Math 330 - Additional Material

Student edition with proofs

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4 Logic ⋆

This chapter uses material presented in ch.2 (Logic) and ch.3 (Methods of Proofs) of [2] Bryant, Kirby Course Notes for MAD 2104.

4.1 Statements and Statement Functions

Note 4.1 (Textual variables)**.**

It was mentioned in **(c)** of the introduction to ch.**??** (A First Look at Functions, Sequences and Families) that the input variables and function values of a function need not necessarily numbers, but they can also be textual. For example, the domain of a function may consist of the first names of certain persons.

> A note on textual variables: If the variable is the last name of the person James Joice and valid input for the function $F : p \mapsto$ "Each morning p writes two pages.") then we write interchangeably Joyce or 'Joyce'. Quotes are generally avoided unless they add clarity.

In the above example "Each morning 'Joyce' writes two pages." emphasizes that Joyce is the replacement of a parameter whereas F('Joyce') does not seem to improve the simpler notation F(Joyce) and you will most likely see the expression F(Joyce) = "Each morning 'Joyce' writes two pages." ☐

Definition 4.1 (Statements)**.** A **statement** [1] is a sentence or collection of sentences that is either true or false. We write T or **true** for "true" and F or **false** for "false" and we refer to those constants as **truth values** ☐

Example 4.1. The following are examples of statements:

- **(a)** "Dogs are mammals" (a true statement);
- **(b)** "Roses are mammals. 7 is a number." This is a false statement which also could have been written as a single sentence: "Roses are mammals and 7 is a number";
- **(c)** "I own 5 houses" (a statement because this sentence is either true or false depending on whether I told the truth or I lied);
- **(d)** "The sum of any two even integers is even" (a true statement);
- **(e)** "The sum of any two even integers is even **and** Roses are mammals" (a false statement);
- **(f)** "**Either** the sum of any two even integers is even **or** Roses are mammals" (a true statement). ☐

Example 4.2. The following are **not** statements:

- **(a)** "Who is invited for dinner?"
- **(b)** "$2x = 27$" (the variable x must be bound (specified) to determine whether this sentence is true or false: It is true for $x = 13.5$ and it is false for $x = 33$)
- **(c)** "$x^2 + y^2 = 34$" (both variables x and y must be bound to determine whether this sentence is true or false It is true for $x = 5$ and $y = 3$ and it is false for $x = 7.8$ and $y = 2$)
- **(d)** "Stop bothering me!" ☐

[1] usually called a **proposition** in a course on logic but we do not use this term as in mathematics "proposition" means a theorem of lesser importance.

For the remainder of the entire chapter on logic we define

$$\mathscr{S} \;:=\; \text{the set of all statements} \tag{4.1}$$

$\mathscr{S}$ will appear as the codomain of statement functions.

Be sure to understand the material of ch.**??** (A First Look at Functions, Sequences and Families) on p.**??**) before continuing.

Definition 4.2 (Statement functions (predicates))**.**

We need to discuss some preliminaries before arriving at the definition of a statement function. Let A be a sentence or collection of sentences which contains one or more variables (placeholders) such that, if each of those variables is assigned a specific value, it is either true or false, i.e., it is an element of the set $\mathscr{S}$ of all statements. If A contains n variables $x_1, x_2, \dots, x_n$ and if they are **bound**, i.e., assigned to the specific values $x_1 = x_{10},\ x_2 = x_{20}, \dots, x_n = x_{n0}$, we write $A(x_{10}, x_{20}, \dots, x_{n0})$ for the resulting statement.

To illustrate this let $A :=$ "x is green and y and z like each other". If we know the specific values for the variables x, y, z then this sentence will be true or false. For example A(this lime, Tim, Fred) is true or false depending on whether Tim and Fred do or do not like each other.

There are restrictions for the choice of $x_1 = x_{10}, x_2 = x_{20}, \dots, x_n = x_{n0}$: Associated with each variable x_j in A is a set $\mathscr{U}_j$ which we call the **universe of discourse**, in short, **UoD**, for the jth variable in A. Each value x_{j0} $(j = 1, 2, \dots n)$ must be chosen in such a way that $x_{j0} \in \mathscr{U}_j$. If this is not the case then the expression $A(x_{10}, x_{20}, \dots, x_{n0})$ is called **inadmissible** and we refuse to deal with it.

What was said can be rephrased as follows: We have an assignment $(x_1, x_2, \dots, x_n) \mapsto A(x_1, x_2, \dots, x_n)$ which results in a statement, i.e., an element of $\mathscr{S}$ (see (4.1)) just as long as $x_{j0} \in \mathscr{U}_j$. In other words we have a function

$$A : \mathscr{U}_1 \times \mathscr{U}_2 \times \dots \times \mathscr{U}_n \to \mathscr{S}, \quad (x_1, x_2, \dots, x_n) \mapsto A(x_1, x_2, \dots, x_n) \tag{4.2}$$

in the sense of def. **??**. with the cartesian product of the UoDs for $x_1, \dots, x_n$ as domain and $\mathscr{S}$ as codomain. We call such a function a **statement function** [2] or **predicate**. □

Note 4.2 (Relaxed notation for statement functions)**.**

You should remember that a statement function is a function in the sense of Definition **??** but we we will often use the simpler notation

$A :=$ "some text that contains the placeholders $x_1, x_2, \dots, x_n$ and evaluates to **true** or **false** once all x_j are bound"

together with the specification of each UoD $\mathscr{U}_j$ rather than the formal notation

$$A : \mathscr{U}_1 \times \mathscr{U}_2 \times \dots \times \mathscr{U}_n \to \mathscr{S}, \quad (x_1, x_2, \dots, x_n) \mapsto A(x_1, x_2, \dots, x_n).$$

If A contains two or more variables then the formal notation has an advantage. There is no doubt when looking at an evaluation such as $A(5.5, 7, -3, 8)$ which placeholder in the string corresponds

[2] A statement function is usually called a **proposition function** in a course on logic. As previously mentioned, we do not use the term "proposition" in this document because in most brances of mathematics it refers to a theorem of lesser importance.

to 5.5, which one corresponds to 7 etc. When employing the relaxed notation then we decide this according to the following

Left to right rule for statement functions: If the string A contains n different place holders then the expression $A(x_{10}, x_{20}, \dots x_{n0})$ implies the following: If the name of the first (leftmost) place holder in A is x then each occurrence of x is bound to the value x_{10}. If the name of the first of the remaining place holders in A is y then each occurrence of y is bound to the value x_{20}, ... After $n-1$ steps the remaining placeholders all have the same name, say z and each occurrence of z is bound to the value x_{n0}. If there is any confusion about what is first, what is second, ... then this will be indicated when A is specified or when its variables are bound for the first time.

Example 4.3.

In Definition 4.2 $A =$ "x is green and y and z like each other" was used to illustrate the concept of a statement function. We never showed how to write the actual statement function. We must decide the UoDs for x, y, z and we define them as follows.

UoD for x: $\mathscr{U}_x :=$ all plants and animals in the U.S.,

UoDs for y and z: $\mathscr{U}_y := \mathscr{U}_z :=$ all BU majors in actuarial science.

(a) Here is the formal definition: Let A be the statement function

$$A : \mathscr{U}_x \times \mathscr{U}_y \times \mathscr{U}_z \to \mathscr{S}, \quad (x,y,z) \mapsto A(x,y,z) := \text{“}x \text{ is green and } y \text{ and } z \text{ like each other”}$$

(b) Here is the relaxed definition: Let A be the statement function

$A :=$ "x is green and y and z like each other" with UoDs $\mathscr{U}_x$ for x, $\mathscr{U}_y$ for y and $\mathscr{U}_z$ for z. ☐

The example above and all those below for statement functions of more than a single variable employ the left to right rule. ☐

Adhering to the left to right rule is not a big deal because of the following convention:

We will restrict ourselves in this document from now on to statement functions of one or two variables.

Example 4.4.

Let $A(t) =$ "$t - 4.7$ is an integer″. Then $A : \mathbb{R} \to \mathscr{S}$, $x \mapsto A(x)$ is a one parameter statement function with UoD $\mathbb{R}$ and x as the variable. Note that it is immaterial that we wrote t in the equation and x in the "$\mapsto$" expression because with deal with a dummy variable and we have employed its name consistently in both cases. We have

- **(a)** $A(\text{Honda}) =$ "(Honda' $- 4.7$ is an integer″ is inadmissible because a car brand is not part of our universe of discourse.
- **(b)** If $u_0 \in \mathscr{U}$ then $A(u_0) =$ "$u_0 - 4.7$ is an integer″ is a statement which evaluates to true or false depending on that fixed but unknown value of u_0.
- **(c)** If $n \in \mathscr{U}$ then $A(n)$ is the statement(!) "$n - 4.7$ is an integer". It does not matter that this expression looks exactly like the original A: The expression $A(n)$ implies that the parameter inside the sentence collection A which happens to be named "n" has been bound to a fixed (but unspecified) value also denoted by n. ☐

Example 4.5.

Let $B(x,y) :=$ "$x^2 - y + 2 = 11$". Then $B : \mathbb{R}\times]1,100[\ \to \mathscr{S},\ (x,y) \mapsto B(x,y)$ is a two parameter statement function with UoD $\mathbb{R}$ for x and UoD $]1,100[$ for the variable y. Then

(a) $B(4,-2) =$ "$4^2 - (-2) + 2 = 11$" (a false statement) because x is the leftmost item in B.

(b) $B(z,10) =$ "$z^2 - 10 + 2 = 11$" (true or false depending on z).

(c) **BE CAREFUL:** If $x, y \in \mathbb{R}$ then $B(y,x) =$ "$y^2 - x + 2 = 11$" and **NOT** "$x^2 - y + 2 = 11$" because the "evaluate left to right" rule matters, not any similarity or even coincidence between the symbols inside the sentence collection and in the evaluation $B(\cdot,\cdot)$ $\square$

Example 4.6. The following are predicates:

(a) $P :=$ "$2x = 27$" (see example 4.2**(b)**), UoD $\mathscr{U} := \{x \in \mathbb{R} : x > 10\}$

(b) $Q :=$ "$x^2 + y^2 = 34$" (example 4.2**(c)**), UoD $\mathscr{V} := \{(x,y) : x, y \in \mathbb{R} \text{ and } x < y\}$

(c) $R :=$ "$x^2 + y^2 = 34$ and $xy > 100$", UoDs are $\mathscr{W}_x := \mathscr{W}_y := [-50, 25]$.

Note the following for **(c)**: $R(-30,20)$ evaluates to a false statement because $(-30)\cdot 20 > 100$ is false. $R(30,20)$ does not evaluate to any kind of statement: It is an inadmissible expression because $30 \notin \mathscr{W}_x$.

(d) The sentence "Stop bothering x!" is **not** a statement function because this imperative will not be true or false even if x is bound to a specific value. $\square$

Example 4.7.

Let $B :=$ "$x + 7 = 16$ and d is a dog". Let $\mathscr{U}_x := \mathbb{N}$ and $\mathscr{U}_d := \{d : d \text{ is a vegetable or animal }\}$.

B becomes a statement function of two variables x and d if we specify that the UoD for x is $\mathscr{U}_x$ and the UoD for d is $\mathscr{U}_d$

Assume for the following that Robby is an animal.

(a) $B(9, \text{Robby})$ is the statement "$9 + 7 = 16$ and Robby is a dog". It is true in case Robby is a dog and false in case Robby is not a dog.

(b) $B(20, \text{Robby})$ is the statement "$20 + 7 = 16$ and Robby is a dog" which is false regardless of what Robby might be because $20 + 7 = 16$ by itself is false.

