Formula Collection for Math 454 Final Exam – Not all items are relevant!

Abbreviations: • rv = r.v. = random variable; r.elem = random element • meas = measure • mble = measurable • fn = function • prob = probability • distr = distribution • conv = convergence • cont = continuous/continuity • Riem-$\int$-ble = Riemann integrable • w.r.t = with respect to • RN = Radon–Nikodým • cond.exp = conditional expectation • BM = Brownian motion • BAM = binom. asset model • BS = Black–Scholes • PF = portfolio • arbPF = arbitrage portfolio • self–fin = self–financing • MPoR = market price of risk • r–n = risk–neutral • r–n val = risk–neutral validation • ZCB = zero coupon bond • SDE/ODE/PDE = stochastic/ordinary/partial differential eqn • F–K = Feynman–Kac

(a) • power set $2^\Omega = \{$ all subsets of $\Omega\}$ • $\forall\, x \dots$: For all $x \dots$ $\boxdot$ $\exists\, x$ s.t. ... There is an x such that ...
$\boxdot$ $\exists!\, x$ s.t. ... There is a unique x s.t. ... $\boxdot$ $p \Rightarrow q$ If p is true then q is true $\boxdot$ $p \Leftrightarrow q$ p iff q, i.e., p true if and only if q true
• Intervals: $]a,b[= \{x \in \mathbb{R} : a < x < b$, $]a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \leq b$, $[a,b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \leq x \leq b$, etc.
• countable set A: can be sequenced: $\boxdot$ $A = \{a_1, a_2, \dots, a_n\}$ (finite set) $\boxdot$ $A = \{_1, a_2, \dots\}$ ("countably infinite" set)
$\boxdot$ $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is uncountable • family $(x_i)_{i \in I}$: index set I may be uncountable • $\bigcup_{i\in J} A_i = \{x : \exists\, i_0 \in J$ s.t. $x \in A_{i_0}\}$ • $\bigcap_{i \in J} A_i = \{x : \forall\, i \in J\, x \in A_i\}$. • Can use $A \uplus B$ for $A \cup B$ if disjoint sets • **De Morgan:** $\boxdot$ $\left(\bigcup_k A_k\right)^{\complement} = \bigcap_k A_k^{\complement}$ $\boxdot$ $\left(\bigcap_k A_k\right)^{\complement} = \bigcup_k A_k^{\complement}$ • **Distributivity:** $\boxdot$ $\bigcup_j (B \cap A_j) = B \cap \bigcup_j A_j$ $\boxdot$ $\bigcap_{j\in I}(B \cup A_j) = B \cup \bigcap_j A_j$
• Cartesian products: $\left| X_1 \times \dots \times X_n \right| = \left|X_1\right| \cdots \left|X_n\right|$ • Indicator fn $\mathbf{1}_A(\omega) = ?$ • preimages of $f : X \to Y$:
Arbitrary index set J and $B, B_j \subseteq Y$: $\boxdot$ $f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$ $\boxdot$ $f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$
$\boxdot$ $f^{-1}(B^{\complement}) = \left(f^{-1}(B)\right)^{\complement}$ $\boxdot$ $B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$ • $A \subseteq \Omega \Rightarrow 1_A(\omega) = 1$ if $\omega \in A$ and 0 else

(b) Integrals $\int_A f d\mu$: • monotone & dominated conv. & other laws for $\int f d\mu$ • Jensen ineq valid for what μ?
• simple fn g: $\int g d\mu = ?$ • $\int h(\vec{x}) d\vec{x}$ (Riemann) vs $\int h d\lambda^d$ • $\mathfrak{B}^d = \sigma\{$ d–dim rectangles $\}$ • For what φ is $A \mapsto \int_A \varphi(\omega)\mu(d\omega)$ a prob meas? • ILMD method used how? for what theorems?
• Use Fubini for both $\int_A f(\vec{y}) d\vec{y}$ and $\int f d\lambda^d$ to compute multidim $\int$. • 1_A Riem-$\int$-ble $\Rightarrow \lambda^d(A)$ defined how?
• mble $f : (\Omega, \mathfrak{F}, \mu) \to (\Omega', \mathfrak{F}')$ img meas $\mu_f(A')$ (= ???) on $\mathfrak{F}'$. Distrib P_X of a r.elem X defined how?
• $f : \Omega \to \mathbb{R}$, μ, ν σ–finite meas on $(\Omega, \mathfrak{F})$ s.t. $\nu(A) = \int_A f d\mu$, $\Rightarrow$ f =(def) RN derivative $\frac{d\nu}{d\mu}$. • meas $\nu \ll \mu \Leftrightarrow [\mu(A) = 0 \Rightarrow \nu(A) = 0]$; $\nu \sim \mu \Leftrightarrow [\mu(A) = 0 \Leftrightarrow \nu(A) = 0]$; • $[\nu \ll \mu$ and μ, ν σ–finite $] \Leftrightarrow [$ RN derivative $\frac{d\nu}{d\mu}$ exists $]$.
• $\int_\Omega g \circ f(\omega)\mu(d\omega) = \int_{\Omega'} g(\omega')\mu_f(d\omega')$ (what are f, g w.r.t. μ?) • $\int f(\omega)\frac{d\nu}{d\mu}(\omega)\mu(d\omega) = \int f(\omega)\nu(d\omega)$ ($f, \mu, \nu, \frac{d\nu}{d\mu}$ =?);
• $f \leqq g$ μ–a.e. $\Leftrightarrow$ $\int_A f d\mu \leqq \int_A g d\mu$ for all $A \in \mathfrak{F}$ • Y functionally dependent on X $\Leftrightarrow$ $\sigma(\_\_\_) \subseteq \sigma(\_\_\_)$ in what sense?

