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Math 488P/588 Additional Material

Additional Material on sets and functions

This write-up provides some additional background on sets and functions. The first part lists some useful properties of direct and indirect images of a function.

0.1 Notation

Notation 0.1. **a)** If two subsets A and B of a space Ω are disjoint then we often write $A \uplus B$ rather than $A \cup B$ or $A + B$. Both $\complement A$ and $A^{\complement}$ denote the complement $\Omega \setminus A$ of A.
b) $\mathbb{R}^+$ denotes the interval $]0, +\infty[$, $\mathbb{R}_+$ denotes the interval $[0, +\infty[$,

c) The set $\mathbb{N} = \{1, 2, 3, \cdots\}$ of all natural numbers excludes the number zero. We write $\mathbb{N}_0$ or $\mathbb{Z}_+$ or $\mathbb{Z}_{\geq 0}$ for $\mathbb{N} \uplus \{0\}$
d) We generally write Ef or $E(f)$ rather than $E[f]$ for the expected value of a random variable f because it is very hard to distinguish $E[|f|]$ from $E\|f\|$ but we shall occasionally use "$[\cdot]$" with expectations in case there are no absolute values around the expression that the expectation is taken of.
i) Let E be a topological space (a set which allows the concept of continuity). $\mathscr{C}(E)$ or $\mathscr{C}^0(E)$ denotes the set of all continuous real valued functions on E,

0.2 Direct images and indirect images (preimages) of a function

Here are references, preimages: [1] Kupferman, Raz: Lecture Notes in Probability (Hebrew University). And another one: [2] Author unknown: mazur-330-func-1.pdf - Introduction to Functions Ch.2. And number 3: [3] Author unknown: mazur-330-func-2.pdf - Properties of Functions Ch.2. And number 4: [4] Author unknown: mazur-330-sets-1.pdf - Ch.1: Introduction to Sets and Functions And number 3: [5] Author unknown: mazur-330-sets-2.pdf - Ch.4: Applications of Methods of Proof

Definition 0.1. Let X, Y be two non-empty sets and $f : X \to Y$ be an arbitrary function with domain X and codomain Y. Let $A \subseteq X$ and $B \subseteq Y$. Let

$$\text{1)} \quad f(A) = \{f(x) : x \in A\} \tag{0.1}$$

$$\text{2)} \quad f^{-1}(B) = \{x \in X : f(x) \in B\} \tag{0.2}$$

(0.3)

We call $f(A)$ the ***direct image*** of A under f and we call We call $f^{-1}(B)$ the ***indirect image*** or ***preimage*** of B under f

Notational conveniences:

If we have a set that is written as $\{\dots\}$ then we may write $f\{\dots\}$ instead of $f(\{\dots\})$ and $f^{-1}\{\dots\}$ instead of $f^{-1}(\{\dots\})$. Specifically for $x \in X$ and $y \in Y$ we get $f^{-1}\{x\}$ and $f^{-1}\{y\}$. Many mathematicians will write $f^{-1}(y)$ instead of $f^{-1}\{y\}$ but this writer sees no advantages doing so whatsover. There seemingly are no savings with respect to time or space for writing that alternate form but we are confounding two entirely separate items: a subset $f^{-1}\{y\}$ of X v.s. an element $f^{-1}(y)$ of X. We can perfectly talk about the latter in case that the inverse function f^{-1} of f exists.

In measure theory and probability theory the following notation is also very common: $\{f \in B\}$ rather than $f^{-1}(B)$ and $\{f = y\}$ rather than $f^{-1}\{y\}$

Let $a < b \in \mathbb{R}$. We write $\{a \leq f \leq b\}$ rather than $f^{-1}([a,b])$, $\{a < f < b\}$ rather than $f^{-1}(]a,b[)$, $\{a \leq f < b\}$ rather than $f^{-1}([a,b[)$ and $\{a < f \leq b\}$ rather than $f^{-1}(]a,b])$.