(c) $B(d,F)$ is the statement "$d + 7 = 16$ **and** $F$is a dog": which is true or false depending on the fixed but unspecified values of d and F. Note that d corresponds to the leftmost variable x inside B and not to the second variable d!

(d) $B(x)$ is not a valid expression as we do not allow "partial evaluation" of a predicate. [3] $\square$

4.2 Logic Operations and their Truth Tables

We now resume our discussion of statements.

[3] To indicate that we consider d as fixed but arbitrary and want to interpret "$x + 7 = 16$ and d is a dog" as a statement function of only x as a variable we could have introduced the notation $B(\cdot,d) : x \mapsto B(x,d)$. Similarly, to indicate that we consider x as fixed but arbitrary and want to interpret "$x + 7 = 16$ and d is a dog" as a statement function of only d as a variable we could have introduced the notation $B(x,\cdot) : d \mapsto B(x,d)$. We choose not to overburden the reader with this additional notation. Rather, this situation can be handled by defining two new predicates $C : x \mapsto C(x) :=$ "$x + 7 = 16$ and z is a dog" and $D : d \mapsto D(d) :=$ "$z + 7 = 16$ and d is a dog" and then state that z is not a variable but a fixed (but unspecified) value.

4.2.1 Overview of Logical Operators

Statements can be connected with **logical operators**, also called **connectives**, to form another statement, i.e., something that is either **true** or **false**.

Here is an overview of the important connectives. [4] Their meaning will be explained subsequently, once we define compound statements and compound statement functions.

negation:	$\neg A$	**not** A
conjunction:	$A \wedge B$	A **and** B
double arrow (biconditional):	$A \leftrightarrow B$	A **double arrow** B
logical equivalence:	$A \Leftrightarrow B$	A **if and only if** B
disjunction (inclusive or):	$A \vee B$	A **or** B
exclusive or:	A **xor** B	**either** A **or** B**, exactly one of** A **or** B
arrow:	$A \to B$	A **arrow** B**, if** A **then** B
implication:	$A \Rightarrow B$	A **implies** B**, if** A **then** B

Notation 4.1 (use of symbols vs descriptive English)**.**

(a) In the entire chapter on logic we generally use for logical operators their symbols like "$\neg$" or "$\Rightarrow$" in formulas but we use their corresponding English expressions (**not** and **implies** in this case) in connection with constructs which contain English language.

For example we would write $\neg(A \vee \neg B)$ rather than **not**(A **or not** B) but we would write "$d + 7 = 16$ **and** F is a dog″ rather than "$(d + 7 = 16) \ \wedge \ (F$is a dog)″

(b) Outside chapter 4 symbols are not used at all for logical operators. We use boldface such as "**and**" rather than just plain type face only to make it visually easier to understand the structure of a mathematical construct which employs connectives. ☐

Definition 4.3 (Compound statements)**.**

A statement which does not contain any logical operators is called a **simple statement** and one that employs logical operators is called a **compound statement**.

Similarly statement functions which contain logical operators are called **compound statement functions**. ☐

Example 4.8.

Statements **(e)** and **(f)** of example 4.1 are examples of compound statements.

In **(e)** the two simple statements "The sum of any two even integers is even" and "Roses are mammals" are connected by **and**.

In **(f)** the two simple statements "The sum of any two even integers is even" and "Roses are mammals" are connected by **either** ... **or**. ☐

[4] This order is rather unusual in that usually you would discuss biconditional and logical equivalence operators last, but logical equivalence between two statements A and B is what we think of when saying "A if and only if B" and it helps to understand what this phrase means in the context of logic as early as possible.

4.2.2 Negation and Conjunction, Truth Tables and Tautologies (Understand this!)

We now give the definition of the first two logical operators which were introduced in the table of section 4.2.1.

Definition 4.4 (Negation)**.** The **negation operator** is represented by the symbol "¬" and it reverses the truth value of a statement A, i.e., if A is **true** then $\neg(A)$ is **false** and if A is **false** then $\neg(A)$ is **true**.

(4.3) This is expressed in this "truth table" for $\neg A$:[5]

A	¬ A
F	T
T	F

□

Example 4.9.
Let $A :=$ "Rover is a horse". Then $\neg A =$ "Rover is **not** a horse" and $\neg\neg A = \neg(\neg A) =$ "Rover is a horse" $= A$.

Let us not quibble here about whether $\neg\neg A$ is not in reality the statement "Rover is not not a horse" which admittedly means the same as "Rover is a horse" but looks different.

There is no question about the fact that the T/F values for A and $\neg\neg A$ are the same. Just compare column 1 with column 3.

A	$\neg A$	$\neg(\neg A)$
F	T	F
T	F	T

Note that we did not use any specifics about A. We derived the T/F values for $\neg\neg A$ from those in the second column by applying the definition of the ¬ operator to the statement $B := \neg A$.

In other words we have proved that the statements A and $\neg\neg A$ are **logically equivalent** in the sense that one of them is true whenever the other one is true and vice versa. □

All operators discussed subsequently are **binary operators**, i.e., they connect two input parameters (statements) A, B and four rather than two rows are needed to show what will happen for each of the four combinations A: **false** and B: **false**, A: **false** and B: **true**, A: **true** and B: **false**, A: **true** and B: **true**.

In contrast, the already discussed negation operator "¬" is a **unary operators**, i.e., it has a single input parameter. We will keep referring to "¬" as a connective even though there are no two or more items that can be connected.

Definition 4.5 (Conjunction)**.** The **conjunction operator** is represented by the symbols "∧" or "**and**". The expression A **and** B is **true** if and only if both A and B are **true**.

(4.4) Truth table for A **and** B:

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

The **and** connective generalizes to more than two statements $A_1, A_2, \dots, A_n$ in the obvious manner: $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ is **true** if and only if each one of $A_1, A_2, \dots, A_n$ is **true** and **false** otherwise. □

[5] The definition of a truth table will be given shortly. See Definition 4.6 on p.59.

Definition 4.6 (Truth table)**.** A **truth table** contains the symbols for statements in the header, i.e., the top row and shows in subsequent rows how their truth values relate.

It contains in the leftmost columns statements which you may think of as varying inputs and it contains in the columns to the right compound statements which were built from those inputs by the use of logical operators. We have a row for each possible combination of truth values for the input statements. Such a combination then determines the truth value for each of the other statements.

When we count rows we start with zero for the header which contains the statement names. Row 1 is the first row which contains T/F values.

An example for a truth table is the following table which you encountered in the definition above 4.5 of the conjunction operator:

A	B	$A \wedge B$
F	F	F
F	T	F
T	F	F
T	T	T

Here the input statements are A and B. The compound statement $A \wedge B$ is built from those inputs with the use of the $\wedge$ operator. We have 4 possible T/F combinations for A and B and each one of those determines the truth value of $A \wedge B$. For example, row 2 contains A:F and B:T and from this we obtain F as the corresponding truth value of $A \wedge B$.

Some truth tables have more than two inputs. If there are three statements A, B, C from which the compound statements that interest us are built then there will be $2^3 = 8$ rows to hold all possible combinations of truth values and for n inputs there will be 2^n rows. □

Definition 4.7 (Logically impossible)**.**

The statements A and B in the truth table of Definition 4.6 were of a generally nature and all four T/F combinations had to be considered. If we deal with statements which are more specific but have some variability because they contain place holders [6] then there may be dependencies that rule out certain combinations as nonsensical. For example let x be some fixed but unspecified number and look at a truth table which has the statements $A := A(x) :=$ "$x > 5$" and $B := B(x) :=$ "$x > 7$" as input. It is clearly impossible that A is false and B is true, no matter what value x may have.

We call such combinations **logically impossible** or **contradictory**. We abbreviate "logically impossible" with **L/I**.

Both truth tables indicate that the combination A:F and B:T is logically impossible for $A =$ "$x > 5$" and $B =$ "$x > 7$".

A	B	$A \wedge B$
F	F	F
F	T	L/I
T	F	F
T	T	T

A	B	$A \wedge B$
F	F	F
T	F	F
T	T	T

□

Remark 4.1.

It was mentioned in the definition of logically impossible T/F combinations that there had to be some relationship between the inputs, i.e., some placeholders or some fixed but unspecified constants to make this an interesting definitions.

Consider what happens if you have two statements A and B for which this is not the case. For example, let $A :=$ "All tomatoes are blue" (obviously false) and $B :=$ "Arkansas is a state of the U.S.A." (obviously true).

[6] e.g., if we have a statement function $P : x \mapsto P(x)$ and we look at the statements $P(x_0)$ for which x_0 belongs to the UoD of P or a certain subset thereof

For those two specific statements we know upfront that we have A:F and B:T, so why bother with the other three cases? In other words, the appropriate truth table is either of those two:

A	B	$A \wedge B$
F	F	L/I
F	T	F
T	F	L/I
T	T	L/I

A	B	$A \wedge B$
F	T	F

□

Remark 4.2.

We chose for a more compact notation to place "L/I" into one of the statement columns but be aware that the L/I attribute really belongs to certain combinations of the T/F values of the inputs. In other words,

the L/I attribute belongs to certain rows of the truth table. A more accurate way would be to place L/I into a separate status column and place "N/A" or "-" or nothing into all columns other than those for the inputs:

Status	A	B	$A \wedge B$
	F	F	F
L/I	F	T	-
	T	F	F
	T	T	T

□

Of course more than two input statements can be involved when discussing logical impossibility. The following example will show this.

Example 4.10.

Let U, V, W, Z be the statement functions
$U := x \mapsto U(x) :=$ "$x \in [0, 4]$",
$V := x \mapsto V(x) :=$ "$x \notin \emptyset$",
$W := x \mapsto W(x) :=$ "$x < -1$",
$Z := x \mapsto Z(x) :=$ "$x > 2$"
with UoD $\mathbb{R}$ in each case. Let Q be a statement function that is built from U, V, W, Z with the help of logical operators.

We observe the following:

(a) $V(x)$ is always true because the empty set does not contain any elements.
a'. In other words, there is no x in the UoD for which $V(x)$ is false.
(b) There is no x in the UoD for which $W(x)$ and $Z(x)$ can both be true.

The following rows in the resulting truth table yield an L/I regardless whether we enter a truth value of T or F into anyone of the "•" entries.

$U(x)$	$V(x)$	$W(y)$	$Z(x)$	$Q(x)$
•	F	•	•	L/I
•	•	T	T	L/I

□

Remark 4.3.

As in example 4.10 above let

$U := U(x) :=$ "$x \in [0, 4]$", $V := V(x) :=$ "$x \notin \emptyset$", $W := W(x) :=$ "$x < -1$", $Z := Z(x) :=$ "$x > 2$".

(a) The statement [7] $Q(x) := \neg(U(x) \wedge V(x)) \wedge W(x) \wedge Z(x)$ can never be true, regardless of x.

[7] It is tough to come up with some decent examples of compound statements if the only operators at your disposal so far are negation and conjunction.

To see this directly note again that $V(x)$ is trivially true for any x because the emptyset by definition does not contain any elements. It follows that $U(x) \wedge V(x)$ means "$x \in [0,4]''$ and $Q(x)$ means "$x < 0'' \wedge$ "$x > 4'' \wedge$ "$x < -1'' \wedge$ "$x > 2''$ which is equivalent to "$x < -1'' \wedge$ "$x > 4''$ and certainly false for any x in the UoD, i.e., $x \in \mathbb{R}$.

