sample with sample picks Y_j $(j = 1, \dots, n)$: ⊡ $\overline{Y}, S, S^2$ defined how? ⊡ When are $\overline{Y}, S^2$ independent? When is $\frac{n-1}{\sigma^2} S^2 \sim \chi^2(n-1)$?

⊡ $E[Y_j] = \mu$, $\sigma_{Y_j} = \sigma \;\Rightarrow\; E[\overline{Y}] = ??$, $\sigma_{\overline{Y}} = ??$

(i) Convergence of random variables and limit theorems:

• 4 modes of convergence $Y_n \to Y$; ⊡ Imply what for $P\{a < Y_n \le b\} vs.\ P\{a < Y \le b\}$? ⊡ What mode for CLTs?

⊡ Two laws of large numbers and two CLTs - What do they state? ⊡ a CLT that uses σ^2 + normal distribution ⊡ a CLT that uses S^2 + t–distribution • approximate binom(n, p) ⊡ w. poisson rv (CLT not used) ⊡ w. normal rv (need CLT)

• t–rvs defined with help of χ^2–rvs. HOW? Both use df = degs of freedom. • $U_n \sim t(\text{df} = n) \;\Rightarrow\;$ D–$\lim_n U_n = ??$

• Sample SDs S_n of random samples with $E[Y_j] = \mu$, $Var[Y_j] = \sigma^2 \;\Rightarrow\;$ a.s.–$\lim_n S_n = ??$

• CLT lets us work random samples with non–normal sample picks. HOW?

(j) NOT ON EXAM - MIGHT HELP TO REMEMBER STUFF:

• abstract integrals $\int \dots dP$: their properties determine those of ⊡ $E[Y] = \int_\Omega Y dP = \int_{\mathbb{R}} y dP_Y$ ⊡ $E[g \circ \vec{Y}] = \int_\Omega g \circ \vec{Y} dP = \int_{\mathbb{R}^n} g(\vec{y}) dP_{\vec{Y}}$ ⊡ $Var[Y] = \int_\Omega g(Y) dP$ (with $g(y) = ??$) ⊡ $\bar{Y}$ and S^2 are statistics for a sample $\vec{Y}$. Means what?