Math 330 - Additional Material

Student edition with proofs

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This document contains chapters 1 and 2 of the Math 330 lecture notes.

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1 Before You Start

Errors detected by Math 330 students, Spring 2017:

Date	Topic
2026-02-17	Error in Introduction 2.4(**d**). **Incorrect version:** function $f(x) : D \to C$ has domain C and codomain D. **Correct version:** function $f(x) : D \to C$ has domain D and codomain C. Detected by **Ryan Wang**.
2017-01-26	Error in Definition **??**. **Incorrect version:** A relation is symmetric if $x_1 R x_2$ implies $x_1 R x_2$ for all $x_1, x_2 \in X$. **Correct version:** A relation is symmetric if $x_1 R x_2$ implies $x_2 R x_1$ for all $x_1, x_2 \in X$. Detected by **Brad Whistance**.

1.1 About This Document

Remark 1.1 (The purpose of this document)**.**

The original version of this document was written in 2005 under the title "Introduction to Abstract Math – A Journey to Approximation Theory". Since then parts of it were discarded and others have been added. It now serves as lecture notes for the course "Math 330: Number systems" which is held at the Department of Mathematical Sciences at Binghamton University. Parts of the remainder of this chapter are specifically addressed to the students of this course.

These notes serve at least two purposes:

(a) They contain material on topics that cannot found in sufficient detail or generality in the textbook [1] Beck/Geoghegan: The Art of Proof. That book serves as the primary reference for the first two thirds of the Math 330 course. It is often simply referred to as "B/G" in these notes.

(b) This document covers material which is beyond the scope of [1] B/G such as
- material on lim inf and lim sup
- convergence, continuity and compactness in metric spaces
- applications of Zorn's Lemma

These topics are usually covered in the last third of my Math 330 class.

Prof. Geoghegan has graciously given permission to let me copy definitions, proofs and theorems verbatim from this text. I have indicated for such items how they are referenced there. An example is, e.g., proposition **??** on p.**??** of this document which is stated here for integral domains and shows its origin by the references B/G prop.1.13 and B/G prop.8.14. No proof is given in the B/G student edition for this proposition, and in all such circumstances I too do not furnish a proof to the student unless I have one that is quite different from the one to be found in the instructor's edition. □

Remark 1.2 (Acknowledgements)**.**

Chapters 2 and **??** of this document draw on [2] Bryant, Kirby Course Notes for MAD 2104. Moreover such a document cannot be written with the intent to supplement the [1] B/G book without strongly borrowing from it. □

Remark 1.3 (How to navigate this document I)**.**

Scrutinize the table of contents, including the headings for the subchapters. You will find many entries there tagged with a directive. The following explains the meaning of those tags.

a. "Understand this" directive: When you read "Understand this", you should know the definitions, propositions and theorems without worrying about proofs. It is quite likely that this kind of material will be referenced in more important sections of this document. As of May 2022 only the chapter on logic contains this directive.

b. " ⋆ " directive: Chapters marked " ⋆ " are optional. The student need not worry about learning the material, although it may be referenced in the non optional chapters if doing so benefits the interested reader. This symbol is also used to indicate that a certain statement or maybe just its proof can be skipped. Again, it may be referenced in the non optional chapters to provide some background for the interested reader. Moreover certain definitions are tagged with this symbol, but the reader should NOT skip those definitions: My students are not expected to give precise equivalents of such definitions in quizzes and exams, but they will need to know where to find them to do their homework and to make sense of the propositions and theorems and their proofs.

Notation Alert: All directives discussed above apply to the entire subtree, and a lower level directive overrides the "parent directives". Accordingly, when you do not see any comment, back up in the table of contents: first to the parent entry, then to its parent entry . . . until you find one.

Homework: You will find almost every week reading assignments as part of your homework. The reading is due prior to when it is needed in class, both for this document and the Beck/Geoghegan text. I assume that you did your reading and I will assume in particular that you have learned the definitions, also those tagged with a " ⋆ " symbol, so that I can move along at a fast pace except for some topics that I will focus on in detail.

We use colored boxes like the two above according to the following. Generally speaking,

These boxes contain important definitions or parts thereof.

These boxes contain important theorems and proposiitions or parts thereof.

These boxes contain other kinds of important items that are worthwhile to know. □

There are definitions and theorems that contain two or even three small boxes rather than a big one. There is a technical reason: such boxes do not span pages and will needlessly inflate the page count of the docum

Remark 1.4 (How to navigate this document II)**.**

I believe that, particularly in Math, more words take a lot less time to understand than a skeletal write-up like one often finds in the [1] B/G text. Accordingly, almost all of the material in this document comes with quite detailed proofs. Those proofs are there for you to study.

Some of those proofs, notably those in prop. **??** on p.**??**, make use of "⇔" to show that two sets are equal. You should study this technique but, as you will hear me say many times in class, I recommend that you abstain from using "⇔" between statements in your proofs. You very likely lack the experience to use this technique without errors.

Some of the material was written from scratch, other material was pulled in from a document that was written as early as 10 years ago. I have make an attempt to make the entire document more homogeneous but there will be some inconsistencies. Your help in pointing out to me the most notable trouble spots would be deeply appreciated.

There are differences in style: the original document was written in a much more colloquial style as it was addressed to a younger audience of high school students who had expressed a special interest in studying college level math. □

> This is a living document: material will be added as I find the time to do so. Be sure to check the latest PDF frequently. You certainly should do so when an announcement was made that this document contains new additions and/or corrections.

1.2 How to Properly Write a Proof

Study this brief chapter to understand some of the dos and don'ts when submitting your homework.

To prove the validity of an equation such as $A = Z$, do one of the following:

Method a.

$$\begin{aligned} A &= B \quad \text{(use ....)} \\ &= C \quad \text{(use ....)} \\ &= D \quad \text{(use ....)} \\ &\ldots\ldots\ldots\ldots\ldots \\ &= Z \quad \text{(use ....)} \end{aligned}$$

You then conclude from the transitivity of equality that $A = Z$ is indeed true.

> **Transitivity of equality** means that if A = B and B = C then A = C.

Method b. You transform the left side (L.S.) and the right side (R.S.) separately and show that in each case you obtain the same item, say M:

Left side:

$$\begin{aligned} A &= B \quad \text{(use ....)} \\ &= C \quad \text{(use ....)} \\ &= D \quad \text{(use ....)} \\ &\ldots\ldots\ldots\ldots\ldots \\ &= M \quad \text{(use ....)} \end{aligned}$$

Right side:

$$\begin{aligned} Z &= Y \quad (\text{use } \ldots.) \\ &= X \quad (\text{use } \ldots.) \\ &= W \quad (\text{use } \ldots.) \\ &\ldots\ldots\ldots\ldots\ldots \\ &= M \quad (\text{use } \ldots.) \end{aligned}$$

You rightfully conclude that the proof is done because it follows from $A = M$ and $Z = M$ that $A = Z$.

You are not allowed to structure your proof that A = Z as follows.

Method c.

$$\begin{aligned} A &= Z \quad (\text{that's what you want to prove}) \\ B &= Y \quad (\text{you do with both } A \text{ and } Z \text{ the same operation } \ldots\ldots) \\ C &= X \quad (\text{you do with both } B \text{ and } Y \text{ the same operation } \ldots\ldots) \\ D &= W \quad (\text{you do with both } C \text{ and } X \text{ the same operation } \ldots\ldots) \\ &\ldots\ldots\ldots\ldots\ldots \\ M &= M \quad (\text{you do with both } L \text{ and } N \text{ the same operation } \ldots\ldots) \end{aligned}$$

What is potentially wrong with that last approach?

In the abstract the issue is that when using method **(a)** or **(b)** you take in each step an equation that is true, and you rightfully conclude by the use of transitivity that you have proved what you wanted to be true.

When you use method c, you take an equation that you want to be true ($A = Z$) but have not yet proved that this is so. If this equation is wrong then doing the same thing to both of its sides will potentially lead to a true equation.

Here is a simple example that demonstrates why method c **is not allowed** We will use this method in two different ways to prove that $-2 = 2$.

First proof that $-2 = 2$:

$$\begin{aligned} -2 &= 2 \quad (\text{want to prove}) \\ -2 \cdot 0 &= 2 \cdot 0 \quad (\text{multiply both sides from the right w. } 0) \\ 0 &= 0 \quad (\text{anything times zero} = \text{zero}) \end{aligned}$$

We are done. ■

Second proof that $-2 = 2$:

$$\begin{aligned} -2 &= 2 \quad (\text{want to prove}) \\ (-2)^2 &= 2^2 \quad (\text{square both sides}) \\ 4 &= 4 \quad (\text{minus times minus} = \text{plus}) \end{aligned}$$

We are done. ■

Now you know why you must never use “method c” for a proof. [1]

[1] You will learn later in this document about injective functions which guarantee that if you do an operation (apply a function) to two different items then the results will also be different. If method c was restricted to only such operations then there would not be a problem. In the two “proofs” that show $-2 = 2$ we use operations that are not injective: In the first proof the assignment $x \mapsto 0 \cdot x$ throws everything into the same result zero. The second proof employs the assignment $x \mapsto x^2$ which maps two numbers x, y that differ by sign only to the same squared value $x^2 = y^2$.

1.3 Blank Page after Ch.1

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2 Preliminaries about Sets, Numbers and Functions

Introduction 2.1.

This document strives for mathematically exact definition of proofs, but not in this chapter. Here we want to provide the reader with a refresher of some material that a student with an interest in mathematics should have previously encountered in beginner's calculus, or even in high school. Examples are union, intersection, and inclusion of sets, integers, rational and real numbers, functions $y = f(x)$ of a real–valued variable x with a real–valued function value y, and some facts about differentiation and integration.

Much of this material will be given again in later chapters, with an accuracy that is satisfactory to a mathematician, so why waste some effort here? For example, you will find in this chapter a preliminary definition of the real numbers. See page 33. You will have to wait for the real thing (no pun intended) until p.**??** of chapter **??**!

We will be careful not to use those preliminary definitions before we reach the exact ones when developing the general theory, unless the instances where we do so will be covered a second time with precision. Examples for this are the preliminary definitions of the integers, the rational numbers, and the real numbers. However we will use the concepts defined here in examples, in clarifying remarks, and in exercises to help the student achieve a better understanding of the material.

The most important concept which we use before it is properly defined is that of "finitely many". We all know what it means, but do we know it in such a way that it can be used for the study of abstract math? This author does not think so, and the proper definition of finiteness is deferred until definition **??** on p.**??**, at the beginning of ch.**??**. Thus you will see examples of infinite sets and we will use phrases like "finitely many" and "infinitely many" in examples and preliminary definitions, but we will avoid using the concepts of finiteness and infiniteness when developing the general theory. □

> The students should read this chapter carefully, with the expectation that it contains material that they are not familiar with, as much of it will be used in lecture without comment. Very likely candidates are power sets, a function $f : X \to Y$ where domain X and codomain Y are part of the definition.
>
> We do not expect that the student has a background in proofs by induction and definitions by recursion. Those concepts are introduced here as tools, and the student is expected to familiarize herself/himself with those techniques before the mathematical underpinnings are provided in chapter **??** (The Integers, the Induction Axiom, and the Induction Principles) on p.**??**.

2.1 Sets and Basic Set Operations

Introduction 2.2.

This first subchapter of ch.2 is different from the following ones in that the treatment of sets given here is sufficiently exact for a PhD in math unless s/he works in the areas of logic or axiomatic set theory. The only exception is the end of the chapter where the preliminary definition of the size of a set (Definition 2.12 on p.30) needs to refer to finiteness.

Ask a mathematician how her or his Math is different from the kind of Math you learn in high school, in fact, from any kind of Math you find outside textbooks for mathematicians and theoretical physicists. One of the answers you are likely to get is that Math is not so much about numbers but also about other objects, among them sets and functions. Once you know about those, you can tackle sets of functions, set functions, sets of set functions, ... □

An entire book can be filled with a mathematically precise theory of sets. [2] For our purposes the following "naive" definition suffices:

Definition 2.1 (Sets)**.**

A **set** is a collection of stuff called **members** or **elements** which satisfies the following rules:

- The order in which the elements are written does not matter.
- If an element is listed two or more times, then **it only counts once**!

We write a set by enclosing within curly braces the elements of the set. This can be done by listing all those elements or giving instructions that describe those elements. □

For example, to denote by X the set of all integer numbers between 18 and 24 we can write either of the following:

$$X := \{18, 19, 20, 21, 22, 23, 24\} \quad \text{or} \quad X := \{n : n \text{ is an integer and } 18 \leq n \leq 24\}$$

Both formulas clearly define the same collection of all integers between 18 and 24. On the left the elements of X are given by a complete list, on the right we use instead **setbuilder notation**, i.e., instructions that specify what belongs to the set.

It is customary to denote sets by capital letters and their elements by small letters but this is not a hard and fast rule. You will see many exceptions to this rule in this document.

We write $x_1 \in X$ to denote that an item x_1 is an element of the set X and $x_2 \notin X$ to denote that an item x_2 is not an element of the set X

For the above example we have $20 \in X$, $27 - 6 \in X$, $38 \notin X$, 'Jimmy' $\notin X$.

Example 2.1. The following collection of alphabetic letters is a set:

$$S_1 = \{a, e, i, o, u\}$$

and so is this one:

$$S_2 = \{a, e, e, i, i, i, o, o, o, o, u, u, u, u, u\}$$

Did you notice that those two sets are equal? □

Example 2.2 (No duplicates in sets)**.** The following collection of alphabetic letters is a set:

$$S_1 = \{a, e, i, o, u\}$$

and so is this one:

$$S_2 = \{a, e, e, i, i, i, o, o, o, o, u, u, u, u, u\}$$

Did you notice that those two sets are equal? □

[2] See remark 2.2 ("Russell's Antinomy") below.

Remark 2.1.
The symbol n in the definition of $X = \{n : n \text{ is an integer and } 18 \leq n \leq 24\}$ is a **dummy variable** in the sense that it does not matter what symbol you use. The following sets all are equal to X:

$$\{x : x \text{ is an integer and } 18 \leq x \leq 24\},$$
$$\{\alpha : \alpha \text{ is an integer and } 18 \leq \alpha \leq 24\},$$
$$\{\mathfrak{z} : \mathfrak{z} \text{ is an integer and } 18 \leq \mathfrak{z} \leq 24\} \quad \square$$

Remark 2.2 (Russell's Antinomy)**.**
Care must be taken so that, if you define a set with the use of setbuilder notation, no inconsistencies occur. Here is an example of a definition of a set that leads to contradictions.

$$A := \{B : B \text{ is a set and } B \notin B\} \tag{2.1}$$

What is wrong with this definition? To answer this question let us find out whether or not this set A is a member of A. Assume that A belongs to A. The condition to the right of the colon states that $A \notin A$ is required for membership in A, so our assumption $A \in A$ must be wrong. In other words, we have established "by contradiction" that $A \notin A$ is true. But this is not the end of it: Now that we know that $A \notin A$ it follows that $A \in A$ because A contains **all** sets that do not contain themselves.

In other words, we have proved the impossible: both $A \in A$ and $A \notin A$ are true! There is no way out of this logical impossibility other than excluding definitions for sets such as the one given above. It is very important for mathematicians that their theories do not lead to such inconsistencies. Therefore, examples as the one above have spawned very complicated theories about "good sets". It is possible for a mathematician to specialize in the field of axiomatic set theory (actually, there are several set theories) which endeavors to show that the sets are of any relevance in mathematical theories do not lead to any logical contradictions.

The great majority of mathematicians take the "naive" approach to sets which is not to worry about accidentally defining sets that lead to contradictions and we will take that point of view in this document. $\square$

We sometimes refer in the examples to the sets of numbers $\mathbb{N}$ (natural numbers), $\mathbb{Z}$ (integers), $\mathbb{R}$ (real numbers). If you are not familiar with those set please review briefly Definitions 2.13 and 2.14 at the start of section 2.3 (Numbers). This will come in handy for understanding the following example which demonstrates that some or all elements of a set can be sets themselves.