Alternatively we can use the results from example 4.10 where we found out that $W(x)$ and $Z(x)$ cannot both be true at the same time.

The remaining rows in the resulting truth table yield an F for $Q(x)$ regardless of the truth values of $U(x)$ and $V(x)$ because $W(x) \wedge Z(x)$ is false, hence $Q(x) =$ whatever $\wedge\big(W(x) \wedge Z(x)\big)$ is false for those remaining rows.

$U(x)$	$V(x)$	$W(y)$	$Z(x)$	$Q(x)$
•	•	F	F	F
•	•	F	T	F
•	•	T	F	F

(b) Let $R : x \mapsto R(x) := \neg Q(x)$ be the statement function with UoD $\mathbb{R}$ which represents for each x in the UoD the opposite of Q. Because $Q(x)$ is false for all x, $R(x)$ is true for all x in the universe of discourse for x. ☐

Statements which are true or false under all circumstances like the statements $R(x)$ and $Q(x)$ from the remark above deserve special names.

Definition 4.8 (Tautologies and contradictions)**.** A **tautology** is a statement which is true under all circumstances, i.e., under all combinations of truth values which are not logically impossible.

A **contradiction** is a statement which is false under all circumstances.

We write T_0 for the tautology "$1 = 1$" and F_0 for the contradiction "$1 = 0$". This gives us a convenient way to incorporate statements which are true or false under all circumstances into formulas that build compound statements. ☐

Example 4.11.

Here are some examples of tautologies.

(a) The statements $R(x)$ of remark 4.3 are tautologies.

(b) T_0 is a boring example of a tautology. So is any true statement without any variables such as "$9 + 12 = 21''$ and "a cat is not a cow".

(c) There are formulas involving arbitrary statements which are tautologies. We will show that for any two statements A and B the statement $P := \neg\big(A \wedge \neg A\big)$ is a tautology.

Here are some examples of contradictions.

(d) The statements $Q(x)$ of remark 4.3 are contradictions.

(e) F_0 is a boring example of a contradiction. So is any false statement without any variables such as "$9 + 12 = 50''$ and "a dog is a whale".

(f) There are formulas involving arbitrary statements which are contradictions. We will show that for any two statements A and B the statement $Q := \big(A \wedge \neg A\big) \wedge B$ is a contradiction. ☐

PROOF of **(c)** and **(f)**:

$P = \neg\big(A \wedge \neg A\big)$ (last column) has entries all T, hence P is a tautology.
$Q = \big(A \wedge \neg A\big) \wedge B$ (next to last column) has entries all F, hence Q is a contradiction.

A	B	$\neg A$	$A \wedge \neg A$	$(A \wedge \neg A) \wedge B$	$\neg(A \wedge \neg A)$
F	F	T	F	F	T
F	T	T	F	F	T
T	F	F	F	F	T
T	T	F	F	F	T

■

We now continue with the conjunction operator.

Example 4.12.

In the following let x, y be two (fixed but arbitrary) integers and let $A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$". Be sure to understand that $A(x)$ and $B(y)$ are in fact statements and not predicates, because the symbols x, y are bound from the start and hence cannot be considered variables of the predicates $A :=$ "$x \in \mathbb{N}$" and $B :=$ "$y \in \mathbb{Z}$ **and** $y > 0$".

We will reuse the statements $A(x)$ and $B(y)$ in examples for the subsequently defined logical operators.

(a) If no assumptions are made about a relationship between x and y then all four T/F combinations are possible and, to explore conjunction, we must deal with the full truth table

$A(x)$	$B(y)$	$A(x) \wedge B(y)$
F	F	F
F	T	F
T	F	F
T	T	T

(b) On the other hand, if $x < y$ then the truth of $A(x)$ implies that of $B(y)$ because if y is an integer which dominates some natural number x then we have $y > x \geq 1 > 0$, i.e., y is an integer bigger than zero, i.e., truth of $A(x)$ and falseness of $B(y)$ are incompatible.

It follows that the combination T/F is L/I. We discard the corresponding row and restrict ourselves to the truth table

$A(x)$	$B(y)$	$A(x) \wedge B(y)$
F	F	F
F	T	F
T	T	T

(c) Even better, if $x = y$, i.e., we compare truth/falsehood of $A(x)$ with that of $B(x)$, we only need to worry about the two combinations F/F and T/T for the following reason: The set of positive integers is the set $\{1, 2, \dots\}$ and this is, by definition, the set $\mathbb{N}$ of all natural numbers. This means that the statements "$x \in \mathbb{N}$" and "$y \in \mathbb{Z}$ **and** $y > 0$" are just two different ways of expressing the same thing.

It follows that either both $A(x)$ and $B(x)$ are true or both are false. We discard the logically impossible combinations F/T and T/F and restrict ourselves to the truth table

$A(x)$	$B(x)$	$A(x) \wedge B(x)$
F	F	F
T	T	T

□

4.2.3 Biconditional and Logical Equivalence Operators – Part 1

Definition 4.9 (Double arrow operator (biconditional))**.** The **double arrow operator** [8] is represented by the symbol "$\leftrightarrow$" and read "A double arrow B". $A \leftrightarrow B$ is **true** if and only if either both A and B are **true or** both A and B are **false**.

(4.5) Truth table for $A \leftrightarrow B$:

A	B	$A \leftrightarrow B$
F	F	T
F	T	F
T	F	F
T	T	T

□

Definition 4.10 (Logical equivalence operator)**.** Two statements A and B are **logically equivalent**

[8][2] Bryant, Kirby Course Notes for MAD 2104 calls this operator the **equivalence operator** but we abstain from that terminology because "A is equivalent to B" has a different meaning and is written $A \Leftrightarrow B$.

if the statement $A \leftrightarrow B$ is a tautology, i.e., if the combinations A:**true**, B:**false** and A:**false**, B:**true** both are logically impossible.

We write $A \Leftrightarrow B$ and we say "A if and only if B" to indicate that A and B are logically equivalent.

(4.6) Truth table for $A \Leftrightarrow B$:

A	B	$A \Leftrightarrow B$
F	F	T
F	T	L/I
T	F	L/I
T	T	T

□

The discussion of the $\leftrightarrow$ and $\Leftrightarrow$ operators will be continued in ch.4.2.6 (Biconditional and Logical Equivalence Operators – Part 2) on p.70

4.2.4 Inclusive and Exclusive Or

Definition 4.11 (Disjunction)**.** The **disjunction operator** is represented by the symbols "$\vee$" or "**or**". The expression A **or** B is **true** if and only if either A or B is **true**.

(4.7) Truth table for $A \vee B$:

A	B	$A \vee B$
F	F	F
F	T	T
T	F	T
T	T	T

The **or** connective generalizes to more than two statements $A_1, A_2, \dots, A_n$ in the obvious manner:

$A_1 \vee A_2 \vee \dots \vee A_n$ is **true** if and only if at least one of $A_1, A_2, \dots, A_n$ is **true** and **false** otherwise, i.e., if each of the A_k is **false**. □

Example 4.13.

As in example 4.12 let $x, y \in \mathbb{Z}$ and let

$A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$"

(a) If no assumptions are made about a relationship between x and y then all four T/F combinations are possible and, to explore conjunction, we must deal with the full truth table

$A(x)$	$B(y)$	$A(x) \vee B(y)$
F	F	F
F	T	T
T	F	T
T	T	T

(b) Let $x < y$. We have seen in example 4.12**(b)** that the combination T/F is impossible and we can restrict ourselves to the simplified truth table

$A(x)$	$B(y)$	$A(x) \vee B(y)$
F	F	F
F	T	T
T	T	T

(c) Now let $x = y$. We have seen in example 4.12**(c)** that either both $A(x)$ and $B(y) = B(x)$ are true or both are false. Because the combinations F/T and T/F are impossible we can restrict ourselves to the simplified truth table

$A(x)$	$B(x)$	$A(x) \wedge B(x)$
F	F	F
T	T	T

□

Definition 4.12 (Exclusive or)**.** The **exclusive or operator** is represented by the symbol "**xor**".

[9] A **xor** B is **true** if and only if either A or B is **true** (but not both as is the case for the inclusive or).

(4.8) Truth table for A **xor** B:

A	B	A **xor** B
F	F	F
F	T	T
T	F	T
T	T	F

□

Example 4.14.

As in example 4.12 let $x, y \in \mathbb{Z}$ and let
$A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$"

(a) If no assumptions are made about a relationship between x and y then all four T/F combinations are possible and, to explore conjunction, we must deal with the full truth table

$A(x)$	$B(y)$	$A(x)$ **xor** $B(y)$
F	F	F
F	T	T
T	F	T
T	T	F

(b) Let $x < y$. We have seen in example 4.12**(b)** that the combination T/F is impossible and we can restrict ourselves to the simplified truth table

$A(x)$	$B(y)$	$A(x)$ **xor** $B(y)$
F	F	F
F	T	T
T	T	F

(c) Now let $x = y$. We have seen in example 4.12**(c)** that either both $A(x)$ and $B(y) = B(x)$ are true or both are false. Because the combinations F/T and T/F are impossible we can restrict ourselves to the simplified truth table

$A(x)$	$B(x)$	$A(x)$ **xor** $B(x)$
F	F	F
T	T	F

This last truth table is remarkable. The truth values for $A(x)$ **xor** $B(x)$ are **false** in each row, hence it is a contradiction as defined in Definition 4.8 on p.61. □

Remark 4.4.

Note that the truth values for $A \leftrightarrow B$ are the exact opposites of those for A **xor** B:

$A \leftrightarrow B$ is true exactly when both A and B have the same truth value whereas A **xor** B is true exactly when A and B have opposite truth values. In other words,

$A \leftrightarrow B$ is true whenever $\neg[A$ **xor** $B]$ is true and false whenever $\neg[A$ **xor** $B]$ is false. □

Exercise 4.1. use that last remark to prove that for any two statements A and B the compound statement

$$[A \leftrightarrow B] \;\leftrightarrow\; \neg[A \textbf{ xor } B]$$

is a tautology. □

4.2.5 Arrow and Implication Operators

Definition 4.13 (Arrow operator)**.** The **arrow operator** [10] is represented by the symbol "→".

[9] Some documents such as [2] Bryant, Kirby Course Notes for MAD 2104. also use the symbol ⊕.
[10] [2] Bryant, Kirby Course Notes for MAD 2104 calls this operator the **implication operator** but we abstain from

We read $A \to B$ as "A arrow B" but see remark 4.6 below for the interpretation "if A then B".

(4.9) Truth table for $A \to B$:

A	B	$A \to B$
F	F	T
F	T	T
T	F	F
T	T	T

In other words, $A \to B$ is **false** if and only if A is **true** and B is **false**. □

Definition 4.14 (Implication operator)**.** We say that A **implies** B and we write

$$A \;\Rightarrow\; B \tag{4.10}$$

for two statements A and B if the statement $A \to B$ is a tautology, i.e., if the combination A: **true**, B: **false** is logically impossible.

(4.11) Truth table for $A \Rightarrow B$:

A	B	$A \Rightarrow B$
F	F	T
F	T	T
T	F	L/I
T	T	T

□

Remark 4.5. There are several ways to express $A \Rightarrow B$ in plain english:

Short form:	Interpret this as:
A implies B	The truth of A implies the truth of B
if A then B	if A is true then B is true
A only if B	A is true only if B is true
B if A	B is true if A is true
B whenever A	B is true whenever A is true
A is sufficient for B	The truth of A is sufficient for the truth of B
B is necessary for A	The truth of B is necessary for the truth of A

□

Theorem 4.1 (Transitivity of "⇒")**.**
Let A, B, C be three statements such that $A \Rightarrow B$ and $B \Rightarrow C$. Then $A \Rightarrow C$.

