Math 454 - Spring 2021 - Homework 02

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Update February 21, 2021

Added reading assignment 3 and written assignments.

Solutions

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

SCF2 (Shreve Textbook):
ch.1, ch.2 through Example 2.3.3

MF454 lecture notes:
ch.2, ch.3

Other: Nothing assigned yet

New reading assignments:

Reading assignment 0 - due ASAP:

I decided to teach more material in the context of measures μ rather than confined to probabilities P. As a result I am teaching more of MF454 before switching to SCF2. I already taught MF454 ch.4.1, so please do the following as quickly as you can to be well prepared for lecture.

- Carefully read MF454 ch.4. If a new version of MF454 should be published then look for additions. The references would be off, so it's rather easy to see where I have inserted new material.

Reading assignment 1 - due Monday, February 22:

a. Finish reading assignment zero.
b. A new version of MF454 will have a chapter 4.4: Convergence of measurable functions and integrals. It will be the last chapter on non-finance related material that goes beyond SCF2. Read it carefully as soon as I publish it.

Reading assignment 2 - due: Wednesday, February 24:

a. Carefully read the remainder of SCF2 ch.2.
b. Skim the exercises of SCF2 ch.2. There is a lot of interesting material about conditional expectations as generalized least squares solutions.

Reading assignment 3 - due Friday, February 26:

a. Read SCF2 ch.3.1 and 3.2. Do not get bogged down in the calculations in ch.3.2.5 – 3.2.7. The only aspect you want to remember that there is a discrete time model for stock prices, given by the process S_n (for which you need not remember the formula (3.2.15)); and that this process converges in distribution to the process $S(t)$ given by formula (3.2.16).

Written assignments:

General note on written assignments: I will not collect those assignments for grading but doing them might be helpful for your quizzes and exams.

Do SCF2 exercises 1.5, 1.6, 1.9, 1.10.

SCF2 exercise 1.5:
Let X be a nonnegative random variable with cumulative distribution function $F(x) = P\{X \leqq x\}$. Show that

$$E[X] \;=\; \int_0^\infty \big(1 - F(x)\big)\, dx$$

by showing that

$$\int_\Omega \int_0^\infty 1_{[0,X(\omega)[}(x)\, dx\, dP(\omega) \tag{0.1}$$

is equal to both $E[X]$ and $\int\limits_0^\infty dx\big(1 - F(x)\big)$.

Solution to SCF2 exercise 1.5:

Let $A := \int\limits_\Omega \Phi(\omega)\, dP(\omega)$ and $B := \int\limits_0^\infty \Psi(x)\, dx$, where

$$\begin{aligned}
\Phi(\omega) \;&:=\; \int_0^\infty 1_{[0,X(\omega)[}(x)\, dx \;=\; \int_0^{X(\omega)} 1\, dx \;=\; X(\omega), \\
\Psi(x) \;&:=\; \int_\Omega 1_{[0,X(\omega)[}(x)\, d\, P(\omega)
\end{aligned}$$

We obtain

$$A \;=\; \int_\Omega X(\omega) d\, P(\omega) \;=\; E[X]. \tag{0.2}$$

To compute B, observe that $1_{[0,X(\omega)[}(x) = 1 \;\Leftrightarrow\; 0 \leqq x < X(\omega) \;\Leftrightarrow\; x < X(\omega) \leqq \infty$, thus

$$\Psi(x) \;:=\; \int_{X>x} d\, P(\omega) \;=\; P\{X > x\} \;=\; 1 \;-\; P\{X \leqq x\} \;=\; 1 - F(x),$$

hence

$$B \;=\; \int_0^\infty dx\big(1 - F(x)\big) \tag{0.3}$$

We can switch the order of integration in (**??**), so $A = B$, i.e.,

$$E[X] \;=\; \int_0^\infty dx\big(1 - F(x)\big). \;\blacksquare$$

Solution to SCF2 exercise 1.6:

Will not be given Your text on mathematical probability should show how to compute the MGF of the $N(\mu, \sigma^2)$ distribution, and part 2 is a trivial consequence of part 1 since $u \mapsto e^u \geqq 1$ for $u \geqq 0$ and, according to part 1, $E[\Phi \circ X] = e^{u\mu + \frac{1}{2}\sigma^{@}}$, whereas $\Phi\big(E[X]\big) = \Phi(\mu) = e^{u\mu}$.