Example 2.3.

(a) $\mathscr{A} := \{\,]a, b[: a, b \in \mathbb{R}, 0 < b - a < 2\}$ is the set of all open intervals of length less than 2
(b) $\mathscr{B} := \{K : \text{K is a set of integers }\}$ We will later refer to $\mathscr{B}$ as the power set of $\mathbb{Z}$. [3] $\square$

[3] See Definition 2.10 on p.29.

Definition 2.2 (empty set)**.**

The **empty set** is the set that does not contain any elements. It is uniquely determined by this property.

The symbols $\emptyset$ and $\{\,\}$ are both in use to denote this set. However, we **STRONGLY DISCOURAGE** the use of $\{\,\}$, since since this makes expressions with nested braces hard to read. $\square$

Remark 2.3 (Elements of the empty set and their properties)**.**

You can state anything you like about the elements of the empty sets, as there are none to which this assertion could apply. The following statements all are true:

- **(a)** If $x \in \emptyset$ then x is a positive number.
- **(b)** If $x \in \emptyset$ then x is a negative number.
- **(c)** Define $a \sim b$ if and only if both are integers and $a - b$ is an even number.
  For all $x, y, z \in \emptyset$ it is true that
  (c1) $x \sim x$,
  (c2) if $x \sim y$ then $y \sim x$,
  (c3) if $x \sim y$ and $y \sim z$ then $x \sim z$.
- **(d)** Let A be a set. If $x \in \emptyset$ then $x \in A$.

As you will learn later, **(c1)**+**(c2)**+**(c3)** means that "$\sim$" is an equivalence relation (see Definition **??** on p.**??**) and **(d)** means that the empty set is a subset (see the next definition) of all sets, including itself. $\square$

Definition 2.3 (subsets, supersets and equality of sets)**.**

- **(a)** We say that a set A is a **subset** of the set B and we write $A \subseteq B$ if each element of A also belongs to B. Equivalently we say that B is a **superset** of the set A and we write $B \supseteq A$. We also say that B includes A or A is included by B. Note that $A \subseteq A$ and $\emptyset \subseteq A$ is true for all sets A.
- **(b)** If $A \subseteq B$ but $A \neq B$, i.e., there is at least one $x \in B$ such that $x \notin A$, then we say that A is a **strict subset** or a **proper subset** of B. We write "$A \subsetneq B$" or "$A \subset B$". Alternatively, we say that B is a **strict superset** or a **proper superset** of A and we write "$B \supsetneq A$" or "$B \supset A$".
- **(c)** We say that two sets A and B are **equal** and we write $A = B$, if both $A \subseteq B$ and $B \subseteq A$ $\square$

Remark 2.4.

The following two points are worthwhile mentioning.

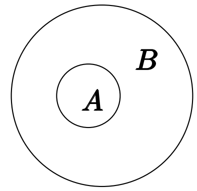

Figure 2.1: Set inclusion: $A \subseteq B,\ B \supseteq A$

(A) Do not write $A \subset B$ to indicate that A is a strict subset of B The reason is that many older texts write $A \subset B$ to express $A \subseteq B$

(B) Note that Definition 2.3 contains the instructions for proving that a set A is a subset of a set B or that both sets are equal:

(1) To prove that $A \subseteq B$, one must show the following: If $x \in A$, then $x \in B$.
(2) To prove that A and B are equal, one must show both that
- if $x \in A$ then $x \in B$,
- if $x \in B$ then $x \in A$.

In not quite so formal terms, $A = B$ iff each element of A also belongs to B and each element of B also belongs to A iff both A and B contain the same elements. Here, "**iff**" is a short for "if and only if": P iff Q for two statements P and Q means that if P is valid then Q is valid and vice versa.

A formal definition of "if and only if" will be given in Definition **??** on p.**??** where we will also introduce the symbolic notation $P \Leftrightarrow Q$. Informally speaking, a statement is something that is either true or false. ☐

Definition 2.4 (Unions and intersections of two sets)**.**

Given are two arbitrary sets A and B. No assumption is made that either one is contained in the other or that either one is not empty!

(a) The **union** $A \cup B$ (pronounced "A union B") is defined as the set of all elements which belong to A or B or both.
(b) The **intersection** $A \cap B$ (pronounced "A intersection B") is defined as the set of all elements which belong to both A and B. ☐

We could have shortened the phrase "all elements which belong to A or B or both" in Definition 2.4**(a)** to "all elements which belong to A or B", and we will almost always do so because it is understood among mathematicians that "or" always means at least one of the choices. If they mean instead exactly one of the choices $\#1, \#2, \dots \#n$, then they will use the phrase "either #1 or #2 or … or $\#n$. See rem**??** on p.**??**. We will also see in a moment that there is a special symbol $A \triangle B$ which denotes the items which belong to either A or B (but not both).

It is obvious how to define unions and intersections of more than two sets: If $A_1, A_2, \dots, A_n$ is a collection of n sets then we define

Definition 2.5 (Unions and intersections of n sets)**.**

Let $A_1, A_2, \dots, A_n$ be arbitrary sets.

(a) The **union** $\bigcup\limits_{j=1}^{n} A_j \ := \ A_1 \cup A_2 \cup \dots \cup A_n$ is defined as the set of all those items which belong to at least one of the sets, i.e.,

$$x \in \bigcup_{j=1}^{n} A_j \quad \Leftrightarrow \quad x \in A_j \text{ for at least one index } j. \tag{2.2}$$

(b) The **intersection** $\bigcap\limits_{j=1}^{n} A_j \ := \ A_1 \cap A_2 \cap \dots \cap A_n$ is defined as the set of all those items which belong to each and everyone of the sets, i.e.,

$$x \in \bigcap_{j=1}^{n} A_j \quad \Leftrightarrow \quad x \in A_j \text{ for each index } j. \ \square \tag{2.3}$$

Definition 2.6 (Disjoint unions)**.**

We call two sets A and B **disjoint** , also **mutually disjoint**, , if $A \cap B = \emptyset$. More generally, we say that a collection of sets $A_1, A_2, \dots, A_n$ is (mutually) disjoint if each pair A_i, A_j for different indices i and j is disjoint. We often write "$\uplus$" (pronounced "disjoint union") rather than "$\cup$" to remind the reader that we are dealing with unions of disjoint sets, i.e., we write

$$A \uplus B \qquad A_1 \uplus A_2 \uplus \dots \uplus A_n \,, \qquad \biguplus_{j=1}^{n} A_j \,,$$

rather than $A \cup B$, $A_1 \cup A_2 \cup \dots \cup A_n$, $\bigcup\limits_{j=1}^{n} A_j$. $\square$

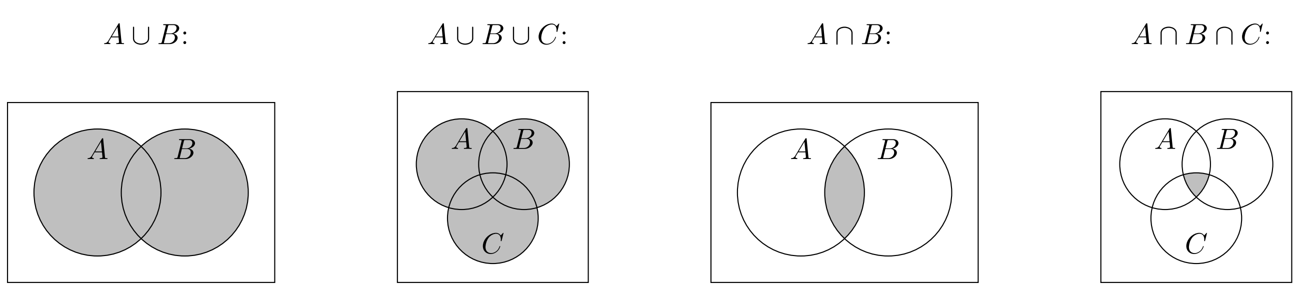

Figure 2.2: Union and intersection of sets

Remark 2.5.

It is obvious from the definition of unions and intersections and the meaning of the phrases " all elements which belong to A or B or both", "all elements which belong to both A and B" and "$A \subseteq B$ if each element of A also belongs to B" that the following is true for arbitrary sets A, B and C.

$$A \cap B \subseteq A \subseteq A \cup B, \tag{2.4}$$

$$A \subseteq B \Rightarrow A \cap B = A \text{ and } A \cup B = B, \tag{2.5}$$

$$A \subseteq B \Rightarrow A \cap C \subseteq B \cap C \text{ and } A \cup C \subseteq B \cup C. \tag{2.6}$$

The symbol $\Rightarrow$ stands for "allows us to conclude that". So $A \subseteq B \Rightarrow A \cap B = A$ means "From the truth of $A \subseteq B$ we can conclude that $A \cap B = A$ is true". Shorter: "From $A \subseteq B$ we can conclude that $A \cap B = A$". Shorter: "If $A \subseteq B$ then it follows that $A \cap B = A$". Shorter: "If $A \subseteq B$ then $A \cap B = A$". More technical: "$A \subseteq B$ implies $A \cap B = A$".

You will learn more about implication in ch.**??** of this document and in ch.3 (Some Points of Logic) of [1] Beck/Geoghegan: The Art of Proof. ☐

Definition 2.7 (Set differences and symmetric differences)**.**

Given are two sets A and B. No assumption is made that either one is contained in the other or that either one is not empty!

The **difference set** or **set difference** $A \setminus B$ (pronounced "A minus B") is defined as the set of all elements which belong to A but not to B:

$$A \setminus B := \{x \in A : x \notin B\} \tag{2.7}$$

The **symmetric difference** $A \triangle B$ (pronounced "A delta B") is defined as the set of all elements which belong to either A or B but not to both A and B:

$$A \triangle B := (A \cup B) \setminus (A \cap B) \quad \square \tag{2.8}$$

Definition 2.8 (Universal set)**.**

There usually is a big set Ω that contains everything we are interested in, and we then deal with all kinds of subsets $A \subseteq \Omega$. Such a set is called a **"universal" set**. ☐

For example, in this document, we often deal with real numbers and our universal set will then be $\mathbb{R}$. [4] If there is a universal set, it makes perfect sense to talk about the complement of a set:

[4] $\mathbb{R}$ is the set of all real numbers, i.e., the kind of numbers that make up the x-axis and y-axis in a beginner's calculus course (see ch.2.3 ("Classification of numbers") on p.32).

Definition 2.9 (Complement of a set)**.**

Let Ω be a universal set. The **complement** $A^{\complement}$. of a set $A \subseteq \Omega$ consists of all elements of Ω which do not belong to A. In other words:

$$A^{\complement} \; = = \; \Omega \setminus A \; = \; \{\omega \in \Omega : \omega \notin A\}\,. \; \square \tag{2.9}$$

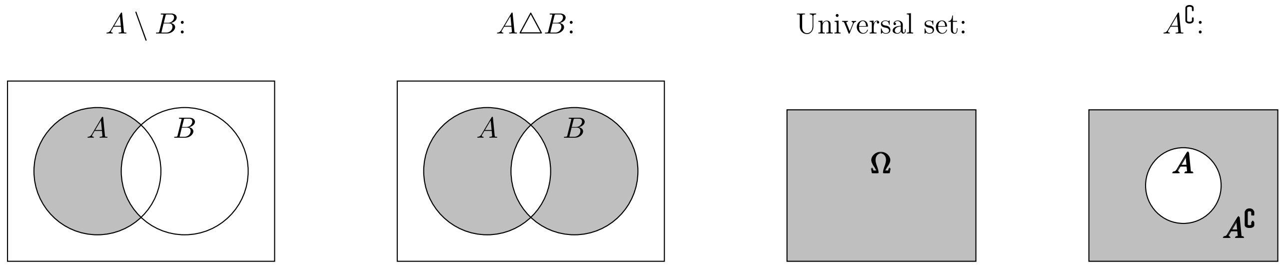

Figure 2.3: Difference, symmetric difference, universal set, complement

Remark 2.6. Note the following: If Ω is a universal set then

$$\Omega^{\complement} \; = \emptyset, \qquad \emptyset^{\complement} \; = \; \Omega. \; \square \tag{2.10}$$

Example 2.4 (Complement of a set relative to the unit interval)**.** Assume we are exclusively dealing with the unit interval, i.e., $\Omega = [0,1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Let $a \in [0,1]$ and $\delta > 0$ and

$$A = \{x \in [0,1] : a - \delta < x < a + \delta\} \tag{2.11}$$

the δ–neighborhood [5] of a (with respect to $[0,1]$ because numbers outside the unit interval are not considered part of our universe). Then the complement of A is

$$A^{\complement} \; = \; \{x \in [0,1] : x \leq a - \delta \text{ or } x \geq a + \delta\}. \; \square$$

You are encouraged to draw some Venn diagrams to visualize the following formulas.

Proposition 2.1.

[5] Neighborhoods of a point will be discussed in the chapter on the topology of $\mathbb{R}^n$ (see (**??**) on p.**??**). In short, the δ–neighborhood of a is the set of all points with distance less than δ from a.

Let A, B, X be subsets of a universal set Ω and assume $A \subseteq X$. Then

(2.12a) $A \cup \emptyset = A; \quad A \cap \emptyset = \emptyset$

(2.12b) $A \cup \Omega = \Omega; \quad A \cap \Omega = A$

(2.12c) $A \cup A^{\complement} = \Omega; \quad A \cap A^{\complement} = \emptyset$

(2.12d) $A \triangle B = (A \setminus B) \uplus (B \setminus A)$

(2.12e) $A \setminus A = \emptyset$

(2.12f) $A \triangle \emptyset = A; \quad A \triangle A = \emptyset$

(2.12g) $X \triangle A = X \setminus A$

(2.12h) $A \cup B = (A \triangle B) \uplus (A \cap B)$

(2.12i) $A \cap B = (A \cup B) \setminus (A \triangle B)$

(2.12j) $A \triangle B = \emptyset$ *if and only if* $B = A$

PROOF: The proof is left as exercise 2.2. See p.58. ■

Next we give a very detailed and rigorous proof of a simple formula for sets. The reader should make an effort to understand it line by line.

Proposition 2.2 (Distributivity of unions and intersections for two sets)**.**

Let A, B, C be sets. Then

(2.13) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C),$

(2.14) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$

PROOF: We only prove (2.13). The proof of (2.14) is left as exercise 2.1.

PROOF of "$\subseteq$": Let $x \in (A \cup B) \cap C$. It follows from (2.4) on p.25 that $x \in (A \cup B)$, i.e., $x \in A$ or $x \in B$ (or both). It also follows from (2.4) that $x \in C$. We must show that $x \in (A \cap C) \cup (B \cap C)$ regardless of whether $x \in A$ or $x \in B$.

Case 1: $x \in A$. Since also $x \in C$, we obtain $x \in A \cap C$, hence, again by (2.4), $x \in (A \cap C) \cup (B \cap C)$, which is what we wanted to prove.

Case 2: $x \in B$. We switch the roles of A and B. This allows us to apply the result of case 1, and we again obtain $x \in (A \cap C) \cup (B \cap C)$.

PROOF of "$\supseteq$": Let $x \in (A \cap C) \cup (B \cap C)$, i.e., $x \in A \cap C$ or $x \in B \cap C$ (or both). We must show that $x \in (A \cup B) \cap C$ regardless of whether $x \in A \cap C$ or $x \in B \cap C$.

Case 1: $x \in A \cap C$. It follows from $A \subseteq A \cup B$ and (2.6) on p.25 that $x \in (A \cup B) \cap C$, and we are done in this case.

Case 2: $x \in B \cap C$. This time it follows from $B \subseteq A \cup B$ that $x \in (A \cup B) \cap C$. This finishes the proof of (2.13).

