Formula Collection for Math 447 Final Exam, Fall 2025 – Not all items are relevant!

Advice: Study this sheet before the exam, so you know where to look if you need something!

Abbreviations: • rv = r.v. = random variable; r.e. = random element • prob = probability • distr = distribution • cond = conditional • cond$\mathbb{P}$ = conditional prob • fn = function • conv = convergence • cont = continuous/continuity • abs = absolute • w.r.t = with respect to • def'd = defined • discr = discrete • indep = independent, independence • $\int$-ble = integrable • meas = measure • cond.$\mathbb{E}$ = conditional expectation • mble = measurable

(1) **(a)** • power set $2^\Omega = \{$ all subsets of $\Omega\}$ • $\forall\, x \dots$: For all $x \dots$ ⊡ $\exists\, x$ s.t. $\dots$ There is an x such that $\dots$ ⊡ $\exists!\, x$ s.t. $\dots$ There is a unique x s.t. $\dots$ ⊡ $p \Rightarrow q$ If p is true then q is true ⊡ $p \Leftrightarrow q$ p iff q, i.e., p true if and only if q true
• Intervals: $]a,b[= \{x \in \mathbb{R} : a < x < b,\ ]a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \leq b,\ [a,b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \leq x \leq b$, etc.
• countable set A: can be sequenced: ⊡ $A = \{a_1, a_2, \dots, a_n\}$ (finite set) ⊡ $A = \{_1, a_2, \dots\}$ ("countably infinite" set)
⊡ $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is uncountable • family $(x_i)_{i \in I}$: index set I may be uncountable • $\bigcup_{i \in J} A_i = \{x : \exists\, i_0 \in J$ s.t. $x \in A_{i_0}\}$ • $\bigcap_{i \in J} A_i = \{x : \forall\, i \in J\, x \in A_i\}$. • Can use $A \uplus B$ for $A \cup B$ if disjoint sets • **Distributivity:** ⊡ $\bigcup_j (B \cap A_j) = B \cap \bigcup_j A_j$ ⊡ $\bigcap_{j \in I} (B \cup A_j) = B \cup \bigcap_j A_j$ • **De Morgan:** ⊡ $\left(\bigcup_k A_k\right)^{\complement} = \bigcap_k A_k^{\complement}$ ⊡ $\left(\bigcap_k A_k\right)^{\complement} = \bigcup_k A_k^{\complement}$
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• Cartesian products: $\left|\, X_1 \times \cdots \times X_n \,\right| = \left|X_1\right| \cdots \left|X_n\right|$ • Formulas f. preimages of $f : X \to Y$:
Arbitrary index set J and $B, B_j \subseteq Y$: ⊡ $f^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$ ⊡ $f^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} f^{-1}(B_j)$
⊡ $f^{-1}(B^{\complement}) = \left(f^{-1}(B)\right)^{\complement}$ ⊡ $B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$ • $A \subseteq \Omega \Rightarrow \mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 else
• partition B_j $(j \in \mathbb{N})$ of Ω, $A \in \Omega \Rightarrow A = \biguplus_j (A \cap B_j) \Rightarrow P(A) = \dots;$

(b) Sums and Riemann, Lebesgue, abstract integrals: Riem-$\int$ = $\int_A f(\vec{y})d\vec{y}$, Leb-$\int$ = $\int_A f d\lambda^d$, abstr-$\int$ = $\int_A f d\mu$:
• $x_n \geq 0$ or $\sum_n x_n$ abs conv $\Rightarrow \sum_n x_n$ satisfies WHAT? • $\mathbb{1}_A$ **=?** • $\mathbf{1}_A$ Riem-$\int$-ble $\Rightarrow \lambda^d(A)$ def'd how? • Borel sets $\mathfrak{B}^d = \sigma\{$ d–dim rectangles $\}$ • meas μ is like λ^d, but $(\Omega, \mathfrak{F})$ replaces $(\mathbb{R}, \mathfrak{B}^d)$ • μ is like prob meas P, but need not obey WHAT? • step function h: $\int h(\vec{y})d\vec{y} =$? • simple function g: $\int g d\lambda^d =$? • $\int_A f(\vec{y})d\vec{y} = \int_A f d\lambda^d$ if both exist • common roots between $\int \dots d\lambda^d$, $\int \dots d\mu$ (thus, $\int \dots dP$) $\Rightarrow$ ALL 3 satisfy ⊡ positive, monotone, linear ⊡ mon. + domin. conv
⊡ Use Fubini to compute multidim $\int$. ⊡ $\left[f \geq 0 \text{ or } \int |f| d☺ < \infty\right] \Rightarrow \left[A \mapsto \int_A f\, d☺ \text{ is } \sigma\text{–additive; here } ☺ = \lambda^d, P, \mu\right]$
⊡ So, when are φ, ψ is $A \mapsto \sum_{\omega \in A} \varphi(\omega)$, $A \mapsto \int_A \psi(\vec{y})d\vec{y}\ \left(= \int_A \psi d\lambda^d\right)$ prob meas? ⊡ Fubini just like for Riem-$\int$

