Math 454 - Spring 2023 - Homework 05

Published: Wednesday, February 15, 2023

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

SCF2 (Shreve Textbook):
ch.1 - 3.7

MF454 lecture notes:
ch.2 - 6, ch.7 through Definition 7.6 (Self-financing portfolio)

Other:
Nothing assigned yet

New reading assignments:

Reading assignment 1 - due Monday, February 20:

- **a.** Carefully read the remainder of MF ch.7.1. This chapter is the cornerstone of the financial market definitions that are used in this course to do option pricing. Reread it a second and a third time as time progesses!
- **b.** Carefully read MF ch.7.2.
- **c.** Carefully read MF ch.7.3 through the end of ch.7.3.1 (The One Period Model) That's lots of pages, but the math is trivial.

Reading assignment 2 - due: Wednesday, February 22:

- **a.** Carefully read the remainder of MF ch.7. Be sure you work through the binomial tree example. It is of great importance by itself, and also to understand the continuous time models.
- **b.** Carefully read MF ch.8.1 and 8.2. Refer to SCF2 ch.4 for the proofs!

Reading assignment 3 - due Friday, February 24:

- **a.** Carefully read MF ch.8.3 and 8.4. Refer to SCF2 ch.4 for the proofs!
- **b.** Study for the midterm!

General note on written assignments: I will not collect those assignments for grading but doing them might be helpful for your quizzes and exams.

Prove the following assignments closed book (once you have looked up the proposition).

Written assignment 1: Do the proof of (optional) Proposition 5.1 (Doob Factorization Lemma). It is a very brief proof that is based on the "standard machine".

Written assignment 2: First review Proposition 5.2, then do the following problem, which was the last problem of Quiz #4:

Let $\Omega := [0,3]$, $\mathfrak{F} :=$ Borel sets of $[0,3]$, and let P be the probability measure on $([0,3], \mathfrak{F})$, defined by $P[a,b[:= (b-a)/3$ for $0 \leqq a < b \leqq 1$. Let $G_1 := [0,1[$ and $G_2 := [1,3]$. Let $\mathfrak{G}$ be the σ–algebra generated by G_1 and G_2. Let $X(\omega) := 4\omega$ for $0 \leqq \omega \leqq 3$. Compute $E[X \mid \mathfrak{G}]$. **Hint**: This is a function of ω!

Do this BEFORE LOOKING AT THE SOLUTION GIVEN BELOW!

Solution:

Note that $E[X \mid \mathfrak{G}](\omega) = 1/P(G_j) \cdot \int_{G_j} 4\omega d\omega$ for $j = 1,2$, and that P is the uniform probability measure

on $[0,3]$, i.e., $P[a,b[= \int_a^b (1/3)dx$, i.e. $dP(x) = (1/3)dx$.

Since $P(G_1) = 1/3$ and $P(G_2) = 2/3$,

$$\omega \leqq 1 \;\Rightarrow E[X \mid \mathfrak{G}](\omega) \;=\; 3\int_0^1 (4x)(1/3)dx \;=\; 4\int_0^1 x dx \;=\; 2x^2\Big|_0^1 \;=\; 2(1-0) = 2\,,$$

$$\omega > 1 \;\Rightarrow E[X \mid \mathfrak{G}](\omega) \;=\; 3/2\int_1^3 (4x)(1/3)dx \;=\; 2\int_1^3 x dx \;=\; (2/2)x^2\Big|_1^3 \;=\; 9-1 = 8\,.$$

Written assignment 3: Write from memory Definition 5.1 of $E[X \mid \mathfrak{G}]$ and as many properties as you remember. Then compare with Theorem 5.4 and Theorem 5.5.

Written assignment 4: Prove closed book the linearity of conditional expectations. Proof: See SCF2.

Written assignment 5: You live in a discrete time financial market with trading times at $t = 0,1,2$ years. The interest rate is constant, $R = 0.1$. At $t = 0$ you put \$100.00 into your bank account (asset $\mathscr{A}^{(0)}$ with portfolio holding H_t^0, bank account price process B_t, and dollar value $V_t = B_t H_t^{(0)}$.)

What is $H_0^{(0)}$, B_0, V_0? What is $H_2^{(0)}$, B_2, V_2?

Solution:

- $H_0^{(0)} = H_2^{(0)} = \$100.00$.
- $B_0 = 1$ and $B_2 = (1 + 1/10)^2$.
- $V_0^{(0)} = B_0 \cdot H_0^{(0)} = \$100.00$, and $V_2^{(0)} = B_2 \cdot H_2^{(0)} = \$(1 + 1/10)^2 \cdot 100.00$.

Written assignment 6: Write from memory the definitions of a self–financing portfolio, an arbitrage portfolio, and the discrete market budget equation.

Written assignment 7: Write from memory the value at time of expiration of a forward contract, a European call, and of a European put agreed upon at a strike price of \$85.50, when the underlying stock has the price process S_t.