Math 454 - Spring 2023 - Homework 11

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Status - Reading Assignments:

Here is the status of the reading assignments you were asked to complete by this date.

SCF2 (Shreve Textbook):
ch.1 - 5.4

MF454 lecture notes:
ch.2 - 13

Other:
Nothing assigned yet

New reading assignments:

Reading assignment 1 - due Monday, April 3:

- **(a)** Review the Itô formula. Be able to write it from memory. Both the onedimensional version (8.15) and the multidimensional version (10.8). Understand how to use the multiplication tables to compute $(dX_t)(dY_t)$ for arbitrarily complicated (scalar) Itô processes X_t and Y_t! There will be a reference sheet, but you will not see those items listed!
- **(b)** Review ch.9.6 (The Greeks and Put–Call Parity). You have to know the meaning of the greeks and connection between forward contracts, forward price, puts and calls. You definitely also should review that material in the SCF2 notes.
- **(c)** • You do not have to memorize the integrability condition and the definition of Z_t in Girsanov's theorems, but you should remember that it asserts the existance of $\widetilde{P}$ such that $\widetilde{P} \sim P$ and $\vec{\Theta}_t dt + \vec{W}_t$ defines a $\widetilde{P}$–Brownian motion. Martingale Representation theorem and Lévy's theorem are easy enough to remember.
- **(d)** • Of particular importance in ch.11 and 12: ◦ Remark 11.2 ◦ Formulation of the generalized Black–Scholes model ◦ What are the martingales in Black–Scholes? ◦ Role of martingales in risk–neutral option pricing ◦ Connection between market price of risk and martingale measures and hedges

Reading assignment 2 - due: Wednesday, April 5:

- **(a)** Prerequisite to MF ch.14: Review Markov processes, also in SCF2.
- **(b)** Carefully read MF ch.14.1 - 14.2. Only six pages, but tough reading!
- **(c)** Read the parts of SCF2 ch.6.1–6,4 that correspond to the material in MF ch.6.1-6.2.

Reading assignment 3 - due Friday, April 7:

- **(a)** Carefully read MF ch.13.3. • The stronger students also should read MF ch.13.4.
- **(b)** Do the Monday reading again!
- **(c)** In case you were wondering: I had meant to write ch.14.3 and ch.14.4. You'll find the corrected reading assignment in HW12.

Written assignment 1: Consider a classical Black–Scholes market, with parameters

$$r = 0.05, \;\; \alpha = 0.15, \;\; \sigma = 2.0, \;\; T = 20\,.$$

Thus, the dynamics of bank account price B_t and of stock price S_t are

$$\begin{aligned} dS_t &= \alpha S_t dt \,+\, \sigma S_t dW_t\,, \\ dB_t &= r B_t dt\,. \end{aligned}$$

Find constants β and γ such that stock price dynamics can be expressed as

$$dS_t = \beta S_t dt \,+\, \gamma S_t d\widetilde{W}_t\,,$$

(This would allow us to do Ttô calculus) on the filtered risk-neutral probability space $(\Omega, \mathfrak{F}, \mathfrak{F}_t, \widetilde{P})$, a stepping stone to compute, e.g., the risk–neutral validation $\widetilde{E}[D_t \Pi_t(\mathcal{X})]$ involving the price of a claim.

Solution: It is much easier to work with the variables r, α, σ rather than their specific values.

SInce $d\widetilde{W}_t = \theta dt + dW_t$, where $\theta = (\alpha - r)/\sigma$, i.e., $\theta \cdot \sigma = \alpha - r$,

$$\begin{aligned} dS_t &= \alpha S_t dt \,+\, \sigma S_t dW_t\,, \\ &= \alpha S_t dt \,+\, \sigma S_t \big[\theta dt + dW_t\big] \,-\, (\sigma S_t \theta) dt \\ &= \alpha S_t dt \,+\, \sigma S_t \, d\widetilde{W}_t \,-\, (\sigma\theta) S_t dt \\ &= \alpha S_t dt \,+\, \sigma S_t \, d\widetilde{W}_t \,-\, (\alpha - r) S_t dt \\ &= \sigma S_t \, d\widetilde{W}_t \,+\, r S_t dt \\ &= r S_t dt \,+\, \sigma S_t \, d\widetilde{W}_t\,. \end{aligned}$$

And that is the solution, valid for all possible values of r, α and σ: $\beta = r$, and $\gamma = \sigma$.