(c) Combinatorial Analysis • Think: Does order matter in your prob space or doesn't it?
• multiplication rule for several factors • # of permutations P_r^n vs # of combinations $\binom{n}{r}$ vs $\binom{n}{r_1, \dots, r_k}$
⊡ $0! = 1, n! = 1 \cdot 2 \cdots n;\ (n \in \mathbb{N})$ ⊡ several interpretations of $\binom{n}{r_1, \dots, r_k}$

• deck of 52 cards: ⊡ 4 suits (clubs, spades, hearts, diamonds) of 13 each: Ace, $2, 3, \dots, 10$, Jack, Queen, King ⊡ so: 4 2's, 4 3's, 4 Aces, 4 Jacks, ... • Roulette: ⊡ slots $0, 00, 1, 2, \dots, 36$ ⊡ 18 black, 18 red; numbers 1 – 36 in 12 rows $\times$ 3 cols

(d) • prob space = sample space $(\Omega, \mathfrak{F}, P)$ • $\mathfrak{F} \subseteq 2^\Omega$ def'd how? • distr P_X of r.e. $X : (\Omega, P) \to \Omega'$ def'd on Ω'(!) how?
• Conveniences: ⊡ $\{X = x\} = X^{-1}(\{x\})$; ⊡ $\{X \in B\} = X^{-1}(B)$; ⊡ if Y is r.v., i.e., $\Omega' \subseteq \mathbb{R}$: $\{a < Y \leq b\} = Y^{-1}\left(]a,b]\right)$
• indep for $2, n$, arbitr. many events $(A_i)_i$ or r.e.s $(X_i)_i$ • $P(A \mid B), P(X \in A \mid B), P(A \mid X \in B), \dots$
• Addition rule; multiplication rule for cond$\mathbb{P}$s; complement rule, total prob, Bayes formula

• discr prob spaces and r.e.s and r.v.s X def'd how? Use $p_X(x)$ for $P\{X \in B\}$ how? • cont r.v.s Y def'd how? Use $f_Y(y)$ for $P\{Y \in B\}$ how? • LOTUS: r.e. $X : \Omega \to \Omega'$ and $g : \Omega' \to \mathbb{R}$ Then $E[g(X)] = \int \dots dP = \int \dots dP_X$.
So, ⊡ for X discrete r.e. $E[g(X)] = ?$ ⊡ for Y cont r.v. $E[g(Y)] = ?$ ⊡ for ANY r.v. Y, $E[Y] = ?$ Review Thm.6.13!
• For a r.v. Y, what are $F_Y(y)$, ϕ_p, $E[Y], \sigma_Y^2 = Var[Y]$, $\sigma_Y =$ SD[Y] μ_k', μ_k, $m_Y(t)$ What if Y is discr, Y is cont?

(e) Specific distributions (of discrete and continuous r.v.s): • Can completely ignore beta distr

• iid sequences of random elements ⊡ Bernoulli trials and sequences ⊡ 0–1 encoded Bernoulli trials

• binom(n,p) : $m_Y(t) = \left(pe^t + q\right)^n$ • neg. binom(p,r): $p(y) = \binom{y-1}{r-1}p^r q^{y-r}, \mu = \frac{r}{p}, \sigma^2 = \frac{r(1-p)}{p^2}$

• poisson(λ) : $m_Y(t) = e^{\lambda(e^t-1)}$ • uniform(θ_1,θ_2): $\sigma^2 = \frac{(\theta_1-\theta_2)^2}{12}$

• Don't worry about $m_Y(t)$ for $Y \sim$ uniform, negbinom, or hypergeom • Don't worry about $E[Y], Var[Y]$ for $Y \sim$ hypergeom(N,R,n) • Except for above: <u>MEMORIZE!</u> $p_Y(y), f_Y(y), E[Y], Var[Y]$, SD$[Y], m_Y(t)$ from ch.9 & 10!

• connection betw. $F_Y(y)$, pth quantile $\phi(p) = \phi_p$, uniform distr • $m_Y(t)$ determines μ'_k HOW?

• $\mathscr{N}(\mu,\sigma^2)$: empirical rule =? • gamma(α,β) vs $\chi^2(\text{df}=\nu)$ vs expon(β) • $a > 0 \Rightarrow P\{|Y| \geq a\} \leq E[|Y|^n/a^n]$ (Markov ineq.) • 2× Tchebysheff – know them both (or deduce from Markov) • mixed probs NOT on exam

• Each distribution: ⊡ **typical application = ?** ⊡ Given $m_Y(t)$, must remember P_Y (i.e. $Y \sim$ WHAT?)

