Math 454 - Spring 2023 - Homework 07

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Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

SCF2 (Shreve Textbook):
ch.1 - 4.5

MF454 lecture notes:
ch.2 - 9

Other:
Nothing assigned yet

New reading assignments:

Reading assignment 1 - due Monday, March 6:

- Study for the midterm!

Reading assignment 2 - due: Wednesday, March 8:

a. Carefully read MF ch.10.
b. Carefully read SCF2 ch.4.6. We will not cover SCF2 ch.4.7 (Brownian bridge).

Reading assignment 3 - due Friday, March 10:

a. Carefully read MF ch.11. It is brief but extremely important, since it provides the tools to construct the analogue to the risk neutral measure employed in the binomial market model. Of pareticular importance is Remark 11.2 about the importance of the Girsanov theorem with respect to mathematical finance!
b. Carefully read MF ch.12.1. Be sure to familiarize yourself with the generalized Black–Scholes market model of Definition 12.1 and the notation used in my lecture notes.

Prove the following assignments closed book (once you have looked up the proposition).

Written assignment 1: Re–write the following integral equations for the process U_t in differential form. Here $g = g_t(\omega)$ is adapted, and α, β are constants.

(a) $U_t \;=\; e^{\alpha t}$
(b) $U_t = \int_0^t g_u dW_u.$ $\square$
(c) $U_t \;=\; e^{\alpha W_t}$
(d) $U_t \;=\; e^{\alpha X_t}$ where the Itô process X has the stochastic differential
$dX_t \;=\; \beta dt + \gamma dW_t.$
(e) $U_t \;=\; X_t^2$ where X has the stochastic differential
$dX_t \;=\; \alpha X_t dt + \sigma X_t dW_t.$ $\square$

Solution to (a)

$$dU_t \;=\; \alpha e^{\alpha t}\, dt \;=\; \alpha U_t dt\,.$$

Solution to (b)

$$dU_t \;=\; g_t dW_t\,.$$

Solution to (c): Apply the Itô formula with $f(t,x) = e^{\alpha x}$. Then $f_t = 0$, $f_x = \alpha e^{\alpha x}$, $f_{xx} = \alpha^2 e^{\alpha x}$:

$$dU_t \;=\; df(t, W_t) \;=\; \alpha e^{\alpha x}\, dW_t \;+\; \frac{1}{2}\alpha^2 e^{\alpha x} dt\,.$$

Solution to (d): As in **(c)**, apply the Itô formula with $f(t,x) = e^{\alpha x}$. Since $dX_t dX_t = \gamma^2 dt$,

$$\begin{aligned} dU_t \;&= \alpha e^{\alpha x}\, dX_t \;+\; \frac{1}{2}\alpha^2 e^{\alpha x} dX_t dX_t \;=\; \alpha\beta e^{\alpha x}\, dt \;+\; \alpha\gamma e^{\alpha x}\, dW_t \;+\; \frac{1}{2}\alpha^2\gamma^2 e^{\alpha x} dt \\ &= \alpha U_t \left[\beta \;+\; \frac{\alpha\gamma^2}{2}\right] dt \;+\; \alpha\gamma U_t\, dW_t\,. \end{aligned}$$

Solution to (e): Apply the Itô formula with $f(t,x) = x^2$. Then $f_t = 0$, $f_x = 2x$, $f_{xx} = 2$.

Further, $dX_t dX_t = \sigma^2 X_t^2 dt$. Thus,

$$\begin{aligned} dU_t \;&= df(t, W_t) \;=\; 2X_t\, dX_t \;+\; \frac{1}{2} \cdot 2 \cdot dX_t dX_t\,. \\ &= 2X_t\big(\alpha X_t dt + \sigma X_t dW_t\big) \;+\; \sigma^2 X_t^2\, dt \\ &= \big(2\alpha X_t \;+\; \sigma^2 X_t^2\big) dt \;+\; \sigma X_t dW_t\,. \end{aligned}$$

Written assignment 2: Prove closed book Theorem 8.7 (SCF2 Theorem 4.4.9 - Itô integral of a deterministic integrand). In lecture we stopped short of proving the normality of Itô integrals. You can find the proof in the SCF2 text.

Written assignment 3: Prove that $\displaystyle\int_0^t W_u dW_u \;=\; \frac{W_t^2}{2} \;-\; t,$ by applying the Itô formula to $f(t,x) = x^2$.

Solution to #3: Since $f_t = 0,\; f_x = 2x,\; f_{xx} = 2,$

$$dW_t^2 \;=\; 2W_t dW_t \;+\; \frac{1}{2} \cdot 2dt \quad \text{thus,} \quad W_t dW_t \;=\; d\left(\frac{1}{2}W_t^2\right) \;-\; dt\,.$$

We indegrate that last equation and obtain

$$\int_0^t W_u dW_u \;=\; \frac{1}{2}(W_t^2 - W_0^2) \;-\; (t-0) \;=\; \frac{W_t^2}{2} \;-\; t\,.$$

Written assignment 4: Work open book(!) in the SCF2 text through the proof of Proposition 8.1 (SCF2 Example 4.4.10 - Vasicek interest rate model). You will not find problems of that computational complexity in the exams of this course, but you may need some skills in that area when going for a quant certification.

Written assignment 5: Work open book(!) in the SCF2 text through the proof of Proposition 8.2 (SCF2 Example 4.4.11 - Cox–Ingersoll–Ross (CIR) interest rate model) This one is tougher than #3.

Written assignment 6: Try to do Exercise 8.4 of the lecture notes closed book before you look at the solution which is given here.

Solution to #6: Define Y by $Y_t = e^{\alpha W_t}$. For $f(t,x) = e^{\alpha x}$ the Itô formula then yields

$$dY_t \;= \tfrac{1}{2}\alpha^2 e^{\alpha W_t} dt + \alpha e^{\alpha W_t} dW_t \;=\; \tfrac{1}{2}\alpha^2 Y_t dt + \alpha Y_t dW_t.$$

Moreover $Y_0 = e^{\alpha W_0} = e^0 = 1$. We integrate both sides and obtain

$$Y_t \;= 1 + \tfrac{1}{2}\alpha^2 \int_0^t Y_u du + \alpha \int_0^t Y_u dW_u.$$

Taking expected values will make the stochastic integral vanish. After moving expectation within the integral sign in the ds-integral and defining m by $m(t) = E[Y_t]$ we obtain the equation

$$m(t) \;= 1 + \tfrac{1}{2}\alpha^2 \int_0^t m(s) ds + 0.$$

This is an integral equation, but if we take the t-derivative we obtain the ODE

$$\dot{m}(t) \;= \tfrac{\alpha^2}{2} m(t), \qquad m(0) \;=\; 1.$$

Solving this standard equation gives us the answer

$$E[e^{\alpha W_t}] \;= E[Y_t] \;=\; m(t) \;=\; e^{\alpha^2 t/2}. \;\blacksquare$$