(c) Cond.Exp: • prob space $(\Omega, \mathfrak{F}, P)$; σ–algebras $\mathfrak{H} \subseteq \mathfrak{G} \subseteq \mathfrak{F}$; $\varphi : \mathbb{R} \to \mathbb{R}$ mble. Then $\boxdot$ $E[c_1 X + c_2 Y|\mathfrak{G}] = c_1 E[X|\mathfrak{G}] + c_2 E[Y|\mathfrak{G}]$. $\boxdot$ If X is $\mathfrak{G}$–measurable, then $E[X \cdot Y|\mathfrak{G}] = X \cdot E[Y|\mathfrak{G}]$. $\boxdot$ $E\big[E[X|\mathfrak{G}] \,\big|\, \mathfrak{H}\big] = E[X|\mathfrak{H}]$. $\boxdot$ X independent of $\mathfrak{G} \Rightarrow E[X|\mathfrak{G}] = E[X]$. $\boxdot$ φ convex $\Rightarrow \varphi\big(E[X \mid \mathfrak{G}]\big) \leqq E[\varphi \circ (X) \mid \mathfrak{G}]$. • countable partition $\Omega = \biguplus [G_j : j \in J]$ s.t. $G_j \in \mathfrak{F}$ and $P(G_j) > 0$ for all j; $\mathfrak{G} := \sigma\{G_j : j \in J\}$; r.v. X; $\boxdot$ Then $E[X \mid \mathfrak{G}](\omega) = \dots$

(d1) 1–dim Stoch. calculus: • simple process $Z_t \Rightarrow$ Itô integral $\int_0^t Z_u dW_u = \dots$ • $dt dW_t = ?$ $dW_t dW_t = ?$;
• Itô isometry =? • Itô processes $dX_t = A_t dt + B_t dW_t$, $dY_t = U_t dt + V_t dW_t$; then $\boxdot$ $dX_t dY_t = ?$ $\boxdot$ $d[X,X]_t = ?$
$\boxdot$ When is $t \mapsto \int_0^t X_u dY_u$ a martingale? • Lévy: contin. paths martingale M_t; $M_0 = 0$; $[M,M]_t = t \Rightarrow M_t = ?$
• 1-dim and multidim Itô formulas • Itô product rule • GBM and general GBM: $\boxdot$ $dX_t = ?$ $\boxdot$ meaning of α_t, σ_t

(d2) multidim Stoch. calculus: • d–dim BM $\vec{W}_t = ?$ • $[W^{(i)}, W^{(j)}]_t = ?$ • $dW_t^{(i)} dW_t^{(j)} = ?$ • $dX_t = \Theta_t dt + \vec{\Delta}_t$ • $d\vec{W}_t$ means? • d–dim Lévy =?

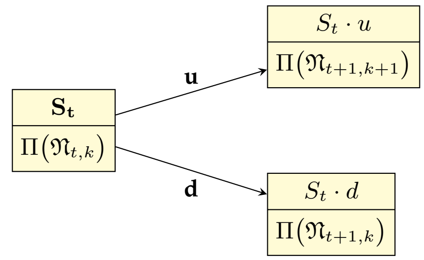
(e) financial markets: • contingent claim $\mathcal{X}$: $\boxdot$ simple if $\mathcal{X} = \Phi(S_T)$; $\boxdot$ $\Pi_t(\mathcal{X})$ = correct (no arbitrage) price of $\mathcal{X}$
• self–fin PF $\vec{H}_t = ??$; arbPF = ?? replicating PF = ?? • pricing principle = ?? • complete market =?