Proposition 0.1. *Some simple properties:*

$$f(\emptyset) = f^{-1}(\emptyset) = 0 \tag{0.4}$$

$$A_1 \subseteq A_2 \subseteq X \Rightarrow f(A_1) \subseteq f(A_2) \tag{0.5}$$

$$B_1 \subseteq B_2 \subseteq Y \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2) \tag{0.6}$$

$$x \in X \Rightarrow f(\{x\}) = \{f(x)\} \tag{0.7}$$

$$f(X) = Y \iff f \text{ is surjective} \tag{0.8}$$

$$f^{-1}(Y) = X \quad \text{always!} \tag{0.9}$$

Proof of all properties is immediate. ■

Proposition 0.2 (f^{-1} is compatible with all basic set ops)**.** *In the following we assume that J is an arbitrary index set, and that $B \subseteq Y$, $B_j \subseteq Y$ for all j.*
The following all are true:

$$f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j) \tag{0.10}$$

$$f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j) \tag{0.11}$$

$$f^{-1}(B^{\complement}) = f^{-1}(B)^{\complement} \tag{0.12}$$

$$f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2) \tag{0.13}$$

(0.14)

Proof of (0.10): Let $x \in X$. Then

$$x \in f^{-1}(\bigcap_{j\in J} B_j) \iff f(x) \in \bigcap_{j\in J} B_j \quad (\textit{def preimage}) \tag{0.15}$$

$$\iff \forall j\ f(x) \in B_j \quad (\textit{def} \cap) \tag{0.16}$$

$$\iff \forall j\ x \in f^{-1}(B_j) \quad (\textit{def preimage}) \tag{0.17}$$

$$\iff x \in \bigcap_{j\in J} f^{-1}(B_j) \quad (\textit{def} \cap) \tag{0.18}$$

Proof of (0.11): Let $x \in X$. Then

$$x \in f^{-1}(\bigcup_{j\in J} B_j) \iff f(x) \in \bigcup_{j\in J} B_j \quad (\textit{def preimage}) \tag{0.19}$$

$$\iff \exists j_0 :\ f(x) \in B_{j_0} \quad (\textit{def} \cup) \tag{0.20}$$

$$\iff \exists j_0 : x \in f^{-1}(B_{j_0}) \quad (\textit{def preimage}) \tag{0.21}$$

$$\iff x \in \bigcup_{j\in J} f^{-1}(B_j) \quad (\textit{def} \cup) \tag{0.22}$$

Proof of (0.12): Let $x \in X$. Then

$$x \in f^{-1}(B^{\complement}) \iff f(x) \in B^{\complement} \quad (\textit{def preimage}) \tag{0.23}$$

$$\iff f(x) \notin B \quad (\textit{def}\ (\cdot)^{\complement}) \tag{0.24}$$

$$\iff x \notin f^{-1}(B) \quad (\textit{def preimage}) \tag{0.25}$$

$$\iff x \in f^{-1}(B)^{\complement} \quad (\cdot)^{\complement}) \tag{0.26}$$

Proof of (0.13): Let $x \in X$. Then

$$x \in f^{-1}(B_1 \setminus B_2) \iff x \in f^{-1}(B_1 \cap B_2^{\complement}) \quad (\textit{def} \setminus) \tag{0.27}$$

$$\iff x \in f^{-1}(B_1) \cap f^{-1}(B_2^{\complement}) \quad (\textit{see (0.10)}) \tag{0.28}$$

$$\iff x \in f^{-1}(B_1) \cap f^{-1}(B_2)^{\complement} \quad (\textit{see (0.12)}) \tag{0.29}$$

$$\iff x \in f^{-1}(B_1) \setminus f^{-1}(B_2) \quad (\textit{def} \setminus) \tag{0.30}$$