Epilogue: The proofs both of "$\subseteq$" and of "$\supseteq$" were **proofs by cases**, i.e., we divided the proof into several cases (to be exact, two for each of "$\subseteq$" and "$\supseteq$"), and we proved each case separately. For example we proved that $x \in (A \cup B) \cap C$ implies $x \in (A \cap C) \cup (B \cap C)$ separately for the cases $x \in A$ and $x \in B$. Since those two cases cover all possibilities for x the assertion "if $x \in (A \cup B) \cap C$ then $x \in (A \cap C) \cup (B \cap C)$" is proven. ■

Proposition 2.3 (De Morgan's Law for two sets)**.**

Let $A, B \subseteq \Omega$. Then the complement of the union is the intersection of the complements, and the complement of the intersection is the union of the complements:

(2.15) ***(a)*** $(A \cup B)^{\complement} = A^{\complement} \cap B^{\complement}$ ***(b)*** $(A \cap B)^{\complement} = A^{\complement} \cup B^{\complement}$

PROOF of **(a)**:

(1) First we prove that $(A \cup B)^{\complement} \subseteq A^{\complement} \cap B^{\complement}$:

Assume that $x \in (A \cup B)^{\complement}$. Then $x \notin A \cup B$, which is the same as saying that x does not belong to either of A and B. That in turn means that x belongs to both $A^{\complement}$ and $B^{\complement}$ and hence also to the intersection $A^{\complement} \cap B^{\complement}$.

(2) Now we prove that $(A \cup B)^{\complement} \supseteq A^{\complement} \cap B^{\complement}$:

Let $x \in A^{\complement} \cap B^{\complement}$. Then x belongs to both $A^{\complement}, B^{\complement}$, hence neither to A nor to B, hence $x \notin A \cup B$. Therefore x belong to the complement of $A \cup B$. This completes the proof of formula **(a)**.

PROOF of **(b)**:

The proof is very similar to that of formula **(a)** and left as an exercise. ■

Formulas **(a)** through **(g)** of the next proposition are very useful. You are advised to learn them by heart and draw pictures to visualize them. You also should examine closely the proof of the next proposition. It shows how a proof which involves 3 or 4 sets can be split into easily dealt with cases.

Proposition 2.4.

Let A, B, C, Ω be sets such that $A, B, C \subseteq \Omega$. Then

(a) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$
(b) $A \triangle \emptyset = \emptyset \triangle A = A$
(c) $A \triangle A = \emptyset$
(d) $A \triangle B = B \triangle A$

Further we have the following for the intersection operation:

(e) $(A \cap B) \cap C = A \cap (B \cap C)$
(f) $A \cap \Omega = \Omega \cap A = A$
(g) $A \cap B = B \cap A$

And we have the following interrelationship between $\triangle$ and $\cap$:

(h) $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$

PROOF:
The proof of **(a)** is very tedious and there is a much more elegant proof, but that one requires knowledge of indicator functions [6] and of base 2 modular arithmetic (see, e.g., [1] B/G (Beck/Geoghegan) ch.6.2).

By definition $x \in U \triangle V$ if and only if either $x \in U$ or $x \in V$, i.e.,
(either) $[\ x \in U$ and $x \notin V\ ]$ or $[\ x \in V$ and $x \notin U\ ]$. Hence,

- $x \in (A \triangle B) \triangle C \;\Leftrightarrow\;$ either $x \in (A \triangle B)$ or $x \in C$
  $\Leftrightarrow$ either $[\ x \in A, x \notin B$ or $x \in B, x \notin A\ ]$ or $x \in C$.
- $x \in A \triangle (B \triangle C) \;\Leftrightarrow\;$ either $x \in A$ or $x \in (B \triangle C)$
  $\Leftrightarrow$ either $[\ x \in B, x \notin C$ or $x \in C, x \notin B\ ]$ or $x \in A$.

Thus, we obtain the following "truth table" for the eight possible combinations:

$x \in \dots$	A	B	C	$A \triangle B$	$B \triangle C$	$(A \triangle B \triangle C)$	$A \triangle (B \triangle C)$
	F	**F**	**F**	**F**	**F**	**F**	**F**
	F	**F**	**T**	**F**	**T**	**T**	**T**
	F	**T**	**F**	**T**	**T**	**T**	**T**
	F	**T**	**T**	**T**	**F**	**F**	**F**
	T	**F**	**F**	**T**	**F**	**T**	**T**
	T	**F**	**T**	**T**	**T**	**F**	**F**
	T	**T**	**F**	**F**	**T**	**F**	**F**
	T	**T**	**T**	**F**	**F**	**T**	**T**

We have a perfect match of set membership in the two rightmost columns. Thus,

$$x \in (A \triangle B \triangle C) \;\Leftrightarrow\; A \triangle (B \triangle C)\,.$$

This proves **(a)**.

The proofs of **(b)**, **(c)**, **(d)** are easy if you work with

$$U \triangle V \;=\; (U \setminus V) \uplus (V \setminus U) \;=\; (U \cap V^{\complement}) \uplus (V \cap U^{\complement})\,.$$

For example the proof of **(c)** is as follows.

$$A \triangle A \;=\; (A \cap A^{\complement}) \uplus (A \cap A^{\complement}) \;=\; \emptyset \uplus \emptyset \;=\; \emptyset\,.$$

The proofs of **(e)**, **(f)**, **(g)** are immediate. The proof of **(h)** can be done by cases, similarly to the proof of **(a)**. ■

Definition 2.10 (Power set)**.**

The **power set**

$$2^{\Omega} \;:=\; \{A : A \subseteq \Omega\}$$

of a set Ω is the set of all its subsets. Note that many older texts also use the notation $\mathfrak{P}(\Omega)$ for the power set. □

[6] Indicator functions will be discussed in ch.?? on p.?? and in ch.?? on p.??.

Example 2.5.

(a) $]3.2, 4.8[\in 2^{[3.2,4.8]}$ because $]3.2, 4.8[$ is a subset of $[3.2, 4.8]$,
but $[3.2, 4.8] \notin 2^{]3.2,4.8[}$ because $]3.2, 4.8[$ is not a subset of $[3.2, 4.8]$.

(b) Let $Z := \{ 5.4, \{19\}, \pi \}$. Then

$$2^Z = \{ \emptyset, \{5.4\}, \{\{19\}\}, \{\pi\}, \{5.4, \{19\}\}, \{5.4, \pi\}, \{\{19\}, \pi\}, \{5.4, \{19\}, \pi\} \} . \square$$

Remark 2.7. Note that $\emptyset \in 2^\Omega$ for all sets Ω, even if $\Omega = \emptyset$, since $2^\emptyset = \{\emptyset\}$. In particular, the power set of the empty set is not empty. $\square$

Definition 2.11 (Partition)**.**

Let Ω be a set and $\mathcal{A} \subseteq 2^\Omega$, i.e., the elements of $\mathcal{A}$ are subsets of Ω.
We call $\mathcal{A}$ a **partition** or a **partitioning** of Ω if

(a) If $A, B \in \mathcal{A}$ such that $A \neq B$ then $A \cap B = \emptyset$. In other words, $\mathcal{A}$ consists of mutually disjoint subsets of Ω (see Definition 2.6),

(b) Each $x \in \Omega$ is an element of some $A \in \mathcal{A}$. $\square$

Remark 2.8.
Let Ω be a set and $\mathcal{A} \subseteq 2^\Omega$. Then $\mathcal{A}$ is a partition of Ω if and only if

$$\text{For each } x \in \Omega, \text{ there exists a UNIQUE } A \in \mathcal{A} \text{ such that } x \in A . \square$$

Example 2.6.

a. For $n \in \mathbb{Z}$ let $A_n := \{n\}$. Then $\mathcal{A} := \{A_n : n \in \mathbb{Z}\}$ is a partition of $\mathbb{Z}$. $\mathcal{A}$ is not a partition of $\mathbb{N}$ because not all its members are subsets of $\mathbb{N}$ and it is not a partition of $\mathbb{Q}$ or $\mathbb{R}$. The reason: $\frac{1}{2} \in \mathbb{Q}$ and hence $\frac{1}{2} \in \mathbb{R}$, but $\frac{1}{2} \notin A_n$ for any $n \in \mathbb{Z}$, hence condition **b** of def.2.11 is not satisfied.

b. For $n \in \mathbb{N}$ let $B_n := [\, n^2, (n+1)^2[= \{x \in \mathbb{R} : n^2 \leq x < (n+1)^2\}$. Then $\mathfrak{B} := \{B_n : n \in \mathbb{N}\}$ is a partition of $[1, \infty[$. $\square$

Definition 2.12 (Size of a set (preliminary))**.**

(a) Let X be a finite set, i.e., a set which only contains finitely many elements. We write $|X|$ for the number of its elements, and we call $|X|$ the **size** of the set X.

(b) For infinite, i.e., not finite sets Y, we define $|Y| := \infty$. $\square$

A lot more will be said about sets once families are defined.

2.2 The Proper Use of Language in Mathematics: Any vs All, etc

Introduction 2.3.

Mathematics must be very precise in its formulations. Such precision is achieved not only by means of symbols and formulas, but also by its use of the English language. We will list some important points to consider early on in this document.

2.2.0.1 All vs. ANY

Assume for the following that X is a set of numbers. Do the following two statements mean the same?

(1) It is true for ALL $x \in X$ that x is an integer.
(2) It is true for ANY $x \in X$ that x is an integer.

You will hopefully agree that there is no difference and that one could rewrite them as follows:

(3) ALL $x \in X$ are integers.
(4) ANY $x \in X$ is an integer.
(5) EVERY $x \in X$ is an integer.
(6) EACH $x \in X$ is an integer.
(7) IF $x \in X$ THEN x is an integer.

Is it then always true that ALL and ANY means the same? Consider

(8a) It is NOT true for ALL $x \in X$ that x is an integer.
(8b) It is NOT true for ANY $x \in X$ that x is an integer.

Completely different things have been said: Statement **(8)** asserts that as few as one item and as many as all items in X are not integers, whereas **(9)** states that no items, i.e., exactly zero items in X, are integers.

My suggestion: Express formulations like **(8b)** differently. You could have written instead

(8c) There is no $x \in X$ such that x is an integer.

2.2.0.2 AND vs. IF ... THEN

Some people abuse the connective AND to also mean IF ... THEN. However, mathematicians use the phrase "p AND q" exclusively to mean that something applies to both p and q. Contrast the use of AND in the following statements:

(9) "Jane is a student AND Joe likes baseball". This phrase means that both are true: Jane is indeed a student and Joe indeed likes baseball.
(10) "You hit me again AND you'll be sorry". **Never, ever use the word AND in this context!** A mathematician would express the above as "IF you hit me again THEN you'll be sorry".

2.2.0.3 OR vs. EITHER ... OR

The last topic we address is the proper use of "OR". In mathematics the phrase

(11) "p is true OR q is true"

is always to be understood as

(12) "p is true OR q is true OR BOTH are true", i.e., at least one of p, q is true.

This is in contrast to everyday language where “p is true OR q is true” often means that exactly one of p and q is true, but not not both.

When referring to a collection of items then the use of “OR” also is inclusive If the items $a, b, c, \dots$ belong to a collection $\mathscr{C}$, e.g., if those items are elements of a set, then

(13) “a OR b OR c OR ...” means that we refer to at least one of $a, b, c, \dots$.

> Note that “OR” in mathematics always is an **inclusive or**, i.e., “A OR B” means “A OR B OR BOTH”. More generally, “A OR B OR ...” means “at least one of A, B, ...”.
>
> To rule out that more than one of the choices is true you must use a phrase like “EXACTLY ONE OF A, B, C, ...” or “EITHER A OR B OR C OR ...”. We refer to this as an **exclusive or**.

2.3 Numbers

We start with an informal classification of numbers. It is not meant to be mathematically exact. We will give exact definitions of the integers, rational numbers and real numbers in chapter **??** (The Real Numbers).

Definition 2.13 (Integers and decimal numerals)**.**

> A **digit** or **decimal digit** Is one of the numbers $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.
>
> We call numbers that can be expressed as a finite string of digits, possibly preceded by a minus sign, **integers**. In particular we demand that an integer can be written without a decimal point.

Examples of integers are

$$3,\ -29,\ 0,\ 3\cdot 10^6,\ -1,\ 2.\bar{9},\ 12345678901234567890,\ -2018. \tag{2.16}$$

Note that $3 \cdot 10^6 = 3000000$ is a finite string of digits and that $2.\bar{9}$ equals 3 (see below about the period of a decimal numeral). We write $\mathbb{Z}$ for the set of all integers.

Numbers in the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of all strictly positive integers are called **natural numbers**.

An integer n is an **even** integer if it is a multiple of 2, i.e., there exists $j \in \mathbb{Z}$ such that $n = 2j$, and it is an **odd** integer otherwise. One can give a strict proof that n is odd if and only if there exists $j \in \mathbb{Z}$ such that $n = 2j + 1$. See prop**??** on p.**??**.

> A **decimal** or **decimal numeral** is a finite or infinite list of digits, possibly preceded by a minus sign, which is separated into two parts by a point, the **decimal point**. □

The list to the left of the decimal point must be finite or empty, but there may be an infinite number of digits to its right. Examples are

$$3.0,\ -29.0,\ 0.0,\ -0.75,\ .\bar{3},\ 2.74\bar{9},\ \pi = 3.141592.....,\ -34.56. \tag{2.17}$$

The bar on top of the rightmost part of a decimal such as "$.\bar{3}$" means that this part should be repeated over and over again, i.e., $.\bar{3} = 0.33333333333\dots$ and $1.234\overline{567} = 1.234567567567\dots$.

We call the barred portion of the decimal digits the **period** of the number and we also talk about **repeating decimals**. The number of digits in the barred portion is called the **period length**. This period length can be bigger than one. For example, the number $1.234\overline{567}$ from above has period length 3 and the number $0.1\overline{45}$ has period length 2.

If the list to the right of the decimal point is of the form

$$d_1 d_2 d_3 \dots d_k 00 \dots,$$

i.e., all digits $d_{k+1}, d_{k+2}, d_{k+3}, \dots$ are zero, then we may remove them from the list. For example, the following all denote the same decimal:

$$-12.34 \;=\; -12.340 \;=\; -12.340000 \;=\; -12.34\bar{0}.$$

The above example shows that any decimal numeral which can be represented by finitely many digits, can also be represented as a repeating decimal (with period $\bar{0}$).

Any integer can be transformed into a decimal numeral of same value by appending the pattern ".0" to its right. Hence the first three integers of (2.16) are equal in value to the first three decimals of (2.17). The mathematician says that we **identify** the integer $\pm d_1 d_2 d_3 \dots d_k$ and the decimal numeral $\pm d_1 d_2 d_3 \dots d_k.0$, i.e., we do not distinguish those expressions and we consider them as equal, just as we would "six" and "half a dozen".

We are ready to give an informal definition of the most important kind of numbers. The formal, axiomatic, definition will be given in axiom **??** on p.**??**.

Definition 2.14 (Real numbers)**.**

> We call any kind of number which can be represented as a decimal numeral, a **real number**. We write $\mathbb{R}$ for the set of all real numbers. It follows from what was remarked at the end of Definition 2.13 that integers, in particular natural numbers, are real numbers. Thus we have the following set relations:
>
> (2.18) $$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}. \quad \square$$

We next define rational numbers. The formal definition will be given in Definition **??** on p.**??**,

Definition 2.15 (Rational numbers)**.**

> A number that is an integer or can be written as a fraction of integers, i.e., as $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ and $n \neq 0$, is called a **rational number**. We write $\mathbb{Q}$ for the set of all rational numbers. $\square$

Examples of rational numbers are

$$\tfrac{3}{4},\ -0.75,\ -\tfrac{1}{3},\ .\bar{3},\ \tfrac{7}{1},\ 16,\ \tfrac{13}{4},\ -5,\ 2.99\bar{9},\ -37\tfrac{2}{7}.$$

Note that a mathematician does not care whether a rational number is written as a fraction

$$\frac{\text{numerator}}{\text{denominator}}$$

or as a decimal numeral. The following all are representations of one third:

$$0.\bar{3} \;=\; .\bar{3} \;=\; 0.33333333333\ldots \;=\; \tfrac{1}{3} \;=\; \tfrac{-1}{-3} \;=\; \tfrac{2}{6}, \tag{2.19}$$

and here are several equivalent ways of expressing the number minus four:

$$-4 \;=\; -4.000 \;=\; -3.\bar{9} \;=\; -\tfrac{12}{3} \;=\; \tfrac{4}{-1} \;=\; \tfrac{-4}{1} \;=\; \tfrac{12}{-3} \;=\; -\tfrac{400}{100}. \tag{2.20}$$

If $q \in \mathbb{Q}$ then there are unique integers n and d such that $q = \frac{n}{d}$ and

(a) $d \in \mathbb{N}$,
(b) d is minimal: there are no numbers $n' \in \mathbb{Z}$ and $d' \in \mathbb{N}$ such that $q = \frac{n'}{d'}$ and $d' < d$.