(e1) BAM (binomial asset model):
• Notation: $\widetilde{P}$ = risk–neutral probab (P = real world probab) on $(\Omega, \mathfrak{F}, \mathfrak{F}_t)$
• def. BAM: $\boxdot$ trading times $t = 0, 1, 2, \dots, T$; multiperiod BAM if $T > 1$ $\boxdot$ $B_0 = 1$; $B_t = (1+r)^t$ = money market acct price; $D_t = 1/B_t$ = discount $\boxdot$ $S_0 = s$ = const; $\boxdot$ $u, d, p_u, p_d, \tilde{u}, \tilde{d}$ def'd HOW? $\boxdot$ compute S_{t+1} from S_t HOW?

• PF $\vec{H}_t = (H_t^B, H_t^S)$: ⊡ $\vec{H}_t$ bought at $t-1$, sold at t ⊡ $V_t = V_t^{\vec{H}} = H_t^B B_t + H_t^S S_t$ = sales value of $\vec{H}_t$ ⊡ $x_t = H_t^B B_{t-1}$ = money in the bank at t_1; ⊡ $y_t = H_t^S$ = stock shares bought at $t-1$; • no arb $\Leftrightarrow$ WHAT? • martingales (P? $\widetilde{P}$?) are ...

• hedge PF for simple claim $\mathcal{X} = \Phi(S_T)$ at $t+1$ is $H_{t+1}^B = (1+r)^{-t} x_{t+1}$ and $H_{t+1}^S = y_{t+1}$, where x_{t+1}, y_{t+1} for the node $\mathfrak{N}_{t,k}$ (remember: $\vec{H}_t$ = purchased at time $t-1$!) in the tree excerpt shown below are, if $\Pi\big(\mathfrak{N}_{t,k}\big)$ = option price $\Pi_t(\mathcal{X})(\omega)$ $\Leftrightarrow$ $S_t(\omega) = s u^k d^{t-k}$ $\Leftrightarrow$ exactly k upward moves and $t-k$ downward moves

$$x_{t+1} = \frac{1}{1+R} \cdot \frac{u\Pi\big(\mathfrak{N}_{t+1,k}\big) - d\Pi\big(\mathfrak{N}_{t+1,k+1}\big)}{u-d},$$
$$y_{t+1} = \frac{1}{s} \cdot \frac{\Pi\big(\mathfrak{N}_{t+1,k+1}\big) - \Pi\big(\mathfrak{N}_{t+1,k}\big)}{u-d}.$$



⊡ Is one period BAM complete? ⊡ Is multiperiod BAM complete?

(e2) contin time financial markets with only 1 riskless asset and self–fin PF $\vec{H}_t$: • PF Value $V_t = V_t^{\vec{H}} = H_t^B B_t + \sum_{j>0} H_t^{(j)} S_t^{(j)}$ • $X_t := H_t^B B_t = V_t - \sum_{j>0} H_t^{(j)} S_t^{(j)}$ • budget eqn: $dV_t = H_t^B dB_t + \sum_{j>0} H_t^{(j)} dS_t^{(j)}$

• If only 1 stock price S_t: ⊡ can write $Y_t := H_t^S$ ⊡ $V_t = X_t + H_t^S S_t = X_t + Y_t S_t$ ⊡ budget eqn: $dV_t = H_t^B dB_t + Y_t dS_t$

(e3) Black–Scholes market • model of (classical) BS market: ⊡ constant $r, \alpha, \sigma \in \mathbb{R}$ meaning =?. ⊡ $dB_t =$? ⊡ $dD_t =$? ⊡ $dS_t =$? ⊡ $(t,x) \mapsto \pi(t,x)$ is C^2 means ? ⊡ $\Phi(S_T) = \pi$(WHAT?, WHAT?) • hedge $\vec{H}_t$ for simple claim $\Phi(S_T)$ $\Rightarrow$ ⊡ $d\big(e^{-rt} V_t^{\vec{H}}\big) = (\alpha - r) H_t^S e^{-rt} S_t dt + \sigma H_t^S e^{-rt} S_t dW_t = H_t^S d\big(e^{-rt} S_t\big)$ ⊡ delta–hedging rule: H_t^S =? ⊡ BS PDE: $\pi_t(t,x) + rx\,\pi_x\big(t,x\big) + \frac{1}{2}\sigma^2 x^2 \pi_{xx}(t,x) = r\,\pi\big(t,x\big)$; π must satisfy $\pi(T,x) = \Phi$(WHAT?) • put–call parity means ...