PROOF:

$A \Rightarrow B$ means that the combination A:T, B:F is logically impossible because otherwise $A \to B$ would have a truth value of F and we would not have a tautology. Hence we can drop row 5 from the truth table on the right. Similarly we can drop row 7 because it contains the combination B:T, C:F which contradicts our assumption that $B \Rightarrow C$. But those are the only rows for which $A \to C$ yields **false** because only they contain the combination A:T, C:F. It follows that $A \to C$ is a tautology, i.e., $A \Rightarrow C$.

	A	B	C
1	F	F	F
2	F	F	T
3	F	T	F
4	F	T	T
5	T	F	F
6	T	F	T
7	T	T	F
8	T	T	T

■

that terminology because "A implies B" has a different meaning and is written $A \Rightarrow B$.

Theorem 4.2 (Transitivity of "$\to$").
Let A, B, C be three statements.
Then $\big[(A \to B) \wedge (B \to C)\big] \;\Rightarrow\; (A \to C)$.

PROOF: We must show that $\big[(A \to B) \wedge (B \to C)\big] \;\to\; (A \to C)$ is a tautology. We do this by brute force and compute the truth table.

A	B	C	$A \to B$	$B \to C$	P := $(A \to B) \wedge (B \to C)$	$A \to C$	$P \to (A \to C)$
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
T	F	T	F	T	F	T	T
T	T	F	T	F	F	F	T
T	T	T	T	T	T	T	T

We see that the last column with the truth values for $\big[(A \to B) \wedge (B \to C)\big] \;\to\; (A \to C)$ contains **true** everywhere and we have proved that this statement is a tautology. ■

Definition 4.15.
In the context of $A \to B$ and $A \Rightarrow B$ we call A the **premise** or the **hypothesis** [11] and we call B the **conclusion**. [12]
We call $B \to A$ the **converse** of $A \to B$ and we call $\neg B \to \neg A$ the **contrapositive** of $A \to B$.
We call $B \Rightarrow A$ the **converse** of $A \Rightarrow B$ and we call $\neg B \Rightarrow \neg A$ the **contrapositive** of $A \Rightarrow B$. □

Remark 4.6.

- **(a)** The difference between $A \to B$ and $A \Rightarrow B$ is that $A \Rightarrow B$ implies a relation between the premise A and the conclusion B which renders the T/F combination A:T, B:F logically impossible, i.e., the pared down truth table has only **true** entries in the $A \Rightarrow B$ column. In other words, $A \Rightarrow B$ is the statement $A \to B$ in case the latter is a tautology as defined in Definition 4.8 on p.61.
- **(b)** Both $A \to B$ and $A \Rightarrow B$ are interpreted as "if A then B" but we prefer in general to say "A arrow B" for $A \to B$ because outside the realm of logic $A \Rightarrow B$ is what mathematicians use when they refer to "If ... then " constructs to state and prove theorems.

□

Example 4.15.
The converse of "if x is a dog then x is a mammal" is "if x is a mammal then x is a dog". You see that, regardless whether you look at it in the context of $\to$ or $\Rightarrow$, a "**if** . . . **then**" statement can be true whereas its converse will be false and vice versa.

[11] also called the **antecedent**
[12] Another word for conclusion is **consequent** .

The contrapositive of "if x is a dog then x is a mammal" is "if x is not a mammal then x is a not a dog". Switching to the contrapositive did not switch the truth value of the "**if** ... **then**" statement. This is not an accident: see the Contrapositive Law (4.37) on p.74. □

Remark 4.7.

What is the connection between the truth tables for $A \to B$, $A \Rightarrow B$ and modeling "if A then B"?

We answer this question as follows:

(a) If the premise A is guaranteed to be false, you should be allowed to conclude from it anything you like:

Consider the following statements which are obviously false:

F_1 : "The average weight of a 30 year old person is 7 ounces",
F_2 : "The number 12.7 is an integer",
F_3 : "There are two odd integers m and n such that $m + n$ is odd",
F_4 : "All continuous functions are differentiable" [13]

and some that are known to be true:

T_1 : "The moon orbits the earth",
T_2 : "The number 12.7 is not an integer",
T_3 : "If m and n are even integers then $m + n$ is even",
T_4 : "All differentiable functions are continuous"

a1. What about the statement "**if** F_3 **then** T_1": "If There are two odd integers m and n such that $m+n$ is odd then the moon orbits the earth"? This may not make a lot of sense to you, but consider this:

The truth of "**if** F_3 **then** T_1" **is not the same as** the truth of just F_1. No absolute claim is made that the moon orbits the earth. You are only asked to concede such is the case under the assumption that two odd integers can be found whose sum is odd. But we know that no such integers exist, i.e., we are dealing with a vacuous premise and there is no obligation on our part to show that the moon indeed orbits the earth! Because of this we should have no problem to accept the validity of "**if** F_3 **then** T_1". Keep in mind though that knowing that **if** F_3 **then** T_1 will not help to establish the truth or falseness of T_1!

a2. Now what about the statement "**if** F_3 **then** F_2": "If There are two odd integers m and n such that $m+n$ is odd then the number 12.7 is an integer"? The truth of this implication should be much easier to understand than allowing to conclude something false from something false:

When was the last time that someone bragged "Yesterday I did xyz" and you responded with something like "If you did xyz then I am the queen of Sheba" in the serene knowledge that there is no way that this person could have possibly done xyz? You know that you have no burden of proof to show that you are the queen of Sheba because you did not make this an absolute claim: You hedged that such is only the case if it is true that the other person in fact did xyz yesterday.

So, yes, the argument "**if** F_3 **then** F_2". sounds OK and we should accept it as true but, as in the case of "**if** F_3 **then** T_1". this has no bearing on the truth or falseness of F_2.

To summarize, "**if** F **then** B". should be true, no matter what you plug in for B. We thus have obtained the first two rows of a sensible truth table for A → B:

A	B	A → B
F	F	T
F	T	T

[13] A counterexample is the function $f(x) = |x|$ because it is continuous everywhere but not differentiable at $x = 0$.

(b) Is it OK to say that if the premise A is true then we may infer that the conclusion B is also true? Definitely! There is nothing wrong with "**if** T_2 **then** T_4", i.e., the statement "If The number 12.7 is not an integer then all differentiable functions are continuous"

We can add the fourth row but we do not have #3 yet:

A	B	A → B
F	F	T
F	T	T
T	F	??
T	T	T

(c) Is it OK to say that, if the premise A is true, we may say in parallel that A implies B even if the conclusion B is false? No way! Let's assume that Jane is a goldfish. Then A: "Jane is a fish" is true and B: "Jane is a rocket scientist" is false. It is definitely NOT OK to say, under those circumstances, "If Jane is a fish" then Jane is a rocket scientist". Contrast that with this modification that fits case b: "If Jane is a fish' then Jane is **not** a rocket scientist". No one should have a problem with that! We now can complete row #3: T → F is false.

We now have the complete truth table for $A \to B$ and it matches the one in Definition 4.13:

A	B	$A \to B$
F	F	T
F	T	T
T	F	F
T	T	T

The truth table (4.11) for $A \Rightarrow B$ is then derived from that for $A \to B$ by demanding that A and B be such that $A \to B$ cannot be false, i.e, the combination A:F, B:T must be logically impossible:

A	B	$A \Rightarrow B$
F	F	T
F	T	T
T	F	L/I
T	T	T

We arrived in this remark at the truth tables for $A \to B$ and $A \Rightarrow B$ based on what seems to be reasonable. But the discipline of logic is as exacting a subject as abstract math and the process had to be done in reverse: We first had to **define** $A \to B$ and $A \Rightarrow B$ by means of the truth tables given in Definition 4.13 and Definition 4.14 and from there we justified why these operators appropriately model "if A then B". ☐

Example 4.16.

As in example 4.12 let $x, y \in \mathbb{Z}$ and let
$A(x) :=$ "$x \in \mathbb{N}''$ and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0''$

(a) If no assumptions are made about a relationship between x and y then all four T/F combinations are possible and, to explore conjunction, we must deal with the full truth table

$A(x)$	$B(y)$	$A(x) \to B(y)$
F	F	T
F	T	T
T	F	F
T	T	T

(b) Let $x < y$. We have seen in example 4.12**(b)** that the combination T/F is impossible and we can restrict ourselves to the simplified truth table

$A(x)$	$B(y)$	$A(x) \to B(y)$
F	F	T
F	T	T
T	T	T

(c) Now let $x = y$. We have seen in example 4.12**(c)** that either both $A(x)$ and $B(y) = B(x)$ are true or both are false. Because the combinations F/T and T/F are impossible we can restrict ourselves to the simplified truth table

$A(x)$	$B(x)$	$A(x) \to B(x)$
F	F	T
T	T	T

We see that $A(x) \to B(y)$ is a tautology in case that $x < y$ or $x = y$. $\square$

We have seen that some work was involved to show that the "$A(x) \to B(y)$" statement of the last example is a tautology. How do we interpret this?

> If you show that a "**if** P **then** Q" statement is a tautology then you have demonstrated that a true premise necessarily results in a true conclusion. You have "**proved**" the validity of the conclusion Q from the validity of the hypothesis P.

The next example is a modification of the previous one. We replace the statements $A(x)$ and $B(y)$ with statement functions $x \mapsto A(x), y \mapsto B(y), (x,y) \mapsto C(x,y)$. and replace $A(x) \to B(y)$ with an equivalent $\to$ statement which involves those three statement functions. Our goal is now to show that this new **if** ... **then** statement is a tautology for all x and y which belong to their universes of discourse.

Example 4.17.

Let $\mathscr{U}_x := \mathscr{U}_y := \mathbb{Z}$ be the UoDs for the variables x and y.

Let $A : \mathscr{U}_x \to \mathscr{S}$ with $x \mapsto$ "$x \in \mathbb{N}$",
$B : \mathscr{U}_y \to \mathscr{S}$ with $y \mapsto$ "$y \in \mathbb{Z}$ **and** $y > 0$",
$C : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}$ with $(x,y) \mapsto$ "$x < y$".

Let us try to show that for any x in the UoD of x and y in the UoD of y, i.e., for any two integers x and y, the function value $T(x,y)$ of the statement function

(4.12) $\quad T : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}$ with $(x,y) \mapsto T(x,y) := \left[\big(A(x) \wedge C(x,y)\big) \;\to\; B(y)\right]$ is a tautology.

Note that

(a) The last arrow in (4.12) is the arrow operator $\to$, not the function assignment operator $\mapsto$.

(b) if we can demonstrate that (4.12) is correct then we can replace $\big(A(x) \wedge C(x,y)\big) \;\to\; B(y)$ with $\big(A(x) \wedge C(x,y)\big) \;\Rightarrow\; B(y)$. We interpret this as having proved the (trivial)
Theorem: It is true for all integers x and y that if $x \in \mathbb{N}$ and $x < y$ then $y \in \mathbb{Z}$ **and** $y > 0$.

The trick is of course to think of x and y not as placeholders but as fixed but unspecified integers. Then $A(x), B(y)$ and $C(x,y)$ are ordinary statements and we can build truth tables just as always. Observe that we now have three "inputs" $A(x), B(y)$ and $C(x,y)$ and the full truth table contains nine entries.