We say that this choice of n and d is a representation of q in **lowest terms** or that q is written in lowest terms. For example, the representation of $.\bar{3}$ in lowest terms is $\frac{1}{3}$ and the representation of -4 in lowest terms is $\frac{-4}{1}$.
Note that if $q \in \mathbb{Q}$ is strictly positive and if $\frac{d}{n}$ represents q in lowest terms then $d \in \mathbb{N}$.

There are real numbers which cannot be expressed as integers or fractions of integers.

Definition 2.16 (Irrational numbers)**.**

We call real numbers that are not rational **irrational numbers**. □

They hence fill the gaps that exist between the rational numbers. In fact, there is a simple way (but not easy to prove) of characterizing irrational numbers: Rational numbers are those that can be expressed with at most finitely many digits to the right of the decimal point, including repeating decimals. You can find the underlying theory and exact proofs in ch.**??** (Decimal Expansions of Real and Rational Numbers). Irrational numbers must then be those with infinitely many decimal digits without a continually repeating pattern.

Example 2.7. To illustrate that repeating decimals are in fact rational numbers we convert $x = 0.1\overline{45}$ into a fraction:

$$99x \;= 100x - x \;=\; 14.5\overline{45} - 0.1\overline{45} = \; 14.4$$

It follows that $x = 144/990$, and that is certainly a fraction. □

Remark 2.9. Examples of irrational numbers are $\sqrt{2}$ *and* $\boldsymbol{\pi}$. A proof that $\sqrt{2}$ is irrational (actually that $\sqrt[n]{2}$ is irrational for any integer $n \geq 2$) is given in prop.**??** on p.**??**. □

Remark 2.10.
We will see in ch.**??** (Countable and Uncountable Subsets of the Real Numbers) on p.**??** that, in a sense, there are a lot more irrational numbers than rational numbers, even though $\mathbb{Q}$ is a **"dense" subset** in $\mathbb{R}$ in the following sense: No matter how small an interval $]a,b[= \{x \in \mathbb{R} : a < x < b\}$ of real numbers you choose, it will contain infinitely many rational numbers. □

We summarize what was said sofar about the classification of numbers:

Definition 2.17 (Types of numbers)**.**

$\mathbb{N} := \{1, 2, 3, \dots\}$ denotes the set of **natural numbers**.
$\mathbb{Z} := \{0, \pm1, \pm2, \pm3, \dots\}$ denotes the set of all **integers**.
$\mathbb{Q} := \{n/d : n \in \mathbb{Z}, d \in \mathbb{N}\}$ denotes the set of all **rational numbers**.
$\mathbb{R} := \{$all integers or decimal numbers with finitely or infinitely many decimal digits$\}$ denotes the set of all **real numbers**.
$\mathbb{R} \setminus \mathbb{Q} = \{$all real numbers which cannot be written as fractions of integers$\}$ denotes the set of all **irrational numbers**. There is no special symbol for irrational numbers. Example: $\sqrt{2}$ and π are irrational. □

Here are some customary abbreviations of some often referenced sets of numbers:

$\mathbb{N}_0 := \mathbb{Z}_+ := \mathbb{Z}_{\geq 0} := \{0, 1, 2, 3, \dots\}$ denotes the set of nonnegative integers,
$\mathbb{R}_+ := \mathbb{R}_{\geq 0} := \{x \in \mathbb{R} : x \geq 0\}$ denotes the set of all nonnegative real numbers,
$\mathbb{R}^+ := \mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$ denotes the set of all positive real numbers,
$\mathbb{R}^\star := \mathbb{R}_{\neq 0} := \{x \in \mathbb{R} : x \neq 0\}$. □

The next definition will be extended to objects more general than $\mathbb{R}$ in Definition **??** in ch.**??**.

Definition 2.18 (Translation and dilation of sets of numbers)**.**

For a set of numbers A and numbers λ and b, we define

$$\lambda A + b := \{\lambda a + b : a \in A\}. \tag{2.21}$$

In particular, for $\lambda = \pm1$, we obtain

$$A + b = \{a + b : a \in A\}, \tag{2.22}$$
$$-A = \{-a : a \in A\}. \tag{2.23}$$ □

The following will be generalized in Definition **??** on p.**??** to so called ordered integral domains.

Definition 2.19 (Intervals of Numbers).

For $a, b \in \mathbb{R}$ we have the following intervals.

- $[a,b] := \{x \in \mathbb{R} : a \le x \le b\}$ is the **closed interval** with endpoints a and b.
- $]a,b[:= \{x \in \mathbb{R} : a < x < b\}$ is the **open interval** with endpoints a and b.
- $[a,b[:= \{x \in \mathbb{R} : a \le x < b\}$ and $]a,b] := \{x \in \mathbb{R} : a < x \le b\}$ are **half-open intervals** with endpoints a and b.

The symbol "∞" stands for an object which itself is not a number but is larger than any (real) number, and the symbol "$-\infty$" stands for an object which itself is not a number but is smaller than any number. We thus have $-\infty < x < \infty$ for any number x. This allows us to define the following intervals of "infinite length":

$$\begin{aligned}]-\infty, a] &:= \{x \in \mathbb{R} : x \le a\}, \ \ ]-\infty, a[:= \{x \in \mathbb{R} : x < a\}, \\ ]a, \infty[&:= \{x \in \mathbb{R} : x > a\}, \ \ [a, \infty[:= \{x \in \mathbb{R} : x \ge a\}, \quad]-\infty, \infty[:= \mathbb{R} \end{aligned} \tag{2.24}$$

You should always work with $a < b$. In case you don't, you get

- $[a,a] = \{a\}$; $[a,a[=]a,a[=]a,a] = \emptyset$
- $[a,b] = [a,b[=]a,b[=]a,b] = \emptyset$ for $a \ge b$ □

Notation 2.1 (Notation Alert for intervals of integers or rational numbers).

It is at times convenient to also use the notation $[\dots]$, $]\dots[$, $[\dots[$, $]\dots]$, for intervals of integers or rational numbers. We will subscript them with $\mathbb{Z}$ or $\mathbb{Q}$. For example,

$$\begin{aligned} [\,3, n\,]_{\mathbb{Z}} &= [\,3,\ n] \cap \mathbb{Z} \ = \ \{k \in \mathbb{Z} : 3 \le k \le n\}, \\ ]-\infty, 7\,]_{\mathbb{Z}} &=]-\infty, 7\,] \cap \mathbb{Z} \ = \ \{k \in \mathbb{Z} : k \le 7\} \ = \ \mathbb{Z}_{\le 7}, \\ ]a,b[_{\mathbb{Q}} &=]a,b[\cap \mathbb{Q} \ = \ \{q \in \mathbb{Q} : a < q < b\}. \end{aligned}$$

An interval which is not subscripted always means an interval of real numbers, but we will occasionally write, e.g., $[a,b]_{\mathbb{R}}$ rather than $[a,b]$, if the focus is on integers or rational numbers and an explicit subscript helps to avoid confusion. □

Definition 2.20 (Absolute value).

For a real number x we define its **absolute value** as

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases} \quad \square$$

Example 2.8. $|3| = 3$; $|-3| = 3$; $|-5.38| = 5.38$. $\square$

Remark 2.11.
For any real number x we have

$$\sqrt{x^2} = |x|. \quad \square \tag{2.25}$$

Remember that for any number a it is true that

$$a \cdot a = (-a)(-a) = a^2, \qquad \text{e.g.,} \quad 2^2 = (-2)^2 = 4,$$

or that, expressed in form of square roots, for any number $b \geq 0$

$$(+\sqrt{b})(+\sqrt{b}) = (-\sqrt{b})(-\sqrt{b}) = b.$$

Assumption 2.1 (Square roots are always assumed nonnegative)**.**

We will always assume that "$\sqrt{b}$" is the **positive** value unless the opposite is explicitly stated. $\square$

Example: $\sqrt{9} = +3$, not -3. $\square$

Proposition 2.5 (The Triangle Inequality for real numbers)**.**

The following inequality is used all the time in mathematical analysis to show that the size of a certain expression is limited from above:

$$\textit{Triangle Inequality}: \quad |\boldsymbol{a} + \boldsymbol{b}| \leq |\boldsymbol{a}| + |\boldsymbol{b}| \tag{2.26}$$

This inequality is true for any two real numbers a and b.

PROOF:
It is easy to prove this: just look separately at the three cases where both numbers are nonnegative, both are negative or where one of each is positive and negative. ■

The next definition should be familiar to anyone who has worked with matrix algebra.

Definition 2.21 (Kronecker symbol)**.**

★

For $i, j \in \mathbb{N}$, the **Kronecker symbol** δ_{ij}, also called the **Kronecker delta**, is defined as follows.

$$\delta_{ij} := \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \quad \square \end{cases}$$

2.4 A First Look at Functions, Sequences and Families

The material on functions presented in this section will be discussed again and in greater detail in chapter **??** (Functions and Relations) on p.**??**.

Introduction 2.4.

You are familiar with functions from calculus. Examples are $f_1(x) = \sqrt{x}$ and $f_2(x, y) = \ln(x - y)$. Sometimes $f_1(x)$ means the entire graph, i.e., the entire collection of pairs $\big(x, \sqrt{x}\big)$ and sometimes it just refers to the function value $\sqrt{x}$ for a "fixed but arbitrary" number x. In case of the function $f_2(x)$: Sometimes $f_2(x, y)$ means the entire graph, i.e., the entire collection of pairs $\big((x, y), \ln(x - y)\big)$ in the plane. At other times this expression just refers to the function value $\ln(x - y)$ for a pair of "fixed but arbitrary" numbers (x, y).

This issue is addressed in the material of ch.**??** on p.**??** which precedes the mathematically precise definition of a function (Definition **??** on p.**??**). You are encouraged to look at it once you have read the remainder of this short section as ch.**??** contains everything you see here.

To obtain a usable definition of a function there are several things to consider. In the following $f_1(x)$ and $f_2(x, y)$ again denote the functions $f_1(x) = \sqrt{x}$ and $f_2(x, y) = \ln(x - y)$.

(a) The source of all allowable arguments (x–values in case of $f_1(x)$ and (x, y)–values in case of $f_2(x, y)$) will be called the **domain** of the function. The domain is explicitly specified as part of a function definition and it may be chosen for whatever reason to be only a subset of all arguments for which the function value is a valid expression. In case of the function $f_1(x)$ this means that the domain must be restricted to a subset of the interval $[0, \infty[$ because the square root of a negative number cannot be taken. In case of the function $f_2(x, y)$ this means that the domain must be restricted to a subset of $\{\ (x, y) : x, y \in \mathbb{R} \textit{ and } x - y > 0\}$ because logarithms are only defined for strictly positive numbers.

(b) The set to which all possible function values belong will be called the **codomain** of the function. As is the case for the domain, the codomain also is explicitly specified as part of a function definition. It may be chosen as any <u>superset</u> of the set of all function values for which the argument belongs to the domain of the function.

For the function $f_1(x)$ this means that we are OK if the codomain is a superset of the interval $[0, \infty[$. Such a set is big enough because square roots are never negative. It is OK to specify the interval $]-3.5, \infty[$ or even the set $\mathbb{R}$ of all real numbers as the codomain. In case of the function $f_2(x, y)$ this means that we are OK if the codomain contains $\mathbb{R}$. Not that it would make a lot of sense, but the set $\mathbb{R} \cup \{$ all inhabitants of Chicago $\}$ also is an acceptable choice for the codomain.

(c) A function $y = f(x)$ is not necessarily something that maps (assigns) numbers or pairs of numbers to numbers. Rather domain and codomain can be a very different kind of animal. In chapter **??** on logic you will learn about statement functions $A(x)$ which assign arguments x from some set $\mathscr{U}$, called the universe of discourse, to statements $A(x)$, i.e., sentences that are either true or false.

(d) Considering all that was said so far one can think of the graph of a function $f(x)$ with domain D and codomain C (see earlier in this note) as the set

$$\Gamma_f := \{\big(x, f(x)\big) : x \in D\}.$$

Alternatively one can characterize this function by the assignment rule which specifies how $f(x)$ depends on any given argument $x \in D$. We write "$x \mapsto f(x)$" to indicate this. You can also write instead $f(x) =$ whatever the actual function value will be.
This is possible if one does not write about functions in general but about specific functions such as $f_1(x) = \sqrt{x}$ and $f_2(x, y) = \ln(x - y)$. We further write

$$f : D \longrightarrow C$$

as a short way of saying that the function $f(x)$ has domain D and codomain C.
In case of the function $f_1(x) = \sqrt{x}$ for which we might choose the interval $X := [\ 2.5, 7\ ]$ as the domain (small enough because $X \subseteq [0, \infty[$) and $Y :=]1, 3[$ as the codomain (big enough because $1 < \sqrt{x} < 3$ for any $x \in X$) we specify this function as

$$\text{either } f_1 : [\ 2.5, 7\ ] \to]1, 3[; \quad x \mapsto \sqrt{x} \qquad \text{or } f_1 : [\ 2.5, 7\ ] \to]1, 3[; \quad f(x) = \sqrt{x}.$$

Let us choose $U := \{(x, y) : x, y \in \mathbb{R} \text{ and } 1 \leq x \leq 10 \text{ and } y < -2\}$ as the domain and $V := [0, \infty[$ as the codomain for $f_2(x, y) = \ln(x - y)$. These choices are OK because $x - y \geq 1$ for any $(x, y) \in U$ and hence $ln(x - y) \geq 0$, i.e., $f_2(x, y) \in V$ for all $(x, y \in U$.
We specify this function as

$$\text{either } f_2 : U \to V, \quad (x, y) \mapsto \ln(x - y) \qquad \text{or } f_2 : U \to V, \quad f(x, y) = \ln(x - y). \ \square$$

We incorporate what we noted above into this preliminary definition of a function.

Definition 2.22 (Preliminary definition of a function)**.**

A **function** f consists of two nonempty sets X and Y and an assignment rule $x \mapsto f(x)$ which assigns any $x \in X$ uniquely to some $y \in Y$. We write $f(x)$ for this assigned value and call it the **function value** of the **argument** x. X is called the **domain** and Y is called the **codomain** of f. We write

$$f : X \to Y, \qquad x \mapsto f(x). \tag{2.27}$$

We read "$a \mapsto b$" as "a is assigned to b" or "a maps to b" and refer to $\mapsto$ as the **maps to operator** or **assignment operator**. The **graph** of such a function is the collection of pairs

$$\Gamma_f := \{\big(x, f(x)\big) : x \in X\}. \ \square \tag{2.28}$$

Remark 2.12.

The name given to the argument variable is irrelevant. Let f_1, f_2, X, Y, U, V be as defined in **(d)** of the introduction to ch.2.4 (A First Look at Functions, Sequences and Families). The function

$$g_1 : X \to Y, \quad p \mapsto \sqrt{p}$$

is identical to the function f_1. The function

$$g_2 : U \longrightarrow V, \quad (t, s) \mapsto \ln(t - s)$$

is identical to the function f_2 and so is the function

$$g_3 : U \longrightarrow V, \quad (s, t) \mapsto \ln(s - t).$$

The last example illustrates the fact that you can swap function names as long as you do it consistently in all places.

