Math 454 - Spring 2023 - Homework 06

Published: Wednesday, February 22, 2023

Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

SCF2 (Shreve Textbook):
ch.1 - 3.7

MF454 lecture notes:
ch.2 - 8.4

Other:
Nothing assigned yet

New reading assignments:

Reading assignment 1 - due Monday, February 27:

- **a.** Carefully read SCF2 ch.4, except for Example 4.4.11 (Cox–Ingersoll–Ross interest rate model). Ch.4.1 through ch.4.4.1 (Formula for Brownian Motion) corresponds to MF ch.8.1-8.4 and has examples and proofs not to be found in the MF lecture notes.
- **b.** Carefully read the remainder of MF ch.8.
- **c.** VERY IMPORTANT: Remember the formulas in differential notation. Much easier!

The Wednesday and Friday reading assignments cover the same material twice.
Beware: VERY DIFFERENT NOTATION!

Reading assignment 2 - due: Wednesday, March 1:

- **a.** Carefully read MF ch.9 through ch.9.5.
- **b.** Carefully read SCF2 ch.4.5 through ch.4.5.4 (Solution to the Black–Scholes–Merton Equation).

Reading assignment 3 - due Friday, March 3:

- **a.** Carefully read the remainder of MF ch.9.
- **b.** Carefully read the remainder of SCF2 ch.4.5.
- **c.** Study for the midterm!

Written assignment 1: Formulate the pricing principle and prove it closed book (under the no arbitrage assumption)

All the following assignments refer to the binomial asset model (BAM).

Written assignment 2: Formulate the dynamics of the bank account price process B_t and of the stock price process S_t.

Written assignment 3: Try to solve this assignment without looking at the solution which is given after the problem.

We have a financial market with one bond and one stock which follows the one period binomial asset model. We assume the interest rate is $R = 0$, so the bond price is $B_0 = B_1 = 1$.

We assume that

$$S_0 \;=\; s \;=\; 100; \quad S_1 \;=\; \begin{cases} \frac{5}{4} \cdot S_0 = 125 & \text{with probability } 0.8, \\ \frac{3}{4} \cdot S_0 = 75 & \text{with probability } 0.2. \end{cases}$$

How do you price a European call at a strike price of 85? If x denotes the number of bond units and y the number of shares in the stock, what is the hedging portfolio (x, y) you establish at $t = 0$ for this contract?

Solution:

The risk–neutral probabilities are $q_u = q_d = \frac{1}{2}$ since $1 = \frac{1}{2} \cdot \frac{5}{4} + \frac{1}{2} \cdot \frac{3}{4}$. Contract values are $\Phi(su) = 125 - 85 = 40$ and $\Phi(sd) = 0$. Thus the options price at time zero is

$$\Pi_0(\mathcal{X}) \;= E^Q[\mathcal{X}] \;=\; q_d \cdot \Phi(sd) \;+\; q_u \cdot \Phi(su) \;=\; \frac{1}{2} \cdot 40 \;=\; \frac{40}{2} \;=\; 20.$$

The quantities involved for setting up the hedge are

$$x \;=\; \frac{1}{1+R} \cdot \frac{u\Phi(sd) - d\Phi(su)}{u-d} \;=\; 1 \cdot \frac{1.25 \cdot 0 \;-\; 0.75 \cdot 40}{0.5} \;=\; -60,$$
$$y \;=\; \frac{1}{s} \cdot \frac{\Phi(su) - \Phi(sd)}{u-d} \;=\; \frac{1}{100} \cdot \frac{40-0}{0.5} \;=\; \frac{80}{100} \;=\; 0.8.$$

Thus the hedging portfolio consists of 0.8 shares of the stock and a short position (loan) of 60 bond units.

For a sanity check: validate that in fact $V_0^H = 20 = \Pi_0(\mathcal{X})$ as must be true according to Prop. 6.4.

$V_0^H \;=\; x \;+\; ys \;=\; -60 \;+\; 0.8 \cdot 100 \;=\; 20.$ ■

Written assignment 3: Work closed book through Example 7.2. Not only the computation of $\Pi_t(\mathcal{X})$ (what we did on Fri 2/24 in lecture), but also the replicating portfolio, i.e., the processes x_t and y_t.