Math 454 - Spring 2023 - Homework 04

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Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

SCF2 (Shreve Textbook):
ch.1 - 2

MF454 lecture notes:
ch.2 - 6.1, Remark 6.2

Other:
Nothing assigned yet

New reading assignments:

Reading assignment 1 - due Monday, February 13:

- **a.** Carefully read the remainder of MF ch.6.1.
- **b.** Carefully read MF ch.6.2.
- **c.** Skim MF ch.6.3 (optional). I'll refer to $\|\vec{x}\|_2 = \sqrt{\sum_{j=1}^{n} {x_j}^2}$ and $\|f\|_{L^2} = \sqrt{\int f^2\,d\mu}$ in lecture, but the material will not be on any quiz or exam.

Reading assignment 2 - due: Wednesday, February 15:

- **a.** Carefully read MF ch.6.4.
- **b.** Skim Shreve ch.3.1-3.2
- **c.** Review Shreve ch.3.3 - 3.4 with the goal to deepen your understanding of MF ch.6.2 and ch.6.4

Reading assignment 3 - due Friday, February 17:

- **a.** Read VERY CAREFULLY MF ch.6.5 and ch.6.6 through Theorem 6.6. If I refer to SCF2 for a (non–optional) proof, then be sure to study the proof in that book! Read SCF2 ch.3.5 concurrently.
- **b.** Skim the remainder of MF ch.6.
- **c.** SCF2 ch.3.6 and 3.7 is of importance for the pricing of "American options". Since we do not address this topic in this class, you may SKIP that material.
- **d.** Read carefully MF ch.7 through Definition 7.6 (Self-financing portfolio). We are finally dealing with the "Finance" part of "Financial Mathematics".

Written assignments:

General note on written assignments: I will not collect those assignments for grading but doing them might be helpful for your quizzes and exams.

Prove the following assignments closed book (once you have looked up the proposition).

Written assignment 1:

In the following we assume that X is real valued and $I = [0, \infty[$.

(1) Let $A = \{2.78 < X_s \leqq 3.14,\ \text{for } 5 \leqq s < 7\}$. True or false, and why?
• $A \in \mathfrak{F}_7^X$ • $A \in \mathfrak{F}_{6.999}^X$,

(2) For some arbitrary $t, h > 0$, let $B = \{X_{t+h} < 0\}$. True or false, and why?
• $B \in \mathfrak{F}_{t+h}^X$ • $B \in \mathfrak{F}_t^X$,

(3) Assume that X has continuous trajectories $s \mapsto X_s(\omega)$. For $0 < h < T$ let $Z_T(\omega) = \int\limits_0^T X_u(\omega)du$ (Riemann integral) True or false, and why?
• Z_T is $\mathfrak{F}_{T-h}^X$–measurable • Z_T is $\mathfrak{F}_T^X$–measurable • Z_T is $\mathfrak{F}_{T+h}^X$–measurable

(4) Assume that X has continuous trajectories $s \mapsto X_s(\omega)$. Let $\tau(\omega) := \inf\{s \geqq 0 : X_s(\omega) \geqq 20\}$ i.e., τ = first time that the trajectory enters the interval $[20, \infty[$. True or false, and why?
• $\{\tau \leqq 8.5\} \in \mathfrak{F}_{8.499}$ • $\{\tau \leqq 8.5\} \in \mathfrak{F}_{8.5}$ • $\{\tau \leqq 8.5\} \in \mathfrak{F}_{8.501}$

(4a) Same as **(4)**, but we replace 8.5 with $t > 0$, and 20 with $\gamma \in \mathbb{R}$. $\tau(\omega) := \inf\{s \geqq 0 : X_s(\omega) \geqq \gamma\}$ True or false, and why?
• $\{\tau \leqq t\} \in \mathfrak{F}_{t-\delta}$ • $\{\tau \leqq 8.5\} \in \mathfrak{F}_t$ • $\{\tau \leqq 8.5\} \in \mathfrak{F}_{t+\delta}$

(5) This one is similar to the last problem on quiz 3. Assume that X has continuous trajectories $s \mapsto X_s(\omega)$. Let $\rho(\omega) := \sup\{s \geqq 0 : X_s(\omega) \geqq 20\}$, i.e., ρ = the <u>last</u> time that the trajectory is inside the interval $[20, \infty[$. True or false, and why? If $0 < \delta < t$, then
• $\{\rho \leqq t\} \in \mathfrak{F}_{t-\delta}$ • $\{\rho \leqq t\} \in \mathfrak{F}_t$ • $\{\rho \leqq t\} \in \mathfrak{F}_{t+\delta}$

For the answer, see MF Example 4.6.

Written assignment 2:

Prove parts **a–d** of Theorem 4.3 (Fundamental properties of the abstract integral). For the answer, see SCF2, the proofs of Theorems 1.3.1 and 1.3.4.

Written assignment 3: Write from memory

• the definitions of convergence μ–a.e. and P–a.s • monotone and dominated convergence theorems

Written assignment 4: Use the standard machine to prove Proposition 4.18:

Let $(\Omega, \mathfrak{F}, \mu)$ be a measure space and let $f \geqq 0$ be an extended real–valued, Borel–measurable function on Ω.

Let ν be the measure defined by $\nu(A) := \int_A f\, d\mu$, and let φ be an extended real–valued, Borel–measurable function on Ω such that $\varphi \geqq 0$ or φ is ν–integrable. Then

$$\int_A \varphi\, d\nu = \int_A \varphi \cdot f\, d\mu, \quad \text{for all } A \in \mathfrak{F}.$$

Written assignment 5: Write from memory the definitions of $\nu \ll \mu$ and $\nu \sim \mu$, and prove Proposition 4.20.

Written assignment 6: Write from memory the definition of the independence of arbitrary families of σ–algebras $\mathscr{H}_i \in \mathfrak{F}$ and of random variables X_i.

Write as many equivalent formulations of two random variables being independent as you remember. Then compare to Theorem 4.16.

Written assignment 7: Write from memory the definition of a conditional expectation $E[X \mid \mathfrak{G}]$ and its properties. Prove closed book that $E[X \mid \mathfrak{G}]$ is linear in X. See Definition 5.1, Theorem 5.4 and Theorem 5.5.