We need not worry about numbers x and y whose combination (x,y) results in the falseness of the premise $A(x) \wedge C(x,y)$ because **false** $\to B(y)$ always results in **true**. In other words we do not worry about any combination of x and y for which at least one of $A(x)$, $C(x,y)$ is false. To phrase it differently we focus on such x and y for which we have that both $A(x)$, $C(x,y)$ are true and eliminate all other rows from the truth table. There are only two cases to consider: either $B(y)$ is **false** or $B(y)$ is **true**:

$A(x)$	$C(x,y)$	$B(y)$	$A(x) \wedge C(x,y)$	$\big(A(x) \wedge C(x,y)\big) \to B(y)$
T	T	F	T	F
T	T	T	T	T

The proof is done if it can be shown that the first row is a logically impossible. We now look at the components $A(x), C(x,y), B(y)$ in context. We have seen in example 4.12**(b)** that the assumed truth of $C(x,y)$ together with that of $A(x)$ is incompatible with $B(y)$ being false. This eliminates the first row from that last truth table and what remains is

$A(x)$	$C(x,y)$	$B(y)$	$A(x) \wedge C(x,y)$	$\big(A(x) \wedge C(x,y)\big) \to B(y)$
T	T	T	T	T

In other words we obtain the value **true** for all non-contradictory combinations in the last column of the truth table and this proves (4.12). ☐

Remark 4.8.

Let us compare example 4.15**(b)** with example 4.17. Besides using statements in the former and predicates in the latter a more subtle difference is that, because x and y were assumed to be known from the outset,

example 4.15**(b)** allowed us to formulate a truth table in which none of the statements had to explicitly refer to the condition $x < y$.

In contrast to this we had to introduce in example 4.17 the predicate $C =$ "$x < y''$ to bring this condition into the truth tables

Was there any advantage of switching from statements to predicates and adding a significant amount of complexity in doing so? The answer is yes but it will only become clear when we introduce quantifiers for statement functions. ☐

We will come back to the subject of proofs in chapter 4.6.1 (Building blocks of mathematical theories) on p.85.

4.2.6 Biconditional and Logical Equivalence Operators – Part 2 (Understand this!)

This chapter continues the discussion of the $\leftrightarrow$ and $\Leftrightarrow$ operators from ch.4.2.3 (Biconditional and Logical Equivalence Operators – Part 1) on p.62.

Remark 4.9.

(a) Equivalence $A \Leftrightarrow B$ provides a "**replacement principle for statements**": Logically equivalent statements are not "semantically identical" but they cannot be distinguished as far as their "logic content", i.e., the circumstances under which they are true or false are concerned.

(b) Note that $A \Leftrightarrow B$ means the same as the following: A is true whenever B is true and A is false whenever B is false because this is the same as saying that, in a truth table that contains entries for A and B, each row either has the value T in both columns or the value F in both columns. This in turn is the same as saying that the column for $A \leftrightarrow B$ has T in each row, i.e., $A \leftrightarrow B$ is a tautology.

b'. There is not much value to **(b)** if A and B are simple statements but things become a lot more interesting if compound statements like $A := \neg(P \wedge Q)$ and $B := \neg P \vee \neg Q$ are looked at. ☐

We illustrate the above remark with the following theorem.

Theorem 4.3 (De Morgan's laws for statements)**.**
Let A and B be statements. Then we have the following logical equivalences:

$$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B, \tag{4.13}$$
$$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B. \tag{4.14}$$

Those formulas generalize to n statements $A_1, A_2, \dots, A_n$ as follows:

$$\neg(A_1 \wedge A_2 \wedge \cdots \wedge A_n) \Leftrightarrow \neg A_1 \vee \neg A_2 \vee \cdots \vee \neg A_n, \tag{4.15}$$
$$\neg(A_1 \vee A_2 \vee \cdots \vee A_n) \Leftrightarrow \neg A_1 \wedge \neg A_2 \wedge \cdots \wedge \neg A_n. \tag{4.16}$$

PROOF of 4.13: Here is the truth table for both $\neg(A \wedge B)$ and $\neg A \vee \neg B$ depending on the truth values of A and B.

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$\big[\neg(A \wedge B)\big] \leftrightarrow \big[\neg A \vee \neg B\big]$
F	F	F	T	T	T	T	T
F	T	F	T	T	F	T	T
T	F	F	T	F	T	T	T
T	T	T	F	F	F	F	T

This proves the validity of 4.13. Note that the last column of the truth table is superfluous because getting T in each row follows from the fact that the rows of the statement to the left and the one to the right of "$\leftrightarrow$" both contain the same entries T-T-T-F. The column has been included because it illustrates what was said in remark 4.9.

PROOF of 4.14: Left as an exercise. ■

Example 4.18.
As in example 4.12 let $x, y \in \mathbb{Z}$ and let
$A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$"

(a) If no assumptions are made about a relationship between x and y then the full truth table needs all four entries and we obtain

$A(x)$	$B(y)$	$A(x) \leftrightarrow B(y)$
F	F	T
F	T	F
T	F	F
T	T	T

(b) Let $x < y$. We have seen in example 4.12 that the combination T/F is impossible and we can restrict ourselves to the simplified truth table

$A(x)$	$B(y)$	$A(x) \to B(y)$
F	F	T
F	T	F
T	T	T

(c) Now let $x = y$. We have seen in example 4.12**(c)** that then either $A(x)$ and $B(y) = B(x)$ must both be true or they must both be false. Because the combinations F/T and T/F are impossible we can restrict ourselves to the simplified truth table

$A(x)$	$B(x)$	$A(x) \leftrightarrow B(x)$
F	F	T
T	T	T

It follows that for any given number x the statement $A(x) \leftrightarrow B(x)$ is always true, irrespective of the truth values of $A(x)$ and $B(x)$. Hence $A(x) \leftrightarrow B(x)$ is a tautology and we can write $A(x) \Leftrightarrow B(x)$ for all x. $\square$

4.2.7 More Examples of Tautologies and Contradictions (Understand this!)

Now that we have all logical operators at our disposal we can give additional examples of tautologies and contradictions.

Example 4.19.
In the following let P, Q, R be three arbitrary statements, let x, y be two (fixed but arbitrary) integers and let $A(x) :=$ "$x \in \mathbb{N}$" and $B(y) :=$ "$y \in \mathbb{Z}$ **and** $y > 0$". (see example 4.12 on p. 62).

(a) Tautologies:

T_0,
$A_1 :=$ "$5 + 7 = 12$",
$A_2 :=$ "Any integer is even **or** odd",
$A_3 := P \vee \neg P$ (Tertium non datur or law of the excluded middle),
$A_4 := P \vee T_0$,
$A_5 := (P \wedge Q) \vee (P \wedge \neg Q)$,
$A_6 := (P \to Q) \leftrightarrow (\neg P \vee Q)$ (Implication is logically equivalent to an **or** statement),
$A_7 := [$"$x < y$"$ \wedge A(x)] \to B(y)$ (see 4.15**(b)** on p.66),
$A_8 := A(x) \leftrightarrow B(x)$ (see 4.15**(c)**).

Note that we can express the fact that A_6, A_7, A_8 are tautologies as follows:

$$(P \to Q) \Leftrightarrow (\neg P \vee Q), \quad [\text{"}x < y\text{"} \wedge A(x)] \Rightarrow B(y), \quad A(x) \Leftrightarrow B(x).$$

(b) Contradictions:

F_0,
$B_1 :=$ "$5 + 7 = 15$",
$B_2 :=$ "There are some non-zero numbers x such that $x = 2x$",
$B_3 := P \wedge \neg P$,
$B_3 := P \wedge F_0$,
$B_4 := F_0 \wedge (P \vee \neg P)$,
$B_5 := [\neg P \vee \neg Q] \wedge [P \wedge Q]$,
$B_6 := A(x)$ **xor** $B(x)$ (see 4.14**(c)** on p. 64). $\square$

Proof that A_3 is a tautology:

P	$\neg P$	$P \vee \neg P$
F	T	T
T	F	T

Proof that A_4 is a tautology:

P	T_0	$P \vee T_0$
F	T	T
T	T	T

Note that even though there are two inputs, P and T_0, there are only two valid combinations of truth values because the only choice for T_0 is **true**.

Proof that A_6 is a tautology:

P	Q	$P \to Q$	$\neg P$	$\neg P \vee Q$	$(P \to Q) \leftrightarrow (\neg P \vee Q)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

■

Remark 4.10.
The interesting tautologies and contradictions are not those involving only specific statements such as $T_0, F_0, A_1, A_2, B_1, B_2$, from above but those statements like A_5, A_6, B_4 and B_5 which specify formulas relating the general statements P, Q and R. □

4.3 Statement Equivalences (Understand this!)

Symbolic logic has a collection of very useful statement equivalences which are given here. They were taken from ch.2 on logic, subchapter 2.4 (Important Logical Equivalences) of [2] Bryant, Kirby Course Notes for MAD 2104.

Theorem 4.4.
Let P, Q, R be statements.

(a) *Identity Laws:*

$$P \wedge T_0 \Leftrightarrow P \tag{4.17}$$
$$P \vee F_0 \Leftrightarrow P \tag{4.18}$$

(b) *Domination Laws:*

$$P \vee T_0 \Leftrightarrow T_0 \tag{4.19}$$
$$P \wedge F_0 \Leftrightarrow F_0 \tag{4.20}$$

(c) *Idempotent Laws:*

$$P \vee P \Leftrightarrow P \tag{4.21}$$
$$P \wedge P \Leftrightarrow P \tag{4.22}$$

(d) *Double Negation Law:*

$$\neg(\neg P) \Leftrightarrow P \tag{4.23}$$

(e) *Commutative Laws:*

$$P \vee Q \Leftrightarrow Q \vee P \tag{4.24}$$
$$P \wedge Q \Leftrightarrow Q \wedge P \tag{4.25}$$

(f) *Associative Laws:*

$$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R) \tag{4.26}$$
$$\textit{hence } (P \vee Q) \vee R \Leftrightarrow P \vee Q \vee R \tag{4.27}$$
$$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R) \tag{4.28}$$
$$\textit{hence } (P \wedge Q) \wedge R \Leftrightarrow P \wedge Q \wedge R \tag{4.29}$$

(g) *Distributive Laws:*

$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R) \tag{4.30}$$
$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R) \tag{4.31}$$

(h) *De Morgan's Laws:* [14]

$$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q \tag{4.32}$$
$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q \tag{4.33}$$

(i) *Absorption Laws:*

$$P \wedge (P \vee Q) \Leftrightarrow P \tag{4.34}$$
$$P \vee (P \wedge Q) \Leftrightarrow P \tag{4.35}$$

(j) *Implication Law:*

$$(P \to Q) \Leftrightarrow (\neg P \vee Q) \tag{4.36}$$

You should remember this formula because the fact that implication can be expressed as an OR statement is often extremely useful when showing that two statements are logically equivalent.

k. *Contrapositive Laws:*

$$(P \to Q) \Leftrightarrow (\neg Q \to \neg P) \tag{4.37}$$
$$(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P) \tag{4.38}$$

l. *Tautology:*

$$(P \vee \neg P) \Leftrightarrow T_0 \tag{4.39}$$

m. *Contradiction:*

$$(P \wedge \neg P) \Leftrightarrow F_0 \tag{4.40}$$

n. *Equivalence:*

$$(P \to Q) \wedge (Q \to P) \Leftrightarrow (P \leftrightarrow Q) \tag{4.41}$$

The proof for only some of the laws stated above are given here. You can prove all others by writing out the truth tables to show that left and right sides of the ... $\Leftrightarrow$ ... statements are indeed logically equivalent.

PROOF of **(h)** (De Morgan's laws):

[14] This is theorem 4.3 (De Morgan's laws for statements).