There are times when we write $f(\cdot)$ rather than f for a function f when this avoids confusion. For example physicists and engineers often write $x = x(t)$ to denote the height x of a particle as a function of time t. In such a case we would write

$$x(\cdot) : [0, \infty[\longrightarrow \mathbb{R}\,; \qquad t \mapsto x(t)$$

rather than

$$x : [0, \infty[\longrightarrow \mathbb{R}\,; \qquad t \mapsto x(t)$$

and refer to $x(\cdot)$ rather than x. $\square$

Now some remarks about inverse functions.

Remark 2.13.

We all know what it means that $f(x) = \sqrt{x}$ has the function $g(x) = x^2$ as its inverse function: f and f^{-1} cancel each other, i.e.,

$$g\big(f(x)\big) \;=\; f\big(g(x)\big) \;=\; x.$$

That certainly is the most important aspect, but there is more. There is an issue with how free one is in the choice of domains and codomains of both functions. Let us replace g with the more familiar f^{-1}, let us write Dom_f and Cod_f for domain and codomain of f and $Dom_{f^{-1}}$ and $Cod_{f^{-1}}$ for domain and codomain of f^{-1}. Thus we have

$$f : Dom_f \longrightarrow Cod_f \qquad \text{and} \qquad f^{-1} : Dom_{f^{-1}} \longrightarrow Cod_{f^{-1}}\,.$$

We want that f^{-1} cancels the effect of f for **all** arguments of f, and we want that f cancels the effect of f^{-1} for **all** arguments of f^{-1}. In other words we want

$$f^{-1}\big(f(x)\big) \;=\; x \;\text{ for all } x \in Dom_f, \tag{2.29}$$

$$f\big(f^{-1}(y)\big) \;=\; y \;\text{ for all } y \in Dom_{f^{-1}}. \tag{2.30}$$

(a) We choose $Dom_f := [0, \infty[$ since that's the biggest set of real numbers for which the square root exists, and let us choose $Cod_f := \mathbb{R}$. Since everything can be squared we choose $Cod_{f^{-1}} := \mathbb{R}$ and $Cod_{f^{-1}} := \mathbb{R}$.
We have a problem. Let $x := -2$. Then $f\big(f^{-1}(-2)\big) = f(4) = 2$, thus (2.30) does not hold. We have to exclude negative numbers from $Dom_{f^{-1}}$, so we try again with $Dom_{f^{-1}} := [0, \infty[$, leaving everything else unchanged. Now (2.30) is satisfied.

(b) Some abstract considerations for the inverse: Let $f : X \to Y,\ x \mapsto f(x)$. Since the inverse should satisfy $f^{-1}\big(f(x)\big) = x$ it must accept items of the form $f(x)$ as arguments, thus its domain must be part of or maybe even all of Cod_f. Likewise, since the f itself should satisfy $f\big(f^{-1}(y)\big) = y$, it must accept items of the form $f^{-1}(y)$ as arguments, thus its domain must be part of or maybe even all of $Cod_{f^{-1}}$.
There are mathematical reasons to demand equality in the above: We want

$$Dom_{f^{-1}} = Cod_f = Y; \qquad \text{and} \qquad Cod_{f^{-1}} = Dom_f = X.$$

Thus, if $f : X \to Y,\ x \mapsto f(x)$ has an inverse then it must be of the form

$$f^{-1} : Y \to X, \quad y \mapsto f^{-1}(y). \ \square$$

We are ready to give the preliminary definition of an inverse function.

Definition 2.23 (Preliminary definition of the inverse function)**.**

Given are two nonempty sets X and Y and a function $f : X \to Y$ with domain X and codomain Y. We say that f has an **inverse function** if it satisfies all of the following conditions which uniquely determine this inverse function, so that we are justified to give it the symbol f^{-1}:

(a) $f^{-1} : Y \to X$, i.e., f^{-1} has domain Y and codomain X.

(b) $f^{-1}\big(f(x)\big) = x$ for all $x \in X$, and $f\big(f^{-1}(y)\big) = y$ for all $y \in Y$. $\square$

You will find a lot more about functions in ch.**??** (Functions (Mappings) and Families). Here is just one example. You will learn there that a function f has an inverse f^{-1} if and only if f is "onto": for each $y \in Y$ there is at least one $x \in X$ such that $f(x) = y$, and if f is "one–one": for each $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$.

Example 2.9.

Be sure you understand the following:

(a) $f : \mathbb{R} \to \mathbb{R};\ x \to e^x$ <u>does not have an inverse</u> $f^{-1}(y) = \ln(y)$ since its domain would have to be the codomain $\mathbb{R}$ of f and $\ln(y)$ is not defined for $y \leq 0$.

(b) $g : \mathbb{R} \to]0, \infty[;\ x \to e^x$ has the inverse $g^{-1} :]0, \infty[\to \mathbb{R};\ g^{-1}(y) = \ln(y)$ since

$$\begin{aligned} Dom_{g^{-1}} &= Cod_g =]0, \infty[, \qquad & Cod_{g^{-1}} &= Dom_g = \mathbb{R}, \\ e^{\ln(y)} &= y \text{ for } 0 < y < \infty, \qquad & \ln(e^x) &= x \text{ for all } x \in \mathbb{R}. \ \square \end{aligned}$$

We now briefly discuss (infinite) sequences, subsequences and finite sequences. The exact definition of sequences and their subsequences will be given in Definition **??** on p.**??**, that of finite sequences in Definition **??** on p.**??**.

Definition 2.24.

Let $n_\star$ be an integer and let there be a uniquely determined item x_j for each integer $j \geq n_\star$. Such an item can be, e.g., a number or a set (the only items we are looking at for now).

In other words, assume that a unique item x_j is assigned to each $j \in [n_\star, \infty[_{\mathbb{Z}}$. We write

$$(x_j)_{j \geq n_\star} \quad \text{or} \quad (x_j)_{j \in [n_\star, \infty[_{\mathbb{Z}}} \quad \text{or} \quad (x_j)_{j=n_\star}^{\infty} \quad \text{or} \quad x_{n_\star}, x_{n_\star+1}, x_{n_\star+2}, \dots$$

for such a collection of items, and we call it a **sequence** with **start index** $n_\star$. We call the set $[n_\star, \infty[_{\mathbb{Z}}$ of indices the **index set** of the sequence.

The symbol j is a dummy variable, same as the name x of the argument of a function $f(x)$. See Remark 2.12 on p.40. □

Example 2.10.

(a) If $u_k = k^2$ for $k \in \mathbb{Z}$, then $(u_k)_{k \geq -2}$ is the sequence of integers

$$4,\ 1,\ 0,\ 1,\ 4,\ 9,\ 16,\ \dots$$

(b) If $A_j = \left[-1 - \frac{1}{j}, 1 + \frac{1}{j}\right] = \left\{x \in \mathbb{R} : -1 - \frac{1}{j} \leq x \leq 1 + \frac{1}{j}\right\}$,
then $(A_j)_{j \geq 3}$ is a sequence of sets, the intervals (of real numbers)

$$\left[-\frac{4}{3}, \frac{4}{3}\right], \quad \left[-\frac{5}{4}, \frac{5}{4}\right], \quad \left[-\frac{6}{5}, \frac{6}{5}\right], \quad \left[-\frac{7}{6}, \frac{7}{6}\right], \quad \dots$$

(c) For $j \in [0, \infty]_{\mathbb{Z}}$, let $z_j := (-1)^j$. Then $(z_j)_{j=0}^{\infty}$ is the sequence of integers

$$1,\ -1,\ 1,\ -1,\ 1,\ -1,\ 1,\ -1,\ \dots \quad \square$$

(d) The symbols naming a sequence are dummy variables, same as the symbols f and x denoting a function $f(x)$. See Remark 2.12 on p.40. Thus, if $u_m = m^2$ and if $Q_j = j^2$, then $(u_m)_{m \geq -2}$ and $(Q_j)_{j \geq -2}$ are the same sequence of integers as the sequence from **(a)**,

$$(u_k)_{k \geq -2} \quad = \quad 4,\ 1,\ 0,\ 1,\ 4,\ 9,\ 16,\ \dots \quad \square$$

Remark 2.14.

Sequences can be considered as functions which take the indices as arguments:

One can think of a sequence $(x_i)_{i \geq n_\star}$ in terms of the assignment $i \mapsto x_i$, and the sequence can then be interpreted as the function

$$x(\cdot) : [n_\star, \infty[_{\mathbb{Z}} \longrightarrow \text{suitable codomain}; \quad i \mapsto x(i) := x_i\,,$$

where that "suitable codomain" depends on the nature of the items x_i. In other words, **Sequences are functions with domain = index set** $= [n_\star, \infty[_{\mathbb{Z}}$.

In Example 2.10**(a)**, we could choose $\mathbb{Z}$ as codomain. We could also choose either of $[0,\infty]_{\mathbb{Z}}$, $\mathbb{Q}$, $\mathbb{R}$, since each of those sets contains all "function" values u_k that belong to the sequence. On the other hand, the set $]0,\infty[$ of all strictly positive real numbers does not qualify since it does not contain the sequence member $u_0 = 0$.

In Example 2.10**(b)**, we could choose $2^{\mathbb{R}}$, the power set of $\mathbb{R}$, as codomain.

In Example 2.10**(c)**, any set that contains the set $\{-1, 1\}$ is a suitable codomain. Observe that the sequence $(z_j)_{j=0}^{\infty}$ is an infinite collection of tagged items, one for each index $j \in [0,\infty]_{\mathbb{Z}}$. However, the set $\{z_j : j \in [0,\infty]_{\mathbb{Z}}\}$ of all values this sequence can attain, only contains two values. We have

$$\{z_j : j \in [0,\infty]_{\mathbb{Z}}\} \;=\; \{-1, 1\}\,,$$

since duplicate members of a set are ignored. ☐

Definition 2.25.

We occasionally admit an "ending index" $n^\star$ instead of ∞, i.e., there will be an indexed item x_j, for each $j \in [n_\star, n^\star]_{\mathbb{Z}}$. We then talk of a **finite sequence**, and we write

$$(x_n)_{n_\star \leq n \leq n^\star} \quad \text{or} \quad (x_j)_{j=n_\star}^{n^\star} \quad \text{or} \quad x_{n_\star}, x_{n_\star+1}, \dots, x_{n^\star}$$

for such a finite collection of items. If we refer to a sequence $(x_n)_n$ without qualifying it as finite then we imply that we deal with an **infinite sequence**, $(x_n)_{n=n_\star}^{\infty}$.

If one pares down the full set of indices $\{n_\star,\ n_\star+1,\ n_\star+2, \dots\}$ to a subset

$$\{n_1,\ n_2,\ n_3, \dots\} \qquad \text{such that} \qquad n_\star \;\leq\; n_1 \;<\; n_2 \;< n_3 \;<\; \dots$$

then we call the corresponding "thinned out" sequence $(x_{n_j})_{j \in \mathbb{N}}$ a **subsequence** $(x_n)_{n \geq n_\star}$.

If this subset of indices is finite, i.e., we have

$$n_\star \;\leq\; n_1 \;<\; n_2 \;<\; \dots \;<\; n_K \qquad \text{for some suitable } K \in \mathbb{N}\,,$$

then we call $(x_{n_j})_{j=1}^{K}$ a **finite subsequence** of the original sequence. ☐

Remark 2.15.

Keep the sequence $\big((-1)^j\big)_{j=0}^{\infty}$ in mind when considering the following which we only state for infinite sequences, but which also applies to subsequences and finite sequences.

Do not confuse a sequence $(x_n)_{n \geq n_*}$ with the set $\{x_n : n \geq n_*\}$ of its values!
The sequence is a function $n \mapsto x_n$ with domain $[n_*, \infty]_{\mathbb{Z}}$, the set $\{x_n : n \geq n_*\}$ merely is the (smallest possible) codomain of that function.

The sequence $(x_n)_{n \geq n_*}$ always determines the set $\{x_n : n \geq n_*\}$, but the opposite is not true. For example, if you know that the values belonging to the sequence $(x_n)_{n \geq 0}$ constitute the set $\{-1, 1\}$ then you do not know whether

$$x_n = (-1)^n \quad \text{or} \quad x_n = (-1)^{n+1} \quad \text{or} \quad x_n = \begin{cases} 1 & \text{if } n \in [0, 100[_{\mathbb{Z}}, \\ -1 & \text{if } n \in [100, \infty[_{\mathbb{Z}}, \end{cases} \quad \text{or} \quad x_n = \dots \quad \square$$

The members x_k of a sequence $(x_k)_k$ are indexed items in the following sense.

Definition 2.26 (Indexed items)**.**

Given is an expression of the form

$$a_i \, .$$

We say that a_i is **indexed by** or **subscripted by** or **tagged by** i. We call i the **index** or **subsript** of a_i, and we call a_i an **indexed item** . $\square$

Remark 2.16.
Both a_i and i can occur in many different ways. Here is a collection of indexed items:

$$x_7, \ A_\alpha, \ k_T, \ \mathfrak{H}_{2/9}, \ f_x, \ x_t, h_{\mathscr{A}}, \ , i_{\mathbb{R}}, \ H_{2\pi}$$

Some of the indices in this collection are highly unusual. Not only are some of them negative, but they are fractions (e.g., 2/9) or irrational (e.g., 2π). Others don't even look like numbers (e.g., α, T, x, t, $\mathscr{A}$ and $\mathbb{R}$). It is not clear from the information available to us whether those indices are names of variables which represent numbers or whether they represent functions, sets or other mathematical objects. There is one exception: It should be safe to assume that the index $\mathbb{R}$ of $i_{\mathbb{R}}$ denotes the set of all real numbers, since it is hard to imagine that a mathematician would attach a different meaning to that symbol. $\square$

We can turn any set into a "family" by tagging each of its members with an index. As an example, look at the following two indexed versions of the set S_2 from example 2.2 on p. 20:

$$\begin{aligned} F &= (a_1, e_1, e_2, i_1, i_2, i_3, o_1, o_2, o_3, o_4, u_3, u_5, u_9, u_{11}, u_{99}) \\ G &= (a_k, e_{-\sqrt{2}}, e_1, i_{-6}, i_{\mathscr{B}}, i_{\mathbb{R}}, o_7, o_{2/3}, o_{-8}, o_3, u_A, u_B, u_C, u_D, u_E) \end{aligned}$$

We note several things:

(a) F has the kind of indices that we are familiar with: all of them are positive integers.
(b) Some of the indices in F occur multiple times. For example, 3 occurs as an index for i_3, o_3, u_3.
(c) All of the indices in G are unique.
(d) As in remark 2.16, some of the indices are very unusual.

The last point is not much of a problem as mathematicians are used to very unusual notation but point **(b)**, the non-uniqueness of indices, is something that we want to avoid. From now on we ask for the following: The indices of an indexed collection must belong to some set J and each index $i \in J$ must be used exactly once. Remember that this automatically takes care of the duplicate indices problem as a set never contains duplicate values (see Definition 2.1 on p. 20). We also demand that there is a set X such that each indexed item x_i belongs to X.

Those considerations leads us to the definition of a family.

Definition 2.27 (Indexed families)**.**

Let J and X be nonempty sets such that

each $i \in J$ is associated with exactly one indexed item $x_i \in X$.

We write $(x_i)_{i \in J}$ for this collection of indexed items and call it an **indexed family** or **family in** X with **index set** J. The indexed items x_j are called the **members of the family**. ☐

Remark 2.17.

Sequences are families with sets of integers as index sets:
(a) Sequences $(x_j)_{j=n_\star}^{\infty}$ are families with index set $[n_\star, \infty[_{\mathbb{Z}}$.
(b) Finite sequences $(x_j)_{j=n_\star}^{n^\star}$ are families with index set $[n_\star, n^\star]_{\mathbb{Z}}$.
(c) Subsequences $\left(x_{n_j}\right)_{j \in \mathbb{N}}$ are families with index set $\{n_1 < n_2 < \dots\}$; $n_j \in \mathbb{Z}$.
(d) Finite subsequences $(x_j)_{j=1}^{K}$ are families with index set $\{n_1 < \dots < n_K\}$; $n_j \in \mathbb{Z}$. ☐

Example 2.11.

