Formula Collection for the final exam – Not all items are relevant!

"BM" means Brownian motion; P = real world probab, $\widetilde{P}$ = martingale measure on filtered space $(\Omega, \mathfrak{F}, \mathfrak{F}_t)$

(1) **(a)** Most general differential is dX_t for a linear combination X_t of Itô processes: $dX_t = \sum_i A_t^{(i)} dt + \sum_j B_t^{(j)} dW_t$. If $dY_t = \sum_k C_t^{(k)} dt$ (no dW_t terms), then $d(X_t Y_t) = 0$

(b) Learn by heart: • 1-dim and multidim Itô formulas $\boxdot$ Itô product rule $\boxdot$ geometric BM

(c) • Lévy: continuous paths martingale M_t with $M_0 = 0$ and quadratic variation of Brownian motion is a Brownian motion • $\mathfrak{F}_t = \mathfrak{F}_t^W \Rightarrow$ all $\mathfrak{F}_t$ martingales are Ttô integrals

(2) **(a)** measurable $f : (\Omega, \mathfrak{F}, \mu) \to (\Omega', \mathfrak{F}')$ induces image measure $\mu_f(A') = \mu\big(f^{-1}(A')\big)$ on $\mathfrak{F}'$.

(b) random var. $X : (\Omega, \mathfrak{F}, P) \to (\mathbb{R}, \mathfrak{B})$ induces image measure $P_X(B) = P\{X \in B\}$ on Borel sets $\mathfrak{B}$.

$\int_\Omega g \circ f(\omega) \mu(d\omega) = \int_{\Omega'} g(\omega') \mu_f(d\omega')$ (for g Borel measurable)

(3) $f \leqq g$ μ–a.e. $\Leftrightarrow \int_A f d\mu \leqq \int_A g d\mu$ for all $A \in \mathfrak{F}$

(4) Let $(\Omega, \mathfrak{F}, P)$ be a probability space. let $\mathfrak{G}, \mathfrak{H}$ be sub–σ–algebras of $\mathfrak{F}$ such that $\mathfrak{H} \subseteq \mathfrak{G}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be measurable.

(a) $E[c_1 X + c_2 Y | \mathfrak{G}] = c_1 E[X|\mathfrak{G}] + c_2 E[Y|\mathfrak{G}]$. **(b)** If X is $\mathfrak{G}$–measurable, then $E[X \cdot Y | \mathfrak{G}] = X \cdot E[Y|\mathfrak{G}]$. **(c)** $E\big[E[X|\mathfrak{G}] \,\big|\, \mathfrak{H}\big] = E[X|\mathfrak{H}]$.

(d) X independent of $\mathfrak{G} \Rightarrow E[X|\mathfrak{G}] = E[X]$. **(e)** φ convex $\Rightarrow \varphi\big(E[X \mid \mathfrak{G}]\big) \leqq E[\varphi \circ (X) \mid \mathfrak{G}]$.

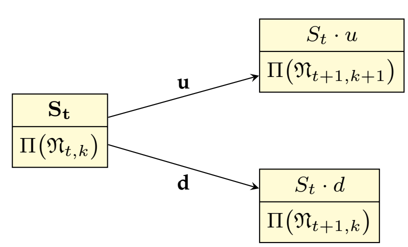
(5) If μ, ν are σ–finite measures on $(\Omega, \mathfrak{F})$ and $f : \Omega \to \mathbb{R}$ satisfies $\nu(A) = \int_A f d\mu$, then f is called the Radon–Nikodým derivative $\frac{d\nu}{d\mu}$.

(6) Probability space $(\Omega, \mathfrak{F}, P)$, a countable partition $\Omega = \biguplus \big[G_j : j \in J\big]$ s.t. $G_j \in \mathfrak{F}$ and $P(G_j) > 0$ for all j. Let $\mathfrak{G} := \sigma\{G_j : j \in J\}$ Let the random variable X be integrable or satisfy $X \geqq 0$. $\boxdot$ Then $E[X \mid \mathfrak{G}](\omega) = \sum_{j \in J} \frac{1}{P(G_j)} E\left[X 1_{G_j}\right] \cdot 1_{G_j}(\omega)$ a.s.

(7) Binomial model: trading times $t = 0, 1, 2, \dots, T$; $\boxdot$ $B_0 = 1; B_t = (1+R)^t$; $\boxdot$ $S_0 = s; S_{t+1} = uS_t$ w. $P\{u \text{ happens }\} = p_u$ and $P\{d \text{ happens }\} = p_d = 1 - p_u$ $\boxdot$ $x_t = H_t^B (1+R)^{t-1}$ = money in the bank at t_1; $y_t = H_t^S$ = stock shares acquired at $t-1$; $\boxdot$ For a contingent claim $\mathcal{X}$, in particular, a simple claim $\Phi(S_T)$, $\Pi_t(\mathcal{X})$ = correct price of the claim in an arbitrage free market

(8) In the multiperiod binomial model, $\mathcal{X}$ can be hedged. The portfolio quantities H_{t+1}^B and H_{t+1}^S are given by $H_{t+1}^B = (1+R)^{-t} x_{t+1}$ and $H_{t+1}^S = y_{t+1}$, where x_{t+1}, y_{t+1} for the node $\mathfrak{N}_{t,k}$ (remember: $\vec{H}_t$ = purchases at time $t-1$!) in the tree excerpt shown below are, if $\Pi\big(\mathfrak{N}_{t,k}\big)$ = option price $\Pi_t(\mathcal{X})(\omega) \Leftrightarrow S_t(\omega) = su^k d^{t-k} \Leftrightarrow$ exactly k upward moves and $t-k$ downward moves

$$x_{t+1} = \frac{1}{1+R} \cdot \frac{u\Pi\big(\mathfrak{N}_{t+1,k}\big) - d\Pi\big(\mathfrak{N}_{t+1,k+1}\big)}{u-d},$$
$$y_{t+1} = \frac{1}{s} \cdot \frac{\Pi\big(\mathfrak{N}_{t+1,k+1}\big) - \Pi\big(\mathfrak{N}_{t+1,k}\big)}{u-d}.$$