See theorem 4.3 on p.71.

PROOF of **(j)** (implication law):
We prove (4.36) using a truth table:

We see that the entries T-T-F-T in the $\neg P \vee Q$ column match those given for $P \to Q$ in Definition 4.13 on p.64 of the arrow operator. This proves the logical equivalence of those statements.

P	Q	$\neg P$	$\neg P \vee Q$
F	F	T	T
F	T	T	T
T	F	F	F
T	T	F	T

PROOF of **(k)** (contrapositive law for $\to$):
We prove (4.37) with the help of the previously given laws **(a)** through **(j)**:

$$(P \to Q) \overset{((\mathbf{j}))}{\Leftrightarrow} (\neg P \vee Q) \overset{((\mathbf{e}))}{\Leftrightarrow} (Q \vee \neg P) \overset{((\mathbf{d}))}{\Leftrightarrow} (\neg(\neg Q) \vee \neg P) \overset{((\mathbf{j}))}{\Leftrightarrow} (\neg Q \to \neg P)$$

■

Example 4.20.
Use the logical equivalences of thm.4.4 to prove that $\neg\big(\neg A \wedge (A \wedge B)\big)$ is a tautology. □

Solution:

	$\neg\big(\neg A \wedge (A \wedge B)\big)$	
$\Leftrightarrow$	$\neg(\neg A) \vee \neg(A \wedge B)$	De Morgan's Law (4.32)
$\Leftrightarrow$	$A \vee (\neg A \vee \neg B)$	De Morgan (4.32) + Double negation (4.23)
$\Leftrightarrow$	$(A \vee \neg A) \vee \neg B$	Associative law (4.26)
$\Leftrightarrow$	$T_0 \vee \neg B$	Tautology (4.39)
$\Leftrightarrow$	T_0	Commutative Law (4.24) + Domination Law (4.19) ■

Example 4.21.
Find a simple expression for the negation of the statement "**if** you come before 6:00 **then** I'll take you to the movies". □

Solution: Let A := "You come before 6:00" and B := "I'll take you to the movies". Our task is to find a simple logical equivalent to $\neg(A \to B)$. We proceed as follows:

$$\neg(A \to B) \overset{((\mathbf{j}))}{\Leftrightarrow} \neg(\neg A \vee B) \overset{((\mathbf{h}))}{\Leftrightarrow} (\neg(\neg A) \wedge \neg B) \overset{((\mathbf{d}))}{\Leftrightarrow} (A \wedge \neg B)$$

This translates into the statement "you come before 6:00 **and** I won't take you to the movies".
■

Remark 4.11. Now that we accept that such logical expressions are DEFINED by their truth tables, we must accept the following: if two logical expressions with two statements A and B as input have the same truth table, then they are logically equivalent and we may interchangeably use one or the other in a proof. □

4.4 The Connection Between Formulas for Statements and for Sets (Understand this!)

Given statements a, b and sets A, B you may have the impression that there are connections between $a \wedge b$ and $A \cap B$, between $a \vee b$ and $A \cup B$, between $\neg a$ and $A^{\complement}$, etc. We will briefly explore this.

In this chapter we switch to small letters for statements and statement functions and use capital letters to denote sets. You have already seen an example in the introduction.

We assume the existence of a universal set $\mathscr{U}$ of which all sets are subsets.

All statements will be of the form $a(x) =$ "$x \in A$" for some set $A \subset \mathscr{U}$. In other words we associate with such a set A the following statement function:

$$a : \mathscr{U} \to \mathscr{S}, \qquad x \mapsto a(x) =: \text{"}x \in A\text{"} \tag{4.42}$$

This relationship establishes a correspondence between the subset A of $\mathscr{U}$ and the predicate $a =$ "$x \in A$" with UoD $\mathscr{U}$. We write $a \cong A$ for this correspondence.

Example 4.22.

Let $a \cong A$ and $b \cong B$.

We have

(a) $T_0 \cong \mathscr{U}$, $F_0 \cong \emptyset$

(b) $\neg a : x \mapsto \neg a(x) = \neg$"$x \in A$" evaluates to a true statement if and only if $x \notin A$, i.e. $x \in A^{\complement}$. Hence $\neg a \cong A^{\complement}$.

(c) $a \wedge b : x \mapsto a(x) \wedge b(x) =$ "$x \in A$ **and** $x \in B$" evaluates to a true statement if and only if $x \in A \cap B$. Hence $a \wedge b \cong A \cap B$.

(d) $a \vee b : x \mapsto a(x) \vee b(x) =$ "$x \in A$ **or** $x \in B$" evaluates to a true statement if and only if $x \in A \cup B$. Hence $a \vee b \cong A \cup B$. ☐

We expand the table of formulas for statements given in thm 4.4 on p.73 of ch.4.3 (Statement equivalences) with a third column which shows the corresponding relation for sets. Having a translation of statement relations to set relations allows you to use Venn diagrams as a visualization aid.

Theorem 4.5.

For a set $\mathscr{U}$ Let p, q, r be statement functions and let $P, Q, R \subseteq \mathscr{U}$ such that $p \cong P$, $q \cong Q$, $r \cong R$. Then we have the following:

***(a)** Identity:*

$$p \wedge T_0 \Leftrightarrow p \tag{4.43}$$
$$P \cap \mathscr{U} = P$$

$$p \vee F_0 \Leftrightarrow p \tag{4.44}$$
$$P \cup \emptyset = P$$

***(b)** Domination:*

$$p \vee T_0 \Leftrightarrow T_0 \tag{4.45}$$
$$P \cup \mathscr{U} = \mathscr{U}$$

$$p \wedge F_0 \Leftrightarrow F_0 \tag{4.46}$$
$$P \cap \emptyset = \emptyset$$

(c) *Idempotency:*	(4.47) $p \vee p \Leftrightarrow p$	$P \cup P = P$
	(4.48) $p \wedge p \Leftrightarrow p$	$P \cap P = P$
(d) *Double Negation:*	(4.49) $\neg(\neg p) \Leftrightarrow p$	$(P^{\complement})^{\complement} = P$
(e) *Commutative:*	(4.50) $p \vee q \Leftrightarrow q \vee p$	$P \cup Q = Q \cup P$
	(4.51) $p \wedge q \Leftrightarrow q \wedge p$	$P \cap Q = Q \cap P$
(f) *Associative:*	(4.52) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$	$(P \cup Q) \cup R = P \cup (Q \cup R)$
	(4.53) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$	$(P \cap Q) \cap R = P \cap (Q \cap R)$
(g) *Distributive:*	(4.54) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$	$P \cup (Q \cap R) = (P \cup Q) \cap (P \cup R)$
	(4.55) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	$P \cap (Q \cup R) = (P \cap Q) \cup (P \cap R)$
(h) *De Morgan:*	(4.56) $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	$(P \cap Q)^{\complement} = P^{\complement} \cup Q^{\complement}$
	(4.57) $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$	$(P \cup Q)^{\complement} = P^{\complement} \cap Q^{\complement}$
(i) *Absorption:*	(4.58) $p \wedge (p \vee q) \Leftrightarrow p$	$P \cap (P \cup Q) = P$
	(4.59) $p \vee (p \wedge q) \Leftrightarrow p$	$P \cup (P \cap Q) = P$
j1. *Implication 1:*	(4.60) $(p \to q) \Leftrightarrow (\neg p \vee q)$	$(P \setminus Q)^{\complement} = P^{\complement} \cup Q$

Interpretation: $p(x) \to q(x)$, *i.e.,* "$x \in P$" $\to$ "$x \in Q$" *is* ***true*** *if and only if* $p(x)$*:T,* $q(x)$*:F is L/I., i.e., if and only if* $x \notin P \cap Q^{\complement} = P \setminus Q$, *i.e.,* $x \in (P \setminus Q)^{\complement}$.

$$(4.61) \qquad p \Rightarrow q$$

j2. *Implication 2:* $P \setminus Q = \emptyset$, *i.e.*, $P \subseteq Q$

Note that we are not dealing with $p \to q$ but with $p \Rightarrow q$ where we assume for all x a relation between p and q which renders $p(x)$:T, $q(x)$:F logically impossible.

k. *Contrapositive:*

$$(4.62) \quad (P \to Q) \Leftrightarrow (\neg Q \to \neg P) \qquad P^{\complement} \cup Q = Q \cup P^{\complement}$$

$$(4.63) \quad (P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P) \qquad P \subseteq Q \Leftrightarrow Q^{\complement} \subseteq P^{\complement}$$

l. *Tautology:*

$$(4.64) \quad (P \vee \neg P) \Leftrightarrow T_0 \qquad P \cup P^{\complement} = \mathscr{U}$$

m. *Contradiction:*

$$(4.65) \quad (P \wedge \neg P) \Leftrightarrow F_0 \qquad P \cap P^{\complement} = \emptyset$$

n1. *Equivalence 1:*

$$(4.66) \quad (p \to q) \wedge (q \to p) \Leftrightarrow (p \leftrightarrow q) \qquad (P^{\complement} \cup Q) \cap (Q^{\complement} \cup P) = \{x : x \textit{ both in } P, Q \textbf{ or } x \textit{ neither in } P \textit{ nor in } Q\}$$

n2. *Equivalence 2:*

$$(4.67) \quad (p \Rightarrow q) \wedge (q \Rightarrow p) \Leftrightarrow (p \Leftrightarrow q) \qquad (P \subseteq Q) \textbf{ and } (Q \subseteq P) \Leftrightarrow (P = Q)$$

PROOF: The set equalities are evident except for the following:
PROOF of Equivalence 1:

$$\begin{aligned}(P^{\complement} \cup Q) \cap (Q^{\complement} \cup P) &= \left[(P^{\complement} \cup Q) \cap Q^{\complement}\right] \cup \left[(P^{\complement} \cup Q) \cap P\right] \\ &= (P^{\complement} \cap Q^{\complement}) \cup (Q \cap Q^{\complement}) \cup (P^{\complement} \cap P) \cup (Q \cap P) \\ &= (P^{\complement} \cap Q^{\complement}) \cup (Q \cap P) \\ &= \{x : x \text{ neither in } P \text{ nor in } Q \textbf{ or } x \text{ both in } P, Q\}.\end{aligned}$$

■

4.5 Quantifiers for Statement Functions

This chapter has been kept rather brief. You can find more about quantifiers in ch.2 on logic, subchapter ch.2.3 (Predicates and Quantifiers) of [2] Bryant, Kirby Course Notes for MAD 2104.

