Math 454 - Spring 2021 - Homework 03

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Solutions

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

SCF2 (Shreve Textbook):
ch.1-2, ch.3.1-3.2

MF454 lecture notes:
ch.2, ch.3, ch.4.1–4.3

Other: Nothing assigned yet

New reading assignments:

Reading assignment 0 - due ASAP:

- Rather than reading SCF2 ch.3.1 and 3.2 you should focus on the material I already have done in class: New additions to MF454 ch.4.3, if any, and MF454 ch.4.4–4.7. On this Friday I will lecture about ch.5.1.

Reading assignment 1 - due Monday, March 1:

a. Finish reading assignment zero.
b. Do the reading that was assigned for Feb.26: Read SCF2 ch.3.1 and 3.2. Do not get bogged down in the calculations in ch.3.2.5 – 3.2.7. The only aspect you want to remember that there is a discrete time model for stock prices, given by the process S_n (for which you need not remember the formula (3.2.15)); and that this process converges in distribution to the process $S(t)$ given by formula (3.2.16).

Reading assignment 2 - due: Wednesday, March 3:

a. Carefully read SCF2 ch.3.3.1–3.3.3.

Reading assignment 3 - due Friday, March 5:

a. Carefully read SCF2 ch.3.3.4–3.4.3.

Written assignments:

> **General note on written assignments:** I will not collect those assignments for grading but doing them might be helpful for your quizzes and exams.

Written assignment 1:
Prove MF454 Remark 4.20(i): The relation $\mu \sim \nu \Leftrightarrow \mu \ll \nu$ and $\nu \ll \mu$ is an equivalence relation on the set of all measures for $(\Omega, \mathfrak{F})$.

Solution to assignment 1:

Obvious. ■

Written assignment 2:
Let $\Omega = \{\omega_1, \omega_2, \dots\}$ be a countable set and P and Q be two probability measures on $(\Omega, 2^\Omega)$, defined by their probability mass functions

$$p_n \;=\; P\{\omega_n\}, \qquad q_n \;=\; Q\{\omega_n\}$$

Assume that $p_n > 0$ and $q_n > 0$ for all $n \in \mathbb{N}$. Compute the Radon–Nikodým derivatives $\frac{dP}{dQ}$ and $\frac{dQ}{dP}$.

They exist. Can you justify why (without actually trying to compute them)?

Solution to assignment 2:

Let $f_j := \frac{dQ}{dP}(\omega_j)$ and $g_j := \frac{dP}{dQ}(\omega_j)$. Fix j. Then

$$\begin{aligned} q_j &= Q\{\omega_j\} \;=\; \int_{\{\omega_j\}} f_j \, dP \;\overset{(\star)}{=}\; f_j \int_{\{\omega_j\}} dP \;=\; f_j \, P\{\omega_j\} \;=\; f_j p_j, \\ p_j &= P\{\omega_j\} \;=\; \int_{\{\omega_j\}} g_j \, dQ \;\overset{(\star)}{=}\; g_j \int_{\{\omega_j\}} dQ \;=\; g_j \, Q\{\omega_j\} \;=\; g_j q_j, \end{aligned}$$

In both cases $(\star)$ is valid because the functions f_j and g_j are obviously constant on the singleton $\{\omega_j\}$. It follows that

$$\begin{aligned} \frac{dQ}{dP}(\omega_j) &= f_j = \frac{q_j}{p_j}, \\ \frac{dP}{dQ}(\omega_j) &= g_j = \frac{p_j}{q_j}. \end{aligned}$$

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