(f) Multivariate distributions $P_{\vec{Y}}(B) = P\{\vec{Y} \in B\}$ (B = Borel set in $\mathbb{R}^n$), for a vector $\vec{Y} = (Y_1,\dots,Y_k)$ of r.v.s:

• compare joint CDF $F_{\vec{Y}}(\vec{y})$, PMF $p_{\vec{Y}}(\vec{y})$, and PDF $f_{\vec{Y}}(\vec{y})$ with marginal CDFs F_{Y_j}, PMFs p_{Y_j} and PDFs f_{Y_j} to see whether the rvs are independent. HOW? (What assumption must $\{(y_1,y_2)$: the PDF or PMF is $> 0\}$ satisfy?) • use cond PMF $p_{Y_1|Y_2}(y_1|y_2)$; cond PDF $f_{Y_1|Y_2}(y_1|y_2)$ to compute $E[Y_1 \mid Y_2]$, $Var[Y_1 \mid Y_2]$ HOW? • $E[g(Y_1,\dots,Y_k \mid Y=y] =??$

• $E\big[\,E[Y_1 \mid Y_2]\,\big] =$ (11.57) • $Var[Y_1] =$ (11.60) • $Cov[X,Y] = 0$ vs. X, Y independent. Which ⇒ which?

• connection betw sums of indep rvs and their MGFs ... • Given a small 2-dim table (say, 3×4 entries) for a joint PMF, <u>be able to compute</u> marginal and cond distributions and cond.$\mathbb{E}$s and cond variances.

• multinomial distr: ⊡ def. $X_1,\dots,X_n$ multinom. seq; ⊡ def. $\vec{Y} \sim$ multinom w. $n, p_1,\dots,p_k$ ⊡ $E[Y_j]$, $Var[Y_j]$, $Cov[Y_i,Y_j] = \dots$ ⊡ typical use case = WHAT? • Order stats (for continuous, iid rvs): ⊡ Find $F_{Y(1)}$ and $F_{Y(n)}$ directly; differentiate to get PDFs ⊡ For 2 or more indices: compute PDF by finding a corresponding multinomial sequence

(g) Functions (transforms) of rvs $U = h \circ Y$: • Method of distrib functions always works, whereas Method of transformations needs injectivity on $suppt(f_Y)$ • Multidim method of transformations w. Jacobians (will be on final!)

(h) Sampling (ch.8 & 13): • "sample" sometimes = the random vector (sampling action) $\vec{Y}$ and sometimes = "the" realization $\vec{y} = \vec{Y}(\omega)$ ⊡ Random sample vs SRS: which is iid? ⊡ Neither need be on a normal rv. • random sample on rv Y means $Y_j \sim Y$ (Y_j = sample picks) • $\overline{Y}, S, S^2$ (also written $\overline{Y}_n, S_n, S_n^2$) defined how? ⊡ When are $\overline{Y}, S^2$ independent? When is $\frac{n-1}{\sigma^2}S^2 \sim \chi^2(n-1)$? ⊡ $E[Y_j] = \mu,\ \sigma_{Y_j} = \sigma \Rightarrow E[\overline{Y}] =??,\ \sigma_{\overline{Y}} =??$

(i) Convergence of random variables and limit theorems:

• 4 modes of convergence $Y_n \to Y$; ⊡ Imply what for $P\{a < Y_n \leq b\} vs. P\{a < Y \leq b\}$? ⊡ What mode for CLTs?

• $Y_n \xrightarrow{\mathbf{P}} Y$ means $\forall\, \varepsilon > 0\ \lim\limits_{n\to\infty} \mathbb{P}\{\omega \in \Omega : |Y_n(\omega) - Y(\omega)| > \varepsilon\} = 0$ • Two laws of large numbers and two CLTs - What do they state? ⊡ a CLT that uses σ^2 + normal distribution ⊡ a CLT that uses S^2 + t–distribution • 2 ways to approximate binom(n,p) ⊡ (1) w. poisson rv (CLT not used; see ch.9) ⊡ (2) w. normal rv (need CLT)

• t–rvs defined with help of χ^2–rvs. HOW? Both use df = degs of freedom. • $U_n \sim t(\text{df}=n) \Rightarrow$ D–$\lim_n U_n =??$

• Sample SDs S_n of random samples with $E[Y_j] = \mu$, $Var[Y_j] = \sigma^2 \Rightarrow$ a.s.–$\lim_n S_n =??$

• CLT lets us work random samples with non–normal sample picks. HOW?