4.5.1 Quantifiers for One–Variable Statement Functions

Definition 4.16 (Quantifiers)**.**
Let $A : \mathscr{U} \to \mathscr{S},\ x \mapsto A(x)$ be a statement function of a single variable x with UoD $\mathscr{U}$ for x.
(a) The **universal quantification** of the predicate A is the statement

$$\text{“For all } x\ A(x)'', \quad \text{written } \forall x A(x). \tag{4.68}$$

The above is a short for “$A(x)$ is true for each $x \in \mathscr{U}$”. We call the symbol $\forall$ the **universal quantifier** symbol.
(b) The **existential quantification** of the predicate A is the statement

$$\text{“For some } x\ A(x)'', \quad \text{written } \exists x A(x). \tag{4.69}$$

The above is a short for “There exists $x \in \mathscr{U}$ such that $A(x)$ is true”. [15] We call the symbol $\exists$ the **existential quantifier** symbol.
(c) The **unique existential quantification** of the predicate A is the statement

$$\text{“There exists unique } x \text{ such that } A(x)'', \quad \text{written } \exists! x A(x). \tag{4.70}$$

The above is a short for “There exists a unique $x \in \mathscr{U}$ such that $A(x)$ is true”. [16] We call the symbol $\exists!$ the **unique existential quantifier** symbol. ☐

Example 4.23.
Let $A : [-3, 3] \to \mathscr{S}$ be the statement function $x \mapsto$ “$x^2 - 4 = 0''$.
Let $C :=\ \forall x A(x)$ $D :=\ \exists x A(x)$ and $E :=\ \exists! x A(x)$. Then
$C =$ “for all $x \in [-3, 3]$ it is true that $x^2 - 4 = 0$”
$D =$ “there is at least one $x \in [-3, 3]$ such that $x^2 - 4 = 0$”
$E =$ “there is exactly one $x \in [-3, 3]$ such that $x^2 - 4 = 0$”
Note that each of C, D, E is in fact a statement because each one is either true or false: Clearly the zeros of the function $f(x) = x^2 - 4$ in the interval $-3 \leq x \leq 3$ are $x = \pm 2$. It follows that D is a true statement and A and C are false statements. ☐

Example 4.24.
Let $\mathscr{U} :=$ { all human beings } be the UoD for the following three predicates:
$S(x) :=$ “x is a student at NYU”,
$C(x) :=$ “x cheats when taking tests”,
$H(x) :=$ “x is honest”,
Let us translate the following three english verbiage statements into formulas:
$A_1 :=$ “All humans are NYU students”,
$A_2 :=$ “All NYU students cheat on tests”,
$A_3 :=$ “Any NYU student who cheats on tests is not honest”.
Solution:

[15] Equivalently, “$A(x)$ is true for some $x \in \mathscr{U}$” or “$A(x)$ is true for at least one $x \in \mathscr{U}$”.
[16] Equivalently, “$A(x)$ is true for exactly one $x \in \mathscr{U}$”.

$A_1 = \forall x\ S(x)$,
$A_2 = \forall x\ [S(x) \to C(x)]$,
$A_3 = \forall x\ [(S(x) \wedge C(x)) \to \neg H(x)]$. □

Example 4.25.

We continue example 4.24.

Let us simplify $A_3 = \forall x\ [(S(x) \wedge C(x)) \to \neg H(x)]$.

It is clear that "$A(x)$ is true for all x" is equivalent to "There is no x such that $A(x)$ is false". In other words, we have for any statement function A the following:

$$\forall x\ A(x) \ \Leftrightarrow\ \neg\big[\ \exists x\ (\neg A(x))\ \big].$$

But A_3 is the form $\forall x\ A(x)$: replace $A(x)$ with $(S(x) \wedge C(x)) \to \neg H(x)$.

It follows that

$$A_3 \ \Leftrightarrow\ \neg\big[\ \exists x\ (\neg\big(S(x) \wedge C(x)) \to \neg H(x)\big))\ \big].$$

What a mess! let us drop the "(x)" everywhere and the above becomes

$$A_3 \ \Leftrightarrow\ \neg\big[\ \exists x\ (\neg\big(S \wedge C) \to \neg H\big))\ \big].$$

We have seen in example 4.21 on p.75 that for any two statements P and Q the equivalence $\neg(P \to Q) \ \Leftrightarrow\ (P \wedge \neg Q)$ is true.

Let us apply this with $P := S \wedge C$ and $Q := \neg H$. We obtain

$$A_3 \ \Leftrightarrow \neg\big[\ \exists x\ \big((S \wedge C) \wedge \neg(\neg H)\big)\ \big].\ \Leftrightarrow\ \neg\big[\ \exists x\ \big(S \wedge C \wedge H\big)\ \big].$$

where we obtained the last equivalence by applying the double negation law to $\neg(\neg H)$ and the associative law for $\wedge$ to remove the parentheses from $(S \wedge C) \wedge H$.

As a last step we bring back the "(x)" terms and obtain

$$A_3 \ \Leftrightarrow \neg\exists x\ [S(x) \wedge C(x) \wedge H(x)].$$

In other words, A_3 means "There is no one who is an NYU student and who cheats on tests and is honest". This should make sense if you remember the original meaning of A_3: "Any NYU student who cheats on tests is not honest". □

4.5.2 Quantifiers for Two–Variable Statement Functions

We now discuss quantifiers for statement functions of two variables. Things become a lot more interesting because we can mix up $\forall$, $\exists$ and $\exists!$.

Unless mentioned otherwise B denotes the statement function of two variables

$$B : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}, \quad x \mapsto B(x,y) \tag{4.71}$$

It follows that the unverses of discourse are $\mathscr{U}_x$ for x and $\mathscr{U}_y$ for y.

We need a quantifier for each variable to bind the expression $B(x,y)$ with placeholders x and y into a statement, i.e., into something that will be true or false. This done by example as follows:

Definition 4.17 (Doubly quantified expressions)**.**

Here is a table of statements involving two quantifiers and their meanings.

(a)	$\forall x \forall y B(x,y)$	"for all $x \in \mathscr{U}_x$ and for all $y \in \mathscr{U}_y$ (we have the truth of) $B(x,y)''$,
(b)	$\forall x \exists y B(x,y)$	"for all $x \in \mathscr{U}_x$ there exists (at least one) $y \in \mathscr{U}_y$ such that $B(x,y)''$,
(c)	$\exists x \forall y B(x,y)$	"there exists (at least one) $x \in \mathscr{U}_x$ such that for all $y \in \mathscr{U}_y$ $B(x,y)''$,
(d)	$\exists! x \forall y B(x,y)$	"there exists exactly one $x \in \mathscr{U}_x$ such that for all $y \in \mathscr{U}_y$ $B(x,y)''$,
(e)	$\exists x \exists y B(x,y)$	"there exists (at least one) $x \in \mathscr{U}_x$ and (at least one) $y \in \mathscr{U}_y$ such that $B(x,y)''$. □

Example 4.26.

Let $\mathscr{U}_x := \mathbb{N}, \mathscr{U}_y := \mathbb{Z}$ and $B : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}, \quad (x,y) \mapsto B(x,y) :=$ "$x + y = 1''$. Then

(a)	$\forall x \forall y B(x,y)$	**false**
(b)	$\forall x \exists y B(x,y)$	**true**: for the given x choose $y := 1 - x$.
(c)	$\exists y \forall x B(x,y)$	**false**
(d)	$\forall y \exists x B(x,y)$	**false**: If you choose $y > 0$ then the only x that satisfies the equation $x + y = 1$ is $x = 1 - y \leq 0$, i.e., $x \notin \mathbb{N}$, the UoD for x.
(e)	$\exists! x \forall y B(x,y)$	**false**
(f)	$\exists x \exists y B(x,y)$	**true**: choose $x := 10$ and $y := -9$.

Understand the different outcomes of **(b)**, **(c)** and **(d)** and remember this:

(1) The order in which the qualifiers are applied is important.
$\forall x \exists y$ generally does not mean the same as $\exists y \forall x$.

(2) Interchanging variable names in the qualifiers is not OK.
$\forall x \exists y$ generally does not mean the same as $\forall y \exists x$.

Proposition 4.1.

Note the following:

$$\forall x \forall y B(x,y) \ \Leftrightarrow\ \forall y \forall x B(x,y) \tag{4.72}$$

$$\exists x \exists y B(x,y) \ \Leftrightarrow\ \exists y \exists x B(x,y) \tag{4.73}$$

$$\forall x \exists y B(x,y) \ \not\Leftrightarrow\ \exists y \forall x B(x,y) \tag{4.74}$$

$$\exists y \forall x B(x,y) \Rightarrow \forall x \exists y B(x,y) \tag{4.75}$$

PROOF: (4.72) and (4.73) follow from **(a)** and **(e)** in def. 4.17 and we saw an example for (4.72) in the previous example.

The last item is not so obvious. We argue as follows: Assume that $\exists y \forall x B(x,y)$ is true. Then there is some $y_0 \in \mathscr{U}_y$ such that $B(x, y_0$ is true for all $x \in \mathscr{U}_x$.

Why does that imply the truth of $\forall x \exists y B(x,y)$, i.e., for all $x \in \mathscr{U}_x$ you can pick some $y \in \mathscr{U}_y$ such that $B(x,y)$ is true? Here is the answer: Pick y_0. This works because, by assumption, $B(x,y_0)$ is true for all $x \in \mathscr{U}_x$. ■

Remark 4.12.

The last part of the proof of (4.75) is worth a closer look:

"$\forall x \exists y \dots$" only tells you that for all x there will be some y which generally depends on x, something we sometimes emphasize using "functional notation" $y = y(x)$.

"$\exists y \forall x \dots$" does more: it postulates the existence of some y_0 which is suitable for each x in its UoD. The assignment $y(x) = y_0$ is constant in x! □

Remark 4.13 (Partially quantified statement functions)**.**

Given a statement function

$$B : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}, \quad x \mapsto B(x, y)$$

with two place holders x and y, we can elect to use only one quantifier for either x or y. If we only quantify x then we only bind x and y still remains a placeholder and if we only quantify y then we only bind y and x still remains a placeholder. □

Example 4.27.

Let $\mathscr{U}_x$:= { all students at this party } and $\mathscr{U}_y$:= { "Linear Algebra", Discrete Mathematics", "Multivariable Calculus", "Ordinary Differential Equations", "Complex Variables", "Graph Theory", "Real Analysis" }.

Let A := "x studies y'' be the two-variable statement function with UoD $\mathscr{U}_x$ for x and UoD $\mathscr{U}_y$ for y, i.e.,

$A : \mathscr{U}_x \times \mathscr{U}_y \to \mathscr{S}, \quad (x, y) \mapsto A(x, y) =$ "x studies y''.

Then $B := \forall x \; A(x, y)$ is the one-variable predicate

$B : \mathscr{U}_y \to \mathscr{S}, \quad y \mapsto B(y) =$ "all students at this party study y"

and $C := \exists! y \; A(x, y)$ is the one-variable predicate

$C : \mathscr{U}_x \to \mathscr{S}, \quad x \mapsto C(x) =$ "x studies exactly one of the courses listed in $\mathscr{U}_y$". □

4.5.3 Quantifiers for Statement Functions of more than Two Variables

Remark 4.14.

Although this document limits its scope to statement functions of one or two variables (see the note before remark 4.4 in ch.4.1 (Statements and statement functions)) we discuss briefly the use of quantifiers for predicates

$$A : \mathscr{U}_1 \times \mathscr{U}_2 \times \dots \times \mathscr{U}_n \to \mathscr{S}, \quad (x_1, x_2, \dots, x_n) \mapsto A(x_1, x_2, \dots, x_n).$$

with n place holders.

Each one of those variables needs to be bound by one of the quantifiers $\forall, \exists, \exists!$ in order to obtain a statement, i.e., something that is either true or false. □

Example 4.28 (Continuity vs uniform continuity)**.**

This example demonstrates the effect of switching a $\forall$ quantifier with an $\exists$ quantifier for a predicate with four variables. You will learn later that one quantification corresponds to ordinary continuity and the other corresponds to uniform continuity of a function. Do not worry if you do not understand how this example relates to continuity. The only point of interest here is the use of the quantifiers.

Let $a < b$ be two real numbers and let $f :]a,b[\to \mathbb{R}$ be a function which maps each x in its domain $]a,b[$ to a real number $y = f(x)$.