Here are some examples of families.

(a) For $r \in \mathbb{R}$, let $B_r := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq r^2\}$. Then $(B_r)_{r \in \mathbb{R}}$ is a family with index set $\mathbb{R}$ and values in $2^{\mathbb{R}}$. (The indexed items are subsets of $\mathbb{R}$!)

Let $\mathscr{B} := \{B_r : r \in \mathbb{R}\}$ be the set of all tagged items B_r of the above family (thus $\mathscr{B}$ is a set of sets). Do not confuse $\mathscr{B}$ and $(B_r)_{r \in \mathbb{R}}$! The family distinguishes, e.g., between the indexed items B_2 and B_{-2} even though they represent the same set, but $\mathscr{B}$ does not, since sets to not contain any duplicate elements!

We have already seen this behavior for sequences, e.g., in Example 2.10**(c)** on p.42, where we looked at the sequence $\left((-1)^j\right)_{j=0}^{\infty}$. This sequence has infinitely many members since there are infinitely many indices $j = 0, 1, 2, \dots$, but its value set, $\{(-1)^j : j \in [0, \infty[_{\mathbb{Z}}\} = \{-1, 1\}$, only possesses two members.

(b) We take the family from **(a)**, but we shrink the index set to $]-\infty, 0]$, i.e., we consider the family $(B_r)_{r\le 0}$. Note that $B_0 = \{0\} and \ B_r = \emptyset$ whenever $r < 0$. Hence, $\{B_r : r \le 0\} = \{\emptyset, \{0\}\}$.

(c) Let the function $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \sin(x)$, and let $y_x := \sin(x)$ for any real number x. Besides the notation, Is there any real difference between the function h and the family $(y_x)_{x\in\mathbb{R}}$ with values in $\mathbb{R}$? Not really. Both objects do the same; they assign to each $x \in \mathbb{R}$ the real number $\sin(x)$.

(d) This last example generalizes to any function $f : X \to Y$ with arbitrary, nonempty sets X and Y. We can associate with f the function $\big(f(x)\big)_{x\in X}$.

(e) Let $I := [0, 10]$ and $(x_i)_{i\in I}$ the $\mathbb{R}$–valued family defined by $x_i := e^i$. The function

$$\varphi : I \to \mathbb{R}; \qquad i \mapsto e^i$$

is equivalent to that family: both describe the assignment of $i \in I$ to the real number e^i.

(f) This last example generalizes to any X–valued family $(x_i)_{i\in I}$ with arbitrary, nonempty index set I and value set X. We can associate with $(x_i)_{i\in I}$ the function

$$\psi : I \to X; \qquad i \mapsto x_i,$$

and both objects convey the same information. □

Remark 2.18.

(a) Examples 2.11**(c)** through 2.11**(f)** and in particular, examples 2.11**(d)** and 2.11**(f)**, illustrate that families are functions just as sequences are functions: We mentioned in Remark 2.14 on p.42 that a sequence $(x_n)_{n=n_\star}^\infty$ with a suitable codomain X, i.e., $x_n \in X$ for all $n \in [n_\star, \infty[_\mathbb{Z}$, can be interpreted as a function with domain $[n_\star, \infty[_\mathbb{Z}$ and codomain X. Likewise:

A family $(x_i)_{i\in J}$ can be interpreted as the function

$$x(\cdot) : J \longrightarrow X; \quad i \mapsto x(i) := x_i.$$

Families in X are functions with domain = index set = J and codomain X .

(b) Same as for sequences, i is a dummy variable: $(x_i)_{i\in J}$ and $(x_k)_{k\in J}$ describe the same family as long as $i \mapsto x_i$ and $k \mapsto x_k$ describe the same function $x(\cdot) : J \to X$. This should not come as a surprise to you if you recall Remark 2.12 on p.40 concerning function arguments and the end of Definition 2.24 on p.42 (sequences).

(c) Do not confuse the family $(x_i)_{i\in J}$ with the set $\{x_i : i \in J\}$ of its function values. We have illustrated this in examples 2.11**(a)** and 2.11**(b)**. □

Example 2.12.

For $1 \le x \le 10$ let $A_x := [-x, 5x]$. Since $A_x \subseteq \mathbb{R}$, $(A_x)_{x\in[1,10]}$ is a family in $2^\mathbb{R}$, the power set of $\mathbb{R}$, with index set $[1, 10]$. If we define $B_z := [-z, 5z]$ and $\mathscr{A}_\alpha := [-\alpha, 5\alpha]$, then both families $(B_z)_{z\in[1,10]}$ and $(\mathscr{A}_\alpha)_{\alpha\in[1,10]}$ are identical to the family $(A_x)_{x\in[1,10]}$ (!) □

This concludes our first look at families. We will have more to say about this topic in Chapter **??** (Families, Sequences, and Functions as Families).

2.5 Cartesian Products

We next define cartesian products of sets. [7] Those mathematical objects generalize rectangles

$$[a_1, b_1] \times [a_2, b_2] \;=\; \{(x,y) : x, y \in \mathbb{R}, a_1 \leq x \leq b_1 \text{ and } a_2 \leq y \leq b_2\}$$

and quads

$$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \;=\; \{(x,y,z) : x, y, z \in \mathbb{R}, a_1 \leq x \leq b_1, a_2 \leq y \leq b_2 \text{ and } a_3 \leq z \leq b_3\}.$$

Definition 2.28 (Preliminary definition: Cartesian Product)**.**

Let X and Y be two sets The set

$$X \times Y \;:= \{(x,y) : x \in X, y \in Y\} \tag{2.31}$$

is called the **cartesian product** of X and Y.
Note that the order is important: (x,y) and (y,x) are different unless $x = y$.
We write X^2 as an abbreviation for$X \times X$.
This definition generalizes to more than two sets as follows: Let $X_1, X_2, \dots, X_n$ be sets. The set

$$X_1 \times X_2 \cdots \times X_n \;:= \{(x_1, x_2, \dots, x_n) : x_j \in X_j \text{ for each } j = 1, 2, \dots n\} \tag{2.32}$$

is called the cartesian product of $X_1, X_2, \dots, X_n$.
We write X^n as an abbreviation for$X \times X \times \cdots \times X$. ☐

Example 2.13.
The graph Γ_f of a function with domain X and codomain Y (see Definition 2.28) is a subset of the cartesian product $X \times Y$. ☐

Example 2.14.
The domains given in **(a)** and **(d)** of the introduction to ch.2.4 (A First Look at Functions, Sequences and Families) are subsets of the cartesian product

$$\mathbb{R}^2 \;=\; \mathbb{R} \times \mathbb{R} \;=\; \{(x,y) : x, y \in \mathbb{R}\} \;☐$$

.

[7] See ch.**??** (Cartesian Products and Relations) on p.**??** for the real thing and examples.

2.6 Arbitrary Unions and Intersections

In Definition 2.5 on p.24 we had defined unions and intersections of finitely many sets $A_1, A_2, \dots, A_n$ as follows:

$$\bigcup_{j=1}^{n} A_j = \{x : x \in A_j \text{ for at least one } j = 1, 2, \dots, n\},$$
$$\bigcap_{j=1}^{n} A_j = \{x : x \in A_j \text{ for each } j = 1, 2, \dots, n\}.$$

Thus the union of those sets are those items that belong to at least one of those sets, and the intersection of those sets are those items that belong to each one of those sets. This can be generalized to any set of sets [8] or family of sets, finite or not.

Definition 2.29 (Arbitrary unions and intersections)**.**

(A) For a (nonempty) set of sets $\mathscr{A}$, let

$$\bigcup_{B \in \mathscr{A}} B := \bigcup \big[B : B \in \mathscr{A}\big] := \{x : x \in B \text{ for at least one } B \in \mathscr{A}\}, \tag{2.33}$$
$$\bigcap_{B \in \mathscr{A}} B := \bigcap \big[B : B \in \mathscr{A}\big] := \{x : x \in B \text{ for each } B \in \mathscr{A}\}. \tag{2.34}$$

We call $\bigcup\limits_{B \in \mathscr{A}} B$ the **union** and $\bigcap\limits_{B \in \mathscr{A}} B$ the **intersection** of the members of $\mathscr{A}$

(B) For a family $(A_i)_{i \in I}$ of sets A_i, let

$$\bigcup_{i \in I} A_i := \bigcup \big[A_i : i \in I\big] := \{x : x \in A_i \text{ for at least one } i \in I\}, \tag{2.35}$$
$$\bigcap_{i \in I} A_i := \bigcap \big[A_i : i \in I\big] := \{x : x \in A_i \text{ for each } i \in I\}. \tag{2.36}$$

We call $\bigcup\limits_{i \in I} A_i$ the **union** and $\bigcap\limits_{i \in I} A_i$ the **intersection** of the family $(A_i)_{i \in I}$.

Note that use of the "set style" notation $\bigcup \big[B : B \in \mathscr{A}\big], \bigcup \big[A_i : i \in I\big], \dots$ is less common than that of $\bigcup_{B \in \mathscr{A}} B, \bigcup_{i \in I} A_i, \dots$. We find it advantageous if $\mathscr{A}$ or I consists of a rather lengthy expression.

[8] Recall that we encountered, in Example 2.3 on p.21 for example, sets whose elements are sets.

(C) Let $\mathscr{A}$ be a nonempty set of sets, let $(A_i)_{i\in I}$ be a family of sets.

We call the members of $\mathscr{A}$ **disjoint** , also **mutually disjoint**, if $A, A' \in \mathscr{A}$ and $A \neq A'$ implies $A \cap A' = \emptyset$. We call the family $(A_i)_{i\in I}$ **disjoint** , also **mutually disjoint**, if $A_i \cap A_j = \emptyset$ for all $i, j \in J$ such that $i \neq j$.

As done previously, we allow the use of $\biguplus$ instead of $\bigcup$ to indicate disjoint unions:

$$\biguplus_{B\in\mathscr{A}} B := \bigcup_{B\in\mathscr{A}} B\,, \qquad \biguplus_{i\in I} A_i := \bigcup_{i\in I} A_i\,. \tag{2.37}$$

Note that disjointness of sets was already defined in Definition 2.6 on p.24, but only for a finite collection of sets.

(D) Assume that there is Ω, $\mathscr{A}$ such that $\mathscr{A} \subseteq \Omega$ and the members of $\mathscr{A}$ are disjoint.

If $\Omega = \biguplus_{B\in\mathscr{A}} B$, then we call $\mathscr{A}$ a **partition** of Ω.

Assume that there is Ω, $(A_i)_{i\in I}$ such that $A_j \subseteq \Omega$ for all $j \in J$ is a disjoint family.

If $\Omega = \biguplus_{i\in I} A_i$, then we call $(A_i)_{i\in I}$ a **partition** of Ω.

Note that being a partition means that each $x \in \Omega$ belongs to exactly one member of $\mathscr{A}$ (of $(A_i)_{i\in I}$ in case of a family).

Since sequences are special kinds of families with index sets

$$[n_*, \infty[_{\mathbb{Z}} = \{\, n_*,\, n_*+1,\, n_*+2,\, \dots \,\}\,,$$

it is natural to write

$$\bigcup_{i=n_*}^{\infty} A_i := \bigcup_{i\in[n_*,\infty[_{\mathbb{Z}}} A_i\,, \qquad \bigcap_{i=n_*}^{\infty} A_i := \bigcap_{i\in[n_*,\infty[_{\mathbb{Z}}} A_i\,, \quad \square \tag{2.38}$$

Note that any statement concerning arbitrary families of sets such as the definition above covers finite lists $A_1, A_2, \dots, A_n$ of sets ($J = \{1, 2, \dots, n\}$) and also sequences $A_1, A_2, \dots,$ of sets ($J = \mathbb{N}$).

Remark 2.19. "At least one" can also be expressed as "some" or "there exists", and "for each" can also be expressed as "for all". Thus one also writes

$$\bigcup_{B\in\mathscr{A}} B = \{x : x \in B \text{ for some } B \in \mathscr{A}\} = \{x : \text{ there exists } B \in \mathscr{A} \text{ such that } x \in B\},$$
$$\bigcap_{B\in\mathscr{A}} B = \bigcap\big[B : B \in \mathscr{A}\big] := \{x : x \in B \text{ for all } B \in \mathscr{A}\}.$$
$$\bigcup_{i\in I} A_i = \{x : x \in A_i \text{ for some } i \in I\}, \{x : \text{ there exists } i \in I \text{ such that } x \in A_i\},$$
$$\bigcap_{i\in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}. \ \square$$

Example 2.15.

In Example 2.3 on p.21 we considered the sets

(a) $\mathscr{A} := \{\,]a,b[: a,b \in \mathbb{R}, 0 < b-a < 2\}$ (all open intervals of length less than 2),

(b) $\mathscr{B} := \{K$:K is a set of integers $\}$ (the power set of $\mathbb{Z}$).

Since each real number x belongs to the set $]x-\frac{1}{2}, x+\frac{1}{2}[$ which is an element of $\mathscr{A}$, it follows that $x \in \bigcup_{B\in\mathscr{A}} B$. Thus $\bigcup\big[B : B \in \mathscr{A}\big] = \mathbb{R}$, the set of all real numbers.

No real number x is an element of $]x+5, x+6[$. Since this interval belongs to $\mathscr{A}$, it is not true that $x \in B$ for each $B \in \mathscr{A}$. It follows that $x \notin \bigcap\big[B : B \in \mathscr{A}\big]$. Thus no real number belongs to $\bigcap_{B\in\mathscr{A}} B$; we conclude that $\bigcap_{B\in\mathscr{A}} B = \emptyset$.

Things are just that simple for $\mathscr{B}$. Every integer m is an element of the set $\mathbb{Z}$ of all integers, which in turn is an element of $\mathscr{B}$ It follows that $m \in \bigcup\big[K : K \in \mathscr{B}\big]$. Thus this union equals $\mathbb{Z}$.

To compute the intersection of the members of $\mathscr{B}$ we note that no integer m belongs to the set $\{m\}$ which is an element of $\mathscr{B}$ since it is a set of integers. Thus it is not true that $m \in K$ for all $K \in \mathscr{B}$, thus $x \notin \bigcap\big[K : K \in \mathscr{B}\big]$. Since this is true for all integers m, it follows that $\bigcap\big[K : K \in \mathscr{B}\big] = \emptyset$. $\square$

Example 2.16.

Here are two more examples for unions and intersections of sets of sets. The proofs are not as easy as those in the previous example. To understand them you need to be familiar with the properties of limits of real numbers on a beginner's calculus level. [9] Let

$$\mathscr{C} := \{\,\big[\,\pi - 3 + \tfrac{1}{n}, \pi + 3 - \tfrac{1}{n}\,\big] : n = 1,2,3,\dots\},$$
$$\mathscr{D} := \{\,\big]\,\pi - 3 - \tfrac{1}{n}, \pi + 3 + \tfrac{1}{n}\,\big[: n = 1,2,3,\dots\}.$$

We claim that $\bigcup_{A\in\mathscr{C}} A = \,]\pi-3, \pi+3[$ and $\bigcap_{A\in\mathscr{D}} A = [\,\pi-3, \pi+3\,]$.

To see this we first observe the following.

[9] Chapter **??** (Convergence and Continuity in $\mathbb{R}$) will teach you convergence in a mathematically precise way.

(1) The sequence $a_n = (\pi - 3) - \frac{1}{n}$ converges from the left to $\pi - 3$, thus a_n is arbitrarily close to $\pi - 3$ and,

if $x < \pi - 3$, then $x < a_n < \pi - 3$ is true for all sufficiently large n.

(2) The sequence $b_n = (\pi - 3) + \frac{1}{n}$ converges from the right to $\pi - 3$, thus b_n is arbitrarily close to $\pi - 3$ and,

if $x > \pi - 3$, then $\pi - 3 < b_n < x$ is true for all sufficiently large n.