■

Proposition 0.3 (Properties of the direct image). *In the following we assume that J is an arbitrary index set, and that $A \subseteq X$, $A_j \subseteq X$ for all j.*
The following all are true:

$$f(\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J} f(A_j) \tag{0.31}$$

$$f(\bigcup_{j\in J} A_j) = \bigcup_{j\in J} f(A_j) \tag{0.32}$$

Proof of (0.10): This follows from the monotonicity of the direct image (see 0.5):

$$\bigcap_{j\in J} A_j \;\subseteq\; A_i\; \forall i\in J \;\Rightarrow f(\bigcap_{j\in J} A_j) \;\subseteq\; f(A_i)\; \forall i\in J \quad (\textit{see } 0.5) \tag{0.33}$$

$$\Rightarrow f(\bigcap_{j\in J} A_j) \;\subseteq\; \bigcap_{i\in J} f(A_i) \quad (\textit{def} \cap) \tag{0.34}$$

$$\tag{0.35}$$

First proof of (0.11)) - "Expert proof":

$$y\in f(\bigcup_{j\in J} A_j) \iff \exists\, x\in X:\; f(x)=y \textit{ and } x\in \bigcup_{j\in J} A_j \quad (???) \tag{0.36}$$

$$\iff \exists\, x\in X \textit{ and } j_0\in J:\; f(x)=y \textit{ and } x\in A_{j_0} \quad (???) \tag{0.37}$$

$$\iff \exists\, x\in X \textit{ and } j_0\in J:\; f(x)=y \textit{ and } f(x)\in f(A_{j_0}) \quad (???) \tag{0.38}$$

$$\iff \exists\, j_0\in J:\; y\in f(A_{j_0}) \quad (???) \tag{0.39}$$

$$\iff y\;\in \bigcup_{j\in J} f(A_j) \quad (???) \tag{0.40}$$

Alternate proof of (0.11)) - Proving each inclusion separately. Unless you have a lot of practice reading and writing proofs whose subject is the equality of two sets you should write your proof the following way:

A. Proof of "$\subseteq$":

$$y\in f(\bigcup_{j\in J} A_j) \;\Rightarrow \exists\, x\in X:\; f(x)=y \textit{ and } x\in \bigcup_{j\in J} A_j \quad (???) \tag{0.41}$$

$$\Rightarrow \exists\, j_0\in J:\; f(x)=y \textit{ and } x\in A_{j_0} \quad (???) \tag{0.42}$$

$$\Rightarrow y=f(x)\in f(A_{j_0}) \quad (???) \tag{0.43}$$

$$\Rightarrow y\;\in \bigcup_{j\in J} f(A_j) \quad (???) \tag{0.44}$$

B. Proof of "$\supseteq$":

This is a trivial consequence from the monotonicity of $A\mapsto f(A)$:

$$A_i\subset \bigcup_{j\in J} A_j\;\forall\, i\in J \;\Rightarrow f(A_i)\subset f\big(\bigcup_{j\in J} A_j\big)\;\forall\, i\in J \quad (???) \tag{0.45}$$

$$\Rightarrow \bigcup_{i\in J} f(A_i)\subset f\big(\bigcup_{j\in J} A_j\big)\;\forall\, i\in J \quad (???) \tag{0.46}$$

■

You see that the "elementary" proof is barely longer than the first one, but it is so much easier to understand!

Proposition 0.4 (Indirect image and fibers of f)**.** *We define on X the equivalence relation*

$$x_1 \;\sim\; x_2 \iff f(x_1) \;=\; f(x_2), \textit{ i.e.,} \tag{0.47}$$

$$[x]_f \;=\{\bar{x}\in X: f(\bar{x})=f(x)\}, \textit{ are the equivalence classes.} \tag{0.48}$$

Then the following is true:

$$x \in X \quad \Rightarrow \quad \Big[[x]_f \;=\; \{\hat{x} \in X : f(\hat{x} = f(x))\} \;=\; f^{-1}\{f(x)\} \tag{0.49}$$

$$A \subseteq X \quad \Rightarrow \quad f^{-1}(f(A)) \;=\; \bigcup_{a \in A} [a]_f. \tag{0.50}$$

Proof of (0.49): The equation on the left is nothing but the definition of the equivalence classes generated by an equivalence relation, the equation on the right follows from the definition of preimages.

Proof of (0.50):

As $f(A) \;=\; f(\bigcup_{x \in A}\{x\}) \;=\; \bigcup_{x \in A}\{f(x)\}$ (see 0.32), it follows that

$$f^{-1}(f(A)) \;= f^{-1}(\bigcup_{x \in A}\{f(x)\}) \tag{0.51}$$

$$= \bigcup_{x \in A} f^{-1}\{f(x)\} \quad \text{(see 0.11)} \tag{0.52}$$

$$= \bigcup_{x \in A} [x]_f \quad \text{(see 0.49)} \tag{0.53}$$

■

Corollary 0.1.

$$A \in X \quad \Rightarrow \quad f^{-1}(f(A)) \;\supseteq\; A. \tag{0.54}$$

Proof:

It follows from $x \sim x$ for all $x \in X$ that $x \in [x]_f$, i.e., $\{x\} \in [x]_f$ for all $x \in X$. But then

$$A \;=\; \bigcup_{x \in A}\{x\} \;\subseteq\; \bigcup_{x \in A}[x]_f \;=\; f^{-1}(f(A)) \tag{0.55}$$

where the last equation holds because of (0.50). ■

Proposition 0.5.

$$B \subset Y \quad \Rightarrow \quad f(f^{-1}(B)) \;=\; B \cap f(X). \tag{0.56}$$

Proof of "$\subseteq$":

Let $y \in f(f^{-1}(B))$. There exists $x_0 \in f^{-1}(B)$ such that $f(x_0) = y$ (def direct image). We have
a) $x_0 \in f^{-1}(B) \;\Rightarrow y = f(x_0) \in B$ (def. of preimage)
b) Of course $x_0 \in X$. Hence $y = f(x_0) \in f(X)$. a and b together imply $y \in B \cap f(X)$.

Proof of "$\supseteq$":

Let $y \in f(X)$ and $y \in B$. We must prove that $y \in f(f^{-1}(B))$. Because $y \in f(X)$ there exists $x_0 \in X$ such that $y = f(x_0)$. Because $y = f(x_0) \in B$ we conclude that $x_0 \in f^{-1}(B)$ (def preimage). Let us abbreviate $A := f^{-1}(B)$. Now it easy to see that

$$x_0 \in f^{-1}(B) = A \quad \Rightarrow \quad y \;=\; f(x_0) \in f(f^{-1}(B)). \tag{0.57}$$

We have shown that if $y \in f(X)$ and $y \in B$ then $y \in f(f^{-1}(B))$. The proof is completed. ■

Remark 0.1. Be sure to understand how the assumption $y \in f(X)$ was used.

Corollary 0.2.

$$B \in Y \;\Rightarrow\; f(f^{-1}(B)) \subseteq B. \tag{0.58}$$

Trivial as $f(f^{-1}(B)) = B \cap f(X) \subset B$. ■

References

[1] *Raz Kupferman.* Lecture Notes in Probability (Hebrew University). *1st edition, 2009.*

[2] *Unknown.* mazur-330-func-1.pdf - Introduction to Functions Ch.2. *1st edition.*

[3] *Unknown.* mazur-330-func-2.pdf - Properties of Functions Ch.2. *1st edition.*

[4] *Unknown.* mazur-330-sets-1.pdf - Ch.1: Introduction to Sets and Functions. *1st edition.*

[5] *Unknown.* mazur-330-sets-2.pdf - Ch.4: Applications of Methods of Proof. *1st edition.*

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symbols
    $A \uplus B$ - *disjoint union*, 1
    $A^{\complement}$ - *complement*, 1
    $\mathscr{C}(E), \mathscr{C}^0(E)$ - *continuous functions*, 1
    $\complement A$ - *complement*, 1
    $\mathbb{N}, \mathbb{N}_0$, 1
    $\mathbb{R}^+$ - $]0, +\infty[$, 1
    $\mathbb{R}_+$ - $[0, +\infty[$, 1
    $\mathbb{Z}_+, \mathbb{Z}_{\geq 0}$, 1
    $f(A)$ - *direct image*, 1
    $f^{-1}(B)$ - *indirect image (preimage)*, 1