Let $\mathscr{U}_\varepsilon := \mathscr{U}_\delta :=]0,\infty[$ and $\mathscr{U}_x := \mathscr{U}_{x'} :=]a,b[$. Let $P : \mathscr{U}_x \times \mathscr{U}_{x'} \times \mathscr{U}_\delta \times \mathscr{U}_\varepsilon \to \mathscr{S}$ be the predicate

$$(x, x', \delta, \varepsilon) \mapsto P(x, x', \delta, \varepsilon) \ := \ \text{"if } |x - x'| < \delta \text{ then } |f(x) - f(x')| < \varepsilon\text{".}$$

Let $A := \forall\varepsilon\ \forall x\ \exists\delta\ \forall x' P(x, x', \delta, \varepsilon)$ Then A being true is equivalent to saying that the function f is continuous at each point $x \in]a,b[$. [17]

Let $B := \forall\varepsilon\ \exists\delta\ \forall x\ \forall x' P(x, x', \delta, \varepsilon)$. Then B being true is equivalent to saying that the function f is uniformly continuous in $]a,b[$. [18]

The difference between A and B is that in statement A the variable δ whose existence is required may depend on both ε and x, i.e., $\delta = \delta(\varepsilon, x$

On the other hand, to satisfy B, a δ must be found which still may depend on ε but it must be suitable for <u>all</u> $x \in]a,b[$, i.e., $\delta = \delta(\varepsilon$. $\square$

Remark 4.15 (Partially quantified statement functions)**.**

What was said in remark 4.13 about partial qualification of two-variable predicates generalizes to more than two variables: If A is a statement function with n variables and we use quantifiers for only $m < n$ of those variables then $n - m$ variables in the resulting expression remain unbound and this expression becomes a statement function of those unbound variables.

For example, if $A(w,x,y,z)$ is a four-variable predicate then $B : (x,z) \mapsto \big[\forall y\ \neg\exists w\ \ A(w,x,y,z)\big]$ defines a two-variable predicate B which inherits the UoDs for x and z from the original statement function A. $\square$

4.5.4 Quantifiers and Negation (Understand this!)

Negation of statements involving quantifiers is governed by

Theorem 4.6 (De Morgan's laws for quantifiers)**.**

Let A be a statement function with UoD $\mathscr{U}$. Then

(a)	$\neg\big(\forall x A(x)\big) \ \Leftrightarrow\ \exists x\ \neg A(x)$	*"It is **not** true that $A(x)$ is true for all x" $\Leftrightarrow$ "There is some x for which $A(x)$ is **not** true"*
(b)	$\neg\big(\exists x A(x)\big) \ \Leftrightarrow\ \forall x\ \neg A(x)$	*"There is **no** x for which $A(x)$ is true" $\Leftrightarrow$ "$A(x)$ is **not** true for all x "*

PROOF of **(a)**: Not given here but you can find it in ch.2 on logic, subchapter 3.11 (De Morgan's Laws for Quantifiers) of [2] Bryant, Kirby Course Notes for MAD 2104.

PROOF of **(b)**: Let $\mathscr{U}_x$ be the UoD for x.

The truth of $\neg\big(\exists x A(x)\big)$ means that $\exists x A(x)$ is false, i.e., $A(x)$ is false for all $x \in \mathscr{U}_x$. This is equivalent to stating that $\neg A(x)$ is true for all $x \in \mathscr{U}_x$ and this is by definition, the truth of $\forall x\ \neg A(x)$. $\blacksquare$

[17] See Definition **??** (ε-δ continuity) on p.**??**.
[18] See Definition **??** (Uniform continuity of functions) on p.**??**.

You can use the formulas above for negation of statements of more than one variable with more than one quantifier using the following method, demonstrated here by example.

Example 4.29.

Negate the statement $\exists x \forall y P(x,y)$, i.e., move the $\neg$ operator of $\neg\exists x \forall y P(x,y)$ to the right past all quantifiers.

The key is to introduce an intermittent predicate $A : x \mapsto A(x) := \big[\forall y P(x,y)\big]$. We obtain

$$\big[\neg\exists x \forall y P(x,y)\big] \;\Leftrightarrow\; \big[\neg\exists x A(x)\big] \overset{(\textbf{(b)})}{\Leftrightarrow} \big[\forall x \neg A(x)\big] \;\Leftrightarrow\; \big[\forall x(\neg\forall y P(x,y))\big]$$
$$\overset{(\textbf{(a)})}{\Leftrightarrow} \big[\forall x(\exists y \neg P(x,y))\big]. \;\square$$

Example 4.30.

As in example 4.29, negate the statement $\exists x \forall y P(x,y)$ but do so using parentheses instead of explicitly defining an intermittent predicate.

Here is the solution:

$$\big[\neg\exists x \forall y P(x,y)\big] \;\Leftrightarrow\; \big[\neg\exists x\big(\forall y P(x,y)\big)\big] \overset{(\textbf{(b)})}{\Leftrightarrow} \big[\forall x \neg\big(\forall y P(x,y)\big)\big] \;\Leftrightarrow\; \big[\forall x(\neg\forall y P(x,y))\big]$$
$$\overset{(\textbf{(a)})}{\Leftrightarrow} \big[\forall x(\exists y \neg P(x,y))\big]. \;\square$$

4.6 Proofs (Understand this!)

We have informally discussed proofs in examples 4.15 and 4.17 of chapter 4.2.5 (Arrow and Implication Operators) on p.64 and seen in two simple cases how a proof can be done by building a single truth table for an **if** . . . **then** statement and showing that it is a tautology. In this chapter we take a deeper look at the concept of "proof".

Many subjects discussed here follow closely ch.3 (Methods of Proofs) of [2] Bryant, Kirby Course Notes for MAD 2104.

4.6.1 Building Blocks of Mathematical Theories

Some of the terminology definitions in notations 4.2 and 4.4 were taken almost literally from ch.3 (Methods of Proofs), subchapter 1 (Logical Arguments and Formal Proofs) of [2] Bryant, Kirby Course Notes for MAD 2104.

Notation 4.2 (Axioms, rules of inferences and assertions)**.**

- **(a)** An **axiom** is a statement that is true by definition. No justification such as a proof needs to be given.
- **(b)** A **rule of inference** is a logical rule that is used to deduce the truth of a statement from the truth of others.
- **(c)** For some statements it is not clear whether they are true for false. Even if a statement is known to be true there might be someone like a student taking a test who is given the task to demonstrate, i.e., prove its truth. In this context we call a statement an **assertion** and we call it a **valid assertion** if it can be shown to be true. An assertion which is not known to be true by anyone is often called a **conjecture**. □

Example 4.31.

Let $A :=$ "all continuous functions are differentiable" (known to be false [19]) and $B :=$ "all differentiable functions are continuous" (known to be true). A homework problem in calculus may ask the students to figure out which of the four statements $A, \neg A, B, \neg B$ are valid assertions and give proofs to that effect. □

Remark 4.16.

(a) Goldbach's conjecture states that every even integer greater than 2 can be expressed as the sum of two primes, i.e., integers p greater than 1 which can be divided evenly by no natural number other than p $(p/p = 1)$ or 1 $(p/1 = p)$. Goldbach came up with this in 1742, more than 250 years ago. No one has been able until now to either prove the validity of this assertion or provide a counterexample to prove its falsehood.

(b) Fermat's conjecture was that there are no four numbers $a, b, c, n \in \mathbb{N}$ such that $n > 2$ and $a^n + b^n = c^n$. [20] This was stated by Pierre de Fermat in 1637 who then claimed that he had a proof. Unfortunately he never got around to write it down. A successful proof was finally published in 1994 by Andrew Wiles. Accordingly, Fermat's conjecture was rechristened Fermat's Last Theorem. □

[19] see remark 4.7 on p.67 in ch.4.2.5 (Arrow and Implication Operators).

[20] We have an elementary counterexample for $n = 2$: $3^2 + 4^2 = 25 = 5^2$.

Notation 4.3 (Proofs)**.**

A **proof** is the demonstration that an assertion is valid. This demonstration must be detailed enough so that a person with sufficient expert knowledge can understand that we do indeed have a statement which is true for all logically possible combinations of T/F values. To show that the arguments given in this demonstration are valid, available tools are

- **(a)** the rules of inference which wil be discussed in section 4.6.2 (Rules of Inference) on p.88
- **(b)** logical equivalences for statements (see ch.4.2.6 (Biconditional and Logical Equivalence Operators – Part 2) on p.ch.70).

> In almost all cases the assertion in question is of the form "**if** P **then** C". Proving it means showing that the statement $P \to C$ is a tautology, i.e., it can be replaced by the stronger $P \Rightarrow C$ statement. The proof then consists of the demonstration that the combination P: **true**, C: **false** can be ruled out as logically impossible. In other words, assuming P: **true**, i.e., the truth of the premise, it must be shown that C: **true**, i.e., the conclusion then also is necessarily true.

Usually a proof is broken down into several "sub-proofs" which can be proved separately and where some or all of those steps again will be broken down into several steps ... You can picture this as a hierarchical upside down tree with a single node at the top. At the most detailed level at the bottom we have the leaf nodes. The proof of the entire statement is represented by that top node. □

Notation 4.4 (Theorems, lemmata and corollaries)**.**

- **(a)** A **theorem** is an assertion that can be proved to be true using definitions, axioms, previously proven theorems, and rules of inference.
- **(b)** A **lemma** (plural: lemmata) is a theorem whose main importance is that it can used to prove other theorems.
- **(c)** A **corollary** is a theorem whose truth is a fairly easy consequence of another theorem. □

Remark 4.17 (Terminology is different outside logic)**.**

The terminology given in the above definitions is specific to the subject of mathematical logic. In other branches of mathematics and hence outside this chapter 4 different meanings are attached to those terms:

Each one of **lemma**, **proposition**, **theorem**, **corollary** is a theorem as defined above in notations 4.2, i.e., a statement that can be proved to be true. We distinguish those terms by comparing them to propositions:

- **(a)** Theorems are considered more important than propositions.
- **(b)** The main purpose of a lemma is to serve as a tool to prove other propositions or theorems.
- **(c)** A corollary is a fairly easy consequence of some lemma, proposition, theorem or other corollary.

□

It was mentioned as a footnote to the definition of a statement (def. 4.1 on p.53) that what we call a statement, [2] Bryant, Kirby calls a proposition and that we deviate from that approach because mathematics outside logic uses "proposition" to denote a theorem of lesser importance.

Any mathematical theory must start out with a collection of undefined terms and axioms that specify certain properties of those undefined terms.

There is no way to build a theory without undefined terms because the following will happen if you try to define every term: You define T_2 in terms of T_1, then you define T_3 in terms of T_2, etc. Two possibilities:

(1) Each of $T_1, T_2, T_3, \dots$ are different and you end up with an infinite sequence of definitions.
(2) At least one of those terms is repeated and there will be a circular chain of definitions.

Neither case is acceptable if you want to specify the foundations of a mathematical system.

Example 4.32.

Here are a few important examples of mathematical systems and their ingredients.

(a) In Euclid's geometry of the plane some of the undefined terms are "point", "line segment" and "line". The five Euclidean axioms specify certain properties which relate those undefined terms. You may have heard of the fifth axiom, Euclid's parallel postulate. It has been reproduced here with small alterations from Wikipedia's "Euclidean geometry" entry: [21]
(It is postulated that) "if a line segment falling on two line segments makes the interior angles on the same side less than two right angles, the two line segments, if produced indefinitely, meet on that side on which are the angles less than the two right angles".

(b) In the so called Zermelo-Fraenkel set theory which serves as the foundation for most of the math that has been done in the last 100 years, the concept of a "set" and the relation "is an element of" ($\in$) are undefined terms.

(c) Chapters 1 and 2 of [1