(9) **(a)** • contract functions for ⊡ Europ call ⊡ Europ put ⊡ Fwd contract ⊡ Fwd price For_t • market price of risk $(\alpha_t - R_t)/\sigma_t)$ ⊡ Bank acct price $B_0 = 1$, $B_t = e^{\int_0^t R_u\,du}$; Discount $D_t = 1/B_t$; $dB_t = ...$; $dD_t = ...$ ⊡ pricing principle = ...

(b) • portfolio $\vec{H}$: Value $V_t = H_t^B B_t + \sum_{j>0} H_t^{(j)} S_t^{(j)}$ ⊡ money in the bank = $X_t = H_t^B B_t = V_t - \sum_{j>0} H_t^{(j)} S_t^{(j)}$ ⊡ self–fin. portf: $dV_t = H_t^B dB_t + \sum_{j>0} H_t^{(j)} dS_t^{(j)}$ • In Black–Scholes market with only 1 stock price S_t: also write $Y_t := H_t^S$, so ⊡ $V_t = X_t + H_t^S S_t = X_t + Y_t S_t$; ⊡ self–fin. portf: $dV_t = R_t X_t dt + Y_t dS_t$

(c) Greeks in classical Black–Scholes: Delta, Gamma, Rho, Theta, Vega - What are they?

(10) **(a)** • Euro call: ⊡ $c(t,x) = \text{BSM}(\tau, x; K, r, \sigma)$, where ⊡ $\tau = T - t$ ⊡ $\text{BSM}(\tau, x; K, r, \sigma) = xN\big(d_+(\tau,x)\big) - Ke^{-r\,\tau}N\big(d_-(\tau,x)\big)$, where ⊡ $d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\,\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$

(b) • Forward price of a forward contract entered at $t = 0$ with a strike price K: ⊡ K was fair at $t = 0$; Forward price at t would be fair strike price if contract would be entered today (at t)

(c) $\Pi_t(\text{Euro put}) = \Pi_t(\text{Euro call}) - \Pi_t(\text{forward contract})$

(11) 1-dim Black–Scholes: • classic vs generalized & dividends vs no dividends: ⊡ martingale measure = risk–neutral measure = ? ⊡ What are the martingales? For P? For $\widetilde{P}$? • classic Black–Scholes w. const. dividend rate a ($a = 0$ means no dividends): ⊡ $d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log(\frac{x}{K}) + \left(r - a \pm \frac{\sigma^2}{2}\right)\tau\right]$. ⊡ $\text{BSM}(\tau, x; K, r, \sigma) = xe^{-a\tau}N\big(d_+(\tau,x)\big) - e^{-r\tau}K\,N\big(d_-(\tau,x)\big)$
• $\text{For}_S(t,T) = \frac{S_t}{B(t,T)}$; $(0 \le t \le T)$ ⊡ What do those symbols stand for? • Generalized Black–Scholes: Value V_t of self–financing portfolio; $Y_t = H_t^S$ = # of stock shares: ⊡ $d(D_t S_t) = (\sigma_t D_t S_t)d\widetilde{W}_t$ ⊡ $d(D_t V_t) = (Y_t \sigma_t D_t S_t)d\widetilde{W}_t$ ⊡ Above: What is $\widetilde{W}_t$?

(12) multidim Black–Scholes: • 1st & 2nd fundamental theorems of asset pricing = WHAT? ⊡ Assumption f. #1: Existence of $\widetilde{P}$ ⊡ Assumption f. #2: $\vec{\Theta}_t$ exists, and $\mathfrak{F}_t = \mathfrak{F}_t^{\vec{W}}$.

(13) **(a)** • 2-dim SDE: $\vec{W}_t = \big(W_t^{(1)}, W_t^{(2)}\big)$; $\vec{X} = (X_t, Y_t)$ $dX_t = \beta_1\big(t, X_t, Y_t\big)\,dt + \gamma_{11}(\dots)dW_t^{(1)} + \gamma_{12}(\dots)dW_t^{(2)}$; $dY_t = \beta_2(\dots)dt + \gamma_{21}(\dots)dW_t^{(1)} + \gamma_{22}(\dots)dW_t^{(2)}$ ⊡ $0 \leqq u \le t \le T; \vec{a} \in \mathbb{R}^2$, $\vec{X}_u(\omega) = \vec{a} \Rightarrow E\{h(\vec{X}_t) \mid \mathfrak{F}_u\}(\omega) = E\{h(\vec{X}_t) \mid \vec{X}_u = \vec{a}\} = E^{u,\vec{a}}h(\vec{X}_t)$ ⊡ IF [$\vec{X}_t$ solves SDE **and** $r \ge 0; 0 \leqq u \le t \le T; \vec{a} \in \mathbb{R}^2$ **and** $f(u,\vec{a}) := E^{u,\vec{a}}h(\vec{X}_T)$] THEN [f solves PDE $f_t + \beta_1 f_x + \beta_2 f_y + \frac{1}{2}(\gamma_{11}^2 + \gamma_{12}^2)f_{xx} + (\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22})f_{xy} + \frac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2)f_{xy} = rf$, **and** $\forall\vec{x} : f(T,\vec{x}) = h(\vec{x})$]