Likewise, we obtain for $c_n = (\pi + 3) - \frac{1}{n}$ and $d_n = (\pi + 3) + \frac{1}{n}$ that

(3) if $x < \pi + 3$, then $x < c_n < \pi + 3$ is true for all sufficiently large n.

(4) if $x > \pi + 3$, then $\pi + 3 < d_n < x$ is true for all sufficiently large n.

Let us choose some more convenient notation. We define

$$C_n := [\,b_n, c_n\,] = \left[\pi - 3 + \frac{1}{n}, \pi + 3 - \frac{1}{n}\right], \quad C := \bigcup_{A \in \mathscr{C}} A\,,$$

$$D_n := \,]a_n, d_n[\, = \left]\pi - 3 - \frac{1}{n}, \pi + 3 + \frac{1}{n}\right[, \quad D := \bigcap_{A \in \mathscr{D}} A\,.$$

Then

$$\mathscr{C} = \{\, C_n \,:\, n \in \mathbb{N} \}, \text{ thus, } C = \bigcup_{n \in \mathbb{N}} C_n\,, \text{ and we must show } C = \,]\pi - 3, \pi + 3[\,;$$

$$\mathscr{D} = \{\, D_n \,:\, n \in \mathbb{N} \}, \text{ thus, } D = \bigcap_{n \in \mathbb{N}} D_n\,, \text{ and we must show } D = [\,\pi - 3, \pi + 3\,]\,.$$

Note that we rewrote C as a union and D as an intersection of a sequence of sets.

(I): We now show that $\bigcup [A : A \in \mathscr{C}] = \,]\pi - 3, \pi + 3[\,$.

(I.a) Prove that $C \subseteq]\pi - 3, \pi + 3[$.

Let $x \in C$. We must show that $x \in]\pi - 3, \pi + 3[$. it follows from (2.33) that x belongs to some element of $\mathscr{C}$, i.e., there must be some $n \in \mathbb{N}$ such that $x \in C_n$.

From $C_n = \left[\,\pi - 3 + \frac{1}{n}, \pi + 3 - \frac{1}{n}\right]$ we obtain $C_n \subseteq]\pi - 3, \pi + 3[$, hence, $x \in]\pi - 3, \pi + 3[$. We have shown $x \in C \Rightarrow x \in]\pi - 3, \pi + 3[$, and this proves **(I.a)**.

(I.b) Prove that $]\pi - 3, \pi + 3[\subseteq C$.

Let $x \in]\pi - 3, \pi + 3[$. We must show that $x \in C$. Since $\pi - 3 < x < \pi + 3$, it follows from **(2)** and **(3)** above that $\pi - 3 < b_n < x < c_n < \pi - 3$ for all sufficiently large n.

$$\text{Thus, } b_n < x < c_n\,, \text{ thus, } x \in [\,b_n, c_n], \text{ i.e., } x \in C_n$$

is true for all sufficiently large n. Thus there exists an index $n_0 \in \mathbb{N}$ such that $x \in C_{n_0}$. [10] It follows from (2.33) that $x \in C$, and this proves **(I.b)**.

(I.a) and **(I.b)** together yield $\bigcup [A : A \in \mathscr{C}] = \,]\pi - 3, \pi + 3[\,$. We have proved **(I)**.

(II): Here is the proof that $\bigcap [A : A \in \mathscr{D}] = [\,\pi - 3, \pi + 3\,]$.

(II.a) Prove that $[\pi - 3, \pi + 3] \subseteq D$.

Let $x \in [\pi - 3, \pi + 3]$. According to (2.34) we must prove that $x \in A$ for all $A \in \mathscr{D}$, i.e., $x \in]a_n, d_n[$ for all $n \in \mathbb{N}$. This is obviously true, since $[\pi - 3, \pi + 3] \subseteq \,]\pi - 3 - 1/n, \pi + 3 + 1/n[$ and $a_n = \pi - 3 - 1/n$ and $d_n = \pi + 3 + 1/n$.

[10] Of course there are infinitely many such indices, but that is not important for this proof.

(II.b) Prove that $D \subseteq [\pi - 3, \pi + 3]$.

Since $D = \bigcap_{n \in \mathbb{N}}]a_n, d_n[$, we must show, according to (2.34), the following.

(2.39) $$\text{If } x \in]a_n, d_n[\text{ for all } n, \quad \text{then } x \in [\pi - 3, \pi + 3]\,.$$

This is a logical statement of the form

(2.40) $$\text{if } P \text{ is true, then } Q \text{ is true,} \qquad \text{in short, If } P, \text{ then } Q,$$

where P is the **assumption** "$x \in]a_n, d_n[$ for all n", and Q is the **conclusion** "$x \in [\pi - 3, \pi + 3]$". This if ... then statement can also be expressed by its **contrapositive**

(2.41) $$\text{if } Q \text{ is not true, then } P \text{ is not true,} \qquad \text{in short, If not } Q, \text{ then not } P.$$

We claim that both (2.40) and (2.41) are equivalent logical statements in the following sense:

The validity of "if P, then Q" implies that of "if not Q, then not P", and vice versa.

This can be seen as follows.

- Assume that "if P, then Q" is valid.
- Since the truth of P implies the truth of Q, we cannot have both P true and Q false.
- Thus the falseness of Q implies the falseness of P,
- i.e., if Q is not true then P is not true.

Here is the reverse direction: The validity of (2.41) implies that of (2.40).

- Assume that "if Q is not true then P is not true" is valid.
- Since the falseness of Q implies the falseness of P, we cannot have both Q false and P not false.
- Thus the non-falseness of P implies the non-falseness of Q.
- In other words, the truth of P implies the truth of Q, i.e., if P, then Q.

Thus, to prove that $D \subseteq [\pi - 3, \pi + 3]$, we can replace (2.39) by its contrapositive

$$\text{If } x \notin [\pi - 3, \pi + 3] \quad \text{then it is not true that } x \in]a_n, d_n[\text{ for all } n.$$

What does it mean that it is not true that $x \in]a_n, d_n[$ for all n? It means that there must exist an index n (at least one) such that $x \notin]a_n, d_n[$. Thus it suffices to prove

(2.42) $$\text{If } x \notin [\pi - 3, \pi + 3] \quad \text{then there is an index } k \in \mathbb{N} \text{ such that } x \notin]a_k, d_k[\,.$$

So let $x \notin [\pi - 3, \pi + 3]$. Then either $x < \pi - 3$ or $x > \pi + 3$.

First case, $x < \pi - 3$: We have seen in **(1)** [11] that then $x < a_n$ and thus $x \leq a_n$ is true for all sufficiently large n. It follows that $x \notin]a_n, d_n[$ for all such n and, hence, for at least one n.

Second case, $x > \pi + 3$: We have seen in **(4)** that then $x > d_n$ and thus $x \geq d_n$ is true for all sufficiently large n, thus $x \notin]a_n, d_n[$ for all such n and, hence, for at least one n.

In summary, we have shown the validity of (2.42), hence, of (2.42), hence, of $D \subseteq [\pi - 3, \pi + 3]$.

This concludes the proof of **(II.b)** and, thus, **(II)**.

We have learned a few things about logic and proofs which we want to summarize below.

[11] near the beginning of the example

- The statement "if P, then Q" is equivalent to its contrapositive: "if not Q, then not P".
- The method of proving "if P, then Q" by proving the contrapositive is called an **indirect proof by contrapositive**.

Moreover, we used that the opposite of "it" being true for all items is that it is false for at least one item, i.e., that there exists an item for which "it" is false. A moment's reflection tells us the following: The opposite of "it" being true for at least one item, i.e., the opposite of the existence of an item for which "it" is true, is that "it" is false for all items.

We summarize that as follows.

Let P be some property which can be true or false

- If A is the statement "P is true for all x", then not A is the statement "there exists some x for which P is false".
- If B is the statement "there is some x for which P is true", then not B is the statement "P is false for all x". □

2.7 Proofs by Induction and Definitions by Recursion

Introduction 2.5. The integers have a property which makes them fundamentally different from the rational numbers (fractions) and the real numbers: Given any two integers $m < n$, there are only finitely many integers between m and n. To be precise, there are exactly $n - m - 1$ of them. For example, there are only 4 integers between 12 and 17: the numbers $13, 14, 15, 16$. [12]

Therefore, given an integer n, we have the concept of its predecessor, $n - 1$, and its successor, $n + 1$. This has some profound consequences. If we know what to do for a certain starting number $k_0 \in \mathbb{Z}$ (we call this number the base case), and if we can figure out for each integer $k \geq k_0$ what to do for $k + 1$ if only we know what to do for k, then we know what to do for **any** $k \geq k_0$! □

We make use of the above when defining a sequence by **recursion**. Here is a simple example.

Example 2.17.
Let $k_0 = -2$, $x_{k_0} = 5$ (base case), and $x_{k+1} = x_k + 3$ (i.e., we know how to obtain x_{k+1} just from the knowledge of x_k), then we know how to build the entire sequence

$$x_{-2} = 5,\ x_{-1} = x_{-2} + 3 = 8,\ x_0 = x_{-1} + 3 = 11,\ x_1 = x_0 + 3 = 14,\ \dots,$$

The equation $x_{k+1} = x_k + 3$ which tells us how to obtain the next item from the given one is the **recurrence relation** for that recursive definition. □

[12] All of this will be made mathematically precise in ch.**??** on p.**??**.

Example 2.18.

Given is a sequence of sets $A_1, A_2, \dots$. For $n \in \mathbb{N}$ we define $\bigcup\limits_{j=1}^{n} A_j$ and $\bigcap\limits_{j=1}^{n} A_j$ recursively as follows. [13]

$$\bigcup_{j=1}^{1} A_j := A_1, \qquad \bigcup_{j=1}^{n+1} A_j := \Big(\bigcup_{j=1}^{n} A_j\Big) \cup A_{n+1}, \tag{2.43}$$

$$\bigcap_{j=1}^{1} A_j := A_1, \qquad \bigcap_{j=1}^{n+1} A_j := \Big(\bigcap_{j=1}^{n} A_j\Big) \cap A_{n+1}. \tag{2.44}$$

this tells us the meaning of $\bigcup\limits_{j=1}^{n} A_j$ and $\bigcap\limits_{j=1}^{n} A_j$ for any natural number n. For example, $\bigcap\limits_{j=1}^{4} A_j$ is computed as follows.

$$\begin{aligned}
\bigcap_{j=1}^{1} A_j &= A_1,\\
\bigcap_{j=1}^{2} A_j &= \Big(\bigcap_{j=1}^{1} A_j\Big) \cap A_2 = A_1 \cap A_2,\\
\bigcap_{j=1}^{3} A_j &= \Big(\bigcap_{j=1}^{2} A_j\Big) \cap A_3 = (A_1 \cap A_2) \cap A_3,\\
\bigcap_{j=1}^{4} A_j &= \Big(\bigcap_{j=1}^{3} A_j\Big) \cap A_4 = \big((A_1 \cap A_2) \cap A_3\big) \cap A_4. \ \square
\end{aligned}$$

Remark 2.20.

The discrete structure of the integers can also be used as a means to prove a collection of mathematical statements $P(k_0), P(k_0+1), P(k_0+2), \dots$ which is defined for all integers k, starting at $k_0 \in \mathbb{Z}$. Each $P(k)$ might be an equation or an inequality for two numeric expressions that depend on k. It could also be a relation between sets or it could be something entirely different. For example, $P(k)$ could be the statement $\Big(\bigcup\limits_{j=1}^{k} A_j\Big) \cap B = \bigcup\limits_{j=1}^{k} (A_j \cap B)$. An extremely important tool for proofs of this kind is the following principle. Its mathematical justification will be given later in thm.**??** on p.**??**.

Principle of Mathematical Induction

[13] An "official" definition for unions and intersections of arbitrarily many sets (not just for finitely many) will be given in Definition 2.29 on p.48.

Assume that for each integer $k \geq k_0$ there is an associated statement $P(k)$ such that the following is valid:

A. Base case. The statement $P(k_0)$ is true.

B. Induction Step. Assuming that $P(k)$ is true ("**Induction Assumption**"), it can be shown that $P(k+1)$ also is true.

It then follows that $P(k)$ is true for **each** $k \geq k_0$.

Here is an example which generalizes prop.2.2 on p.27.

Proposition 2.6 (Distributivity of unions and intersections for finitely many sets)**.**

Let $A_1, A_2, \dots$ *and* B *be sets. If* $n \in \mathbb{N}$ *then*

$$\left(\bigcup_{j=1}^{n} A_j\right) \cap B = \bigcup_{j=1}^{n} (A_j \cap B), \tag{2.45}$$

$$\left(\bigcap_{j=1}^{n} A_j\right) \cup B = \bigcap_{j=1}^{n} (A_j \cup B). \tag{2.46}$$

PROOF: We only prove (2.45), and this will be done by induction on n, i.e., the number of sets A_j. The proof of (2.46) is left as exercise 2.11

(A) Base case: $k_0 = 1$. The statement $P(1)$ is (2.45) for $n = 1$: $\left(\bigcup_{j=1}^{1} A_j\right) \cap B = \bigcup_{j=1}^{1} (A_j \cap B)$. We must prove that $P(1)$ is true. According to our recursive definition of finite unions which was given in example 2.17 this is the same as $(A_1) \cap B = (A_1 \cap B)$, and this is a true statement. We have proven the base case.

(B) Induction step:

$$\textbf{Induction assumption: } P(k) : \left(\bigcup_{j=1}^{k} A_j\right) \cap B = \bigcup_{j=1}^{k} (A_j \cap B) \text{ is true for some } k \geq 1. \tag{2.47}$$

Under this assumption

$$\text{we must prove the truth of } P(k+1) : \left(\bigcup_{j=1}^{k+1} A_j\right) \cap B = \bigcup_{j=1}^{k+1} (A_j \cap B). \tag{2.48}$$

The trick is to manipulate $P(k+1)$ in a way that allows us to "plug in" the induction assumption. For (2.48) one way to do this is to take the left–hand side and transform it repeatedly until we end up with the right–hand side, and doing so in such a manner that (2.47) will be used at some point.

$$\begin{aligned}
\Big(\bigcup_{j=1}^{k+1} A_j\Big) \cap B &= \Big(\Big(\bigcup_{j=1}^{k} A_j\Big) \cup A_{k+1}\Big) \cap B && \text{we used (2.43)}\\
&= \Big(\Big(\bigcup_{j=1}^{k} A_j\Big) \cap B\Big) \cup (A_{k+1} \cap B) && \text{we used (2.13) on p. 27}\\
&= \bigcup_{j=1}^{k} (A_j \cap B) \cup (A_{k+1} \cap B) && \textbf{we used the induction assumption!}\\
&= \bigcup_{j=1}^{k+1} (A_j \cap B) && \text{we used (2.43)}
\end{aligned}$$

We have managed to establish the truth of $P(k+1)$, and this concludes the proof.

Epilogue: It is crucial that you understand in what way the induction assumption was used to get from the left–hand side of (2.48) to the right–hand side, and that you first had to find a base from which to proceed by proving the base case. ■

Proposition 2.7 (The Triangle Inequality for n real numbers)**.**

Let $n \in \mathbb{N}$ such that $n \geq 2$. Let $a_1, a_2, \dots, a_n \in \mathbb{N}$. Then

$$|a_1 + a_2 + \dots + a_n| \;\leq\; |a_1| + |a_2| + \dots + |a_n| \tag{2.49}$$

PROOF: Note that this proposition generalizes prop.2.5 on p.37 from 2 terms to n terms. The proof will be done by induction on n, the number of terms in the sum.

(A) Base case: For $k_0 = 2$, inequality 2.49 was already shown (see (2.26) on p.37).