• Europ. call: ⊡ $\Phi(x) =$? ⊡ $c(t,x) = \text{BSM}(\tau, x; K, r, \sigma) = xN\big(d_+(\tau,x)\big) - Ke^{-r\tau} N\big(d_-(\tau,x)\big)$, where $0 \le t < T$, $\tau = T - t$, $x > 0$, $d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}} \left[\log \frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$ • fwd contract: ⊡ $\Phi(x) =$? ⊡ $f(t,x) =$?

• Europ. put: ⊡ $\Phi(x) =$? ⊡ $p(t,x) =$? • fwd price For$_t$ defined how? • Greeks: Delta, Gamma, Rho, Theta, Vega are??

• Amer. puts and calls: which one should never be exercised before T?

(f) r–n val (risk–neutral validation): • Girsanov (d–dim): ⊡ Given $T > 0$, $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)$ with d–dim BM $\vec{W}_t$ and $\vec{\Theta}_t = \big(\Theta_t^{(1)}, \dots, \Theta_t^{(d)}\big)$ ⊡ Let $Z_t := \exp\left\{-\int_0^t \vec{\Theta}_u \bullet d\vec{W}_u - \frac{1}{2}\int_0^t \|\vec{\Theta}_u\|_2^2\, du\right\}$ s.t. $E\left[\int_0^T \|\vec{\Theta}_u\|_2^2\, Z_u^2\, du\right] < \infty$. ⊡ Then, $d\widetilde{P} = Z_T dP$ and $d\widetilde{W}_t$ satisfy ...?? • $\mathfrak{F}_t = \mathfrak{F}_t^{\vec{W}}$ $\Rightarrow$ martingale representation thm states WHAT?

• In 1–dim BS market: ⊡ θ_t = MPoR =?? ⊡ $\widetilde{P}$ martingales are ?? Which one is used for r–n val? ⊡ r–n val formula: Def.13.4!! • 1st/2nd fundamental thm of asset pricing = ??

(g) Dividends and ZCBs in 1–dim B–S market: • discrete time dividends with rate $a_j(\omega)$ at t_j: $S_{t_j}(\omega) = \dots$ • continuous time dividends with rate $A_t(\omega)$: ⊡ $dS_t = \dots$ ⊡ If $d\widetilde{P} = Z_T dP$ is Girsanov measure and V_t = value of self–fin PF: Is $D_t S_t$ a $\widetilde{P}$–martingale? Is $D_t V_t$ a $\widetilde{P}$–martingale? • contract function of a ZCB is $\Phi(x) =$??: • $B(t,T)$ defined how?

• r–n val formula for ZCB is (14.39)!! • $B(t,T)$ and For$_S(t,T)$ related how?

(h) SDEs and PDEs (Stochastic and partial Differential Eqns) • differential of an SDE = ? • Each intial condtion yields $P^{u,a}$ + $E^{u,a}$ that ⊡ "play the role" of conditioning on $X_u = a$ ⊡ show Markov property when replacing a with $X_u = X_u(\omega)$: See (15.9), (15.10): $E\{h(X_t) \mid \mathfrak{F}_u\} = E\{h(X_t) \mid X_u\} = E^{u,X_u} h(X_t) = \int_{\mathbb{R}} h(x) P(u, X_u, t, dx)$

• 2–dim SDE w. dynamics $dX_t = \dots$; $dY_t = \dots$; and initial conds when driven by 2–dim $\vec{W}_t = (W_t^{(1)}, W_t^{(2)})$

• F–K for $r = 0$: $\vec{x} := (x,y)$, etc: $(t,x,y) \mapsto g(t,\vec{x}) = E^{t,\vec{x}}\, h\big(X_T, Y_T\big)$ solves PDE ⊡ $g_t + \beta_1 g_x + \beta_2 g_y + \frac{1}{2}(\gamma_{11}^2 + \gamma_{12}^2) g_{xx} + (\gamma_{11}\gamma_{21} + \gamma_{12}\gamma_{22}) g_{xy} + \frac{1}{2}(\gamma_{21}^2 + \gamma_{22}^2) g_{yy} = 0$ ⊡ terminal cond. = ?? • F–K for $r > 0$ = ?? • 1–dim F–K = ?? • Asian option defined how? • Zero coupon yield $Y(t,T)$ defined how?