Math 330 Section 1 - Fall 2025 - Homework 12

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Status - Reading Assignments:

The reading assignments you were asked to complete before the first one of this HW are:

MF lecture notes:
ch.1 - 3, skim ch.4, ch.5 - ch.12.9 (skip ch.8.2 and 10.3)

B/G (Beck/Geoghegan) Textbook:
ch.2.1 – 7, ch.8-13.4

Other material (optional):
B/K lecture notes ch.1.1 and ch.1.2, except ch.1.2.4
Stewart Calculus 7ed - ch.1.7: "The Precise Definition of a Limit"

New reading assignments:

Reading assignment 1 - due Monday, November 3:

- **a.** Read carefully MF ch.12.10. Understand the connection AND DIFFERENCE between the completeness axiom and completeness of $(X,d) = (\mathbb{R}, d_{\|\cdot\|_2})$!
- **b.** Start reviewing convergence and continuity in $(X,d) = (\mathbb{R}, d_{\|\cdot\|_2})$ (Ch.9.3).

Reading assignment 2 - due: Wednesday, November 5:

- Study for the midterm!

Reading assignment 3 - due Friday, November 7:

- **a.** Finish reviewing convergence and continuity in $(X,d) = (\mathbb{R}, d_{\|\cdot\|_2})$ (Ch.9.3).
- **b.** Extra carefully read MF ch.13.1.1.
- **c.** Carefully read MF ch.13.1.2. (Very brief.)
- **d.** The stronger students are encouraged to read MF ch.13.1.3.

Be sure to read pages 2 and 3!

Supplementary instructions for reading MF ch.12:

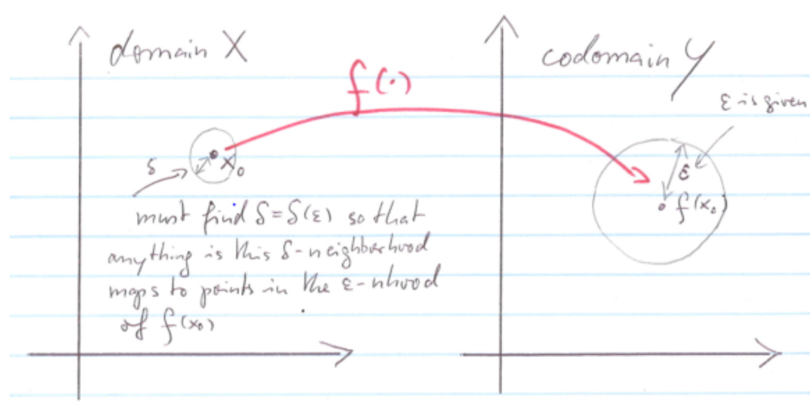
When you read or reread any topics in those chapters then the following is good advice:

a. MF ch.12.1: Draw as many pictures as possible to get a feeling for the abstract concepts. Use the metric spaces $(\mathbb{R}^2, d\big|_{\|\cdot\|_2})$ and $(\mathscr{B}(X,\mathbb{R}), d\big|_{\|\cdot\|_\infty})$ for this. Do these drawings in particular for

- open sets and neighborhoods (ch.12.1.3)
- convergence, expressed with nhoods (the end of def.12.10 in ch.12.1.4)
- metric and topological subspaces (ch.12.1.7): draw an irregular shaped subset $A \subseteq \mathbb{R}^2$ in two pieces $A = A_1 \uplus A_2$ which do not overlap. Draw some points $x_j \in A$ with ε–nhoods (circles with radius ε about x_j) so that some circles are entirely in A, one with $x_j \in A_1$ which reaches into $A^\complement$ but not into A_2, and one with $x_j \in A_2$ which reaches both into $A^\complement$ and A_1. What is $N_\varepsilon^A(x_j)$?
- Contact points, closed sets and closures (ch.12.1.8): Draw subsets $B \subseteq \mathbb{R}^2$ with parts of their boundary (periphery) drawn solid to indicate that points there belong to B and other parts drawn dashed to indicate that those boundary points belong to the complement. What is $\bar{B}$?
  Draw points "completetely inside" B, others "completetely outside" B, and others on the solid and dashed parts of the boundary. Which ones can you approximate from within B by sequences? Which ones can you surround by circles that entirely stay within B, i.e., which ones are interior points of B? Which ones can you surround by circles that entirely stay outside the closure of B, i.e., which ones are entirely within $\bar{B}^\complement$? Use those pictures to visualize the definitions in this chapter and thm 12.6 and thm.12.7.
- Now repeat that exercise with an additional set A which is meant to be a metric subspace of $\mathbb{R}^2$.

b. MF ch.12.2 (Continuity): Draw as many pictures as possible to get a feeling for continuity, especially if you did not take multivariable calculus and are not used to dealing with continuous/differentiable functions of more than one variable. Here is a picture.

Figure 0.1: ε-δ continuity



Written assignments on page 3

Written assignments:

Written assignment 1: Partial credit up to 3 points! Can be turned in repeatedly.

Let $f(x) = x^2$. Prove by use of "ε-δ continuity" that f is continous at $x_0 = 1$. You MUST work with ε-δ continuity (thm.9.7) **NOT WITH SEQUENCE CONTINUITY**, and you cannot use any "advanced" knowledge such as the product of continuous functions being continuous, etc.

Special instructions for assignment 1: Turn in your scratchpaper where you solve for δ (see the hints below).

Hints:

a. What does $d(x, x_0) < \delta$ and $d(f(x), f(x_0) < \varepsilon$ translate to?

b. $x^2 - 1 = (x + 1)(x - 1)$.

c. Do the following on scratch paper: Work your way backward by establishing a relationship between $\varepsilon > 0$ and δ and then "solving for δ" That part should not be in your official proof.

c1. Only small neighborhoods matter (see Proposition 9.24 at the end of the chapter on convergence and continuity.): Given $0 < \varepsilon < 1$ try to find δ that works for such ε. Restrict your search to $\delta < 1$. What kind of bounds do you get for $|\ x^2 - 1|$, $|x + 1|$, $|x - 1|$ if $0 < \delta < 1$? In particular what kind of bounds to you get for $|x + 1|$?

c2. Put all the above together. Show that you obtain $|f(x) - f(x_0)| \leq 3\delta$?. How then do you choose δ when you consider ε as given? You'll get the answer by "solving $|f(x) - f(x_0)| \leq 3\delta$ for δ".

c3. All of the above was done under the assumption that $\delta < 1$. Satisfy it by replacing δ with $\delta' := \min(\delta, 1)$

d. Only now you are ready to construct an acceptable proof: Let $\varepsilon > 0, \delta := \dots$, and $\delta' := \min(\delta, 1)$. Then ....

Written assignment 2: Prove MF Thm. 9.9 (B/G thm.11.12):

If $m \in [0, \infty[_{\mathbb{Z}}$ is not a perfect square, then $\sqrt{m}$ is irrational.

You MUST work with lowest term representations: Modify the proof that $\sqrt{2}$ is irrational. Key is Proposition 9.28 (B/G prop.11.5)! No partial credit for this one!