(B) Induction step: Let us assume that 2.49 is true for some $k \geq 2$. This is our induction assumption. We now must prove the inequality for $k+1$ terms $a_1, a_2, \dots, a_k, a_{k+1} \in \mathbb{N}$. We abbreviate

$$A \;:=\; a_1 + a_2 + \dots + a_k; \qquad B \;:=\; |a_1| + |a_2| + \dots + |a_k|$$

then our induction assumption for k numbers is that $|A| \leq B$. We know from (2.26) that the triangle inequality is valid for the two terms A and a_{k+1}. It follows that $|A + a_{k+1}| \leq |A| + |a_{k+1}|$. We combine those two inequalities and obtain

$$|A + a_{k+1}| \;\leq\; |A| + |a_{k+1}| \;\leq\; B + |a_{k+1}| \tag{2.50}$$

In other words,

$$\big|\big(a_1 + a_2 + \dots + a_k\big) + a_{k+1}\big| \;\leq\; B + |a_{k+1}| \;=\; \big(|a_1| + |a_2| + \dots + |a_k|\big) + |a_{k+1}|, \tag{2.51}$$

and this is (2.49) for $k+1$ rather than k numbers: We have shown the validity of the triangle inequality for $k+1$ items under the assumption that it is valid for k items. It follows from the induction principle that the inequality is valid for any $k \geq k_0 = 2$. ■

To summarize what we did in all of part B: We were able to show the validity of the triangle inequality for $k+1$ numbers under the assumption that it was valid for k numbers.

Remark 2.21 (Why induction works)**.** But how can we from all of the above conclude that the distributivity formulas of prop.2.6 and the triangle inequality of prop.2.7 work for all $n \in \mathbb{N}$ such that $n \geq k_0$? We illustrate this for the triangle inequality.

Step 1:	We know that the statement is true for $k_0 = 2$ because that was proven in the base case.
Step 2:	But according to the induction step, if it is true for $k_0 = 2$, it is also true for the successor $k_0 + 1 = 3$ of 2.
Step 3:	But according to the induction step, if it is true for $k_0 + 1$, it is also true for the successor $(k_0 + 1) + 1 = 4$ of $k_0 + 1$.
Step 4:	But according to the induction step, if it is true for $k_0 + 2$, it is also true for the successor $(k_0 + 2) + 1 = 5$ of $k_0 + 2$.
......	....
Step $53,920$:	But according to the induction step, if it is true for $k_0 + 53,918$, it is also true for the successor $(k_0 + 53,918) + 1 = 53,921$ of $k_0 + 53,918$.
......	....

And now we see why the statement is true for any natural number $n \geq k_0$. □

2.8 Some Preliminaries From Calculus

Remark 2.22.

To understand this remark you need to be familiar with the concepts of continuity, differentiability and antiderivatives (integrals) of functions of a single variable. Just skip the parts where you lack the background.

The following is known from calculus (see [3] Stewart, J: Single Variable Calculus): Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ and let $X :=]a, b[$ be the open (end points a, b are excluded) interval of all real numbers between a and b. Let $x_0 \in]a, b[$ be "fixed but arbitrary".

Let $f :]a, b[\to \mathbb{R}$ be a function which is continuous on $]a, b[$. Then

- **(a)** f is integrable for any $\alpha, \beta \in \mathbb{R}$ such that $a < \alpha < \beta < b$, i.e., the **definite integral** $\int_\alpha^\beta f(u)du$ exists. For a definition of integrability see, e.g., [3] Stewart, J: Single Variable Calculus.
- **(b)** Integration is "linear", i.e., it is additive: $\int_\alpha^\beta \big(f(u) + g(u)\big)du = \int_\alpha^\beta f(u)du + \int_\alpha^\beta g(u)du$, and you also can "pull out" constant $\lambda \in \mathbb{R}$: $\int_\alpha^\beta \lambda f(u)du = \lambda \int_\alpha^\beta f(u)du$.
- **(c)** Integration is "monotonic": If $f(x) \leq g(x)$ for all $\alpha \leq x \leq \beta$ then $\int_\alpha^\beta \big(f(u)\big)du \leq \int_\alpha^\beta g(u)du$.
- **(d)** f has an **antiderivative**: There exists a function $F :]a, b[\to \mathbb{R}$ whose derivative $F'(\cdot)$ exists on all of $]a, b[$ and coincides with f, i.e., $F'(x) = f(x)$ for all $x \in]a, b[$.

(e) This antiderivative satisfies $F(\beta) - F(\alpha) = \int_\alpha^\beta f(u)du$ for all $a < \alpha < \beta < b$ and it is **not** uniquely defined: If $C \in \mathbb{R}$ then $F(\cdot) + C$ is also an antiderivative of f.
On the other hand, if both F_1 and F_2 are antiderivatives for f then their difference $G(\cdot) := F_2(\cdot) - F_1(\cdot)$ has the derivative $G'(\cdot) = f(\cdot) - f(\cdot)$ which is constant zero on $]a, b[$. It follows that any two antiderivatives only differ by a constant.
To summarize the above: If we have one antiderivative F of f then any other antiderivative $\tilde{F}$ is of the form $\tilde{F}(\cdot) = F(\cdot) + C$ for some real number C.
This fact is commonly expressed by writing $\int f(x)dx = F(x) + C$ for the **indefinite integral** (an integral without integration bounds).

(f) It follows from **(e)** that if some $c_0 \in \mathbb{R}$ is given then there is only one antiderivative F such that $F(x_0) = c_0$.
Here is a quick proof: Let G be another antiderivative of f such that $G(x_0) = c_0$. Because $G - F$ is constant we have for all $x \in]a, b[$ that

$$G(x) - F(x) = \text{const} = G(x_0) - F(x_0) = 0,$$

i.e., $G = F$. □

2.9 Exercises for Ch.2

2.9.1 Exercises for Sets

Exercise 2.1. Prove (2.14) of prop.2.2 on p.27.

Exercise 2.2. Prove the set identities of prop.2.1.

Exercise 2.3. Prove that for any three sets A, B, C it is true that $(A \setminus B) \setminus C = A \setminus (B \cup C)$.
Hint: use De Morgan's formula (2.15**(a)**). ■

Exercise 2.4.
Let $X = \{x, y, \{x\}, \{x, y\}\ \}$. True or false?

(a) $\{x\} \in X$ **(c)** $\{\ \{x\}\ \} \in X$ **(e)** $y \in X$ **(g)** $\{y\} \in X$
(b) $\{x\} \subseteq X$ **(d)** $\{\ \{x\}\ \} \subseteq X$ **(f)** $y \subseteq X$ **(h** $\{y\} \subseteq X$ □

For the subsequent exercises refer to example **??** for the preliminary definition of the size $|A|$ of a set A and to Definition **??** (Cartesian Product of Two Sets) for the definition of Cartesian product. You find both in ch.**??** (Cartesian Products and Relations) on p.**??**

Exercise 2.5.
Find the size of each of the following sets:

(a) $A = \{x, y,\ \{x\},\ \{x, y\}\ \}$ **(c)** $C = \{u, v, v, v, u\}$ **(e)** $E = \{\sin(k\pi/2) : k \in \mathbb{Z}\}$
(b) $B = \{1,\ \{0\},\ \{1\}\ \}$ **(d)** $D = \{3z - 10 : z \in \mathbb{Z}\}$ **(f)** $F = \{\pi x : x \in \mathbb{R}\}$ □

Exercise 2.6.

Let $X = \{x, y, \{x\}, \{x, y\}\ \}$ and $Y = \{x, \{y\}\ \}$. True or false?

(a) $x \in X \cap Y$ **(c)** $x \in X \cup Y$ **(e)** $x \in X \setminus Y$ **(g)** $x \in X \Delta Y$
(b) $\{y\} \in X \cap Y$ **(d)** $\{y\} \in X \cup Y$ **(f)** $\{y\} \in X \setminus Y$ **(h)** $\{y\} \in X \Delta Y$ $\square$

Exercise 2.7.
Let $X = \{1, 2, 3, 4\}$ and let $Y = \{x, y\}$.

(a) What is $X \times Y$? **(c)** What is $|X \times Y|$? **(e)** Is $(x, 3) \in X \times Y$? **(g)** Is $3 \cdot x \in X \times Y$?
(b) What is $Y \times X$? **(d)** What is $|X \times Y|$? **(f)** Is $(x, 3) \in Y \times X$? **(h)** Is $2 \cdot y \in Y \times X$? $\square$

Exercise 2.8.
Let $X = \{8\}$. What is $2^{(2^X)}$?

Exercise 2.9.
Let $A = \{1, \{1, 2\}, 2, 3, 4\}$ and $B = \{\{2, 3\}, 3, \{4\}, 5\}$. Compute the following.

(a) $A \cap B$ **(b)** $A \cup B$ **(c)** $A \setminus B$ **(d)** $B \setminus A$ **(e)** $A \triangle B$ $\square$

Exercise 2.10.
Let A, X be sets such that $A \subseteq X$ and let $x \in X$. Prove the following:

(a) If $a \in A$ then $A = (A \setminus \{a\}) \uplus \{a\}$.
(b) If $a \notin A$ then $A = (A \uplus \{a\}) \setminus \{a\}$.
$\square$

2.9.2 Exercises for Proofs by Induction

Exercise 2.11.
Use induction on n to prove (2.46) of prop.2.6 on p.55 of this document: Let $A_1, A_2, \dots$ and B be sets. If $n \in \mathbb{N}$ then $\left(\bigcap\limits_{j=1}^{n} A_j\right) \cup B = \bigcap\limits_{j=1}^{n} (A_j \cup B)$. $\square$

Exercise 2.12. [14]
Let $K \in \mathbb{N}$ such that $K \geq 2$ and $n \in \mathbb{Z}_{\geq 0}$. Prove that $K^n > n$. $\square$

Exercise 2.13.
Let $n \in \mathbb{N}$. Then $n^2 + n$ is even, i.e., this expression is an integer multiple of 2. $\square$

PROOF: The proof is given in this instructor's edition.

The proof is done by induction on n.

The base case ($n_0 = 1$) holds because $1^2 + 1 = 2$, and this is an even number.

Induction step: Let $k \in \mathbb{N}$.

(2.52) Induction assumption: $k^2 + k$ is even, i.e., $k^2 + k = 2j$ for some suitable $j \in \mathbb{Z}$.

[14] Note that this exercise generalizes B/G prop.7.1: If $n \in \mathbb{N}$ then $n < 10^n$. Also note that if you allow K to be a real number rather than an integer then it will not be true for all $K > 1$ and $n \in \mathbb{Z}_{\geq 0}$. For example $K^n > n$ is false for $K = 1.4$ and $n = 2$ (but it is true for $K = 1.5$ and $n = 2$).

We must show that there exists $j' \in \mathbb{Z}$ such that $(k+1)^2 + k + 1 = 2j'$. We have

$$(k+1)^2 + k + 1 \;=\; k^2 + 2k + 1 + k + 1 \;=\; (k^2 + k) + 2(k+1) \;\overset{(2.52)}{=}\; 2j + 2(k+1).$$

Let $j' := j + k + 1$. Then $(k+1)^2 + k + 1 = 2j'$ and this finishes the proof. ■

Exercise 2.14.
Use the result from exercise 2.13 above to prove by induction that $n^3 + 5n$ is an integer multiple of 6 for all $n \in \mathbb{N}$. ☐

PROOF: The proof is given in this instructor's edition.

The proof is done by induction on n.

The base case ($n_0 = 1$) holds because $1^3 + 5 = 6 = 1 \cdot 6$.

Induction step: Let $k \in \mathbb{N}$.

(2.53)
Induction assumption: $k^3 + 5k$ is an integer multiple of 6, i.e., $k^3 + 5k = 6j$ for some $j \in \mathbb{Z}$.

We must show that there exists $j' \in \mathbb{Z}$ such that $(k+1)^3 + 5(k+1) = 6j'$. We know frome exercise 2.13 that $3(k^2 + k) = 3 \cdot 2 \cdot i$ for a suitable $i \in \mathbb{Z}$, hence

$$\begin{aligned}(k+1)^3 + 5(k+1) \;&=\; k^3 + 3k^2 + 3k + 1 + 5k + 5 \;=\; (k^3 + 5k) + 3(k^2 + k) + 6\\ &=\; (k^3 + 5k) + 6i + 6 \;\overset{(2.53)}{=}\; 6(j + i + 1).\end{aligned}$$

Let $j' := j + i + 1$. Then $(k+1)^3 + 5(k+1) = 6j'$ and this finishes the proof. ■

Exercise 2.15.
Let $x_1 = 1$ and $x_{n+1} = x_n + 2n + 1$. Prove by induction that $x_n = n^2$ for all $n \in \mathbb{N}$. ☐

References

[1] Matthias Beck and Ross Geoghegan. The Art of Proof. Springer, 1st edition, 2010.

[2] John Bryant and Penelope Kirby. Course Notes for MAD 2104 Discrete Mathematics I. Florida State University.

[3] James Stewart. Single Variable Calculus. Thomson Brooks Cole, 7th edition, 2012.

List of Symbols

$-A$, 35
$A + b$, 35
$[a, b[,\ ]a, b]$ – half-open intervals , 36
$[a, b]$ – closed interval , 36
$\Rightarrow$ – implication , 25
$\mathfrak{P}(\Omega), 2^{\Omega}$ – power set , 29
$\bigcap \big[A_i : i \in I\big]$, 48
$\bigcap \big[B : B \in \mathscr{A}\big]$, 48
$\bigcap_{B \in \mathscr{A}} B$, 48
$\bigcap_{i \in I} A_i$, 48
$\bigcup \big[A_i : i \in I\big]$, 48
$\bigcup \big[B : B \in \mathscr{A}\big]$, 48
$\bigcup_{B \in \mathscr{A}} B$, 48
$\bigcup_{i \in I} A_i$, 48
$\emptyset$ – empty set, 22
$\pm\infty$ – $\pm$ infinity , 36
$|x|$ – absolute value , 37
$]a, b[_{\mathbb{Q}}$ – interval of rational #s , 36
$]a, b[_{\mathbb{Z}}$ – interval of integers , 36
$]a, b[$ – open interval , 36
$f(\cdot)$ – function , 40
$x \in X$ – element of a set, 20
$x \notin X$ – not an element of a set, 20
$A^{\complement}$ – complement of A , 26
$\lambda A + b$ – translation/dilation , 35
$\mathbb{N}$ – natural numbers, 32
$\mathbb{N}_0$ – nonnegative integers, 35
$\mathbb{Q}$ – rational numbers, 33
$\mathbb{R}$ – real numbers, 33
$\mathbb{R}^{\star}$ – non-zero real numbers, 35
$\mathbb{R}^{+}$ – positive real numbers, 35
$\mathbb{R}_{>0}$ – positive real numbers, 35
$\mathbb{R}_{\geq 0}$ – nonnegative real numbers, 35
$\mathbb{R}_{\neq 0}$ – non-zero real numbers, 35
$\mathbb{R}_{+}$ – nonnegative real numbers, 35
$\mathbb{Z}$ – integers, 32
$\mathbb{Z}$ – integers, 32
$\mathbb{Z}_{\geq 0}$ – nonnegative integers, 35
$\mathbb{Z}_{+}$ – nonnegative integers, 35

$(x_i)_{i \in J}$ – family , 45
$2^{\Omega}, \mathfrak{P}(\Omega)$ – power set , 29
δ_{ij} – Kronecker delta , 38
$|X|$ – size of a set , 30
$\{\}$ – empty set, 22

$A \cap B$ – A intersection B, 23
$A \setminus B$ – A minus B , 25
$A \subset B$ – A is strict subset of B, 22
$A \subseteq B$ – A is subset of B , 22
$A \subsetneq B$ – A is strict subset of B, 22
$A \triangle B$ – symmetric difference of A and B , 25
$A \uplus B$ – A disjoint union B , 24
$B \supset A$ – B is strict superset of A, 22
$B \supsetneq A$ – B is strict superset of A, 22
$f : X \to Y$ – function, 39
$\bigcap_{j=1}^{n} A_j$ – union of A_j , 24
$\bigcup_{j=1}^{n} A_j$ – union of A_j , 24
$\mapsto$ – maps to , 39
$A \cup B$ – A union B , 23
$A \supseteq B$ – A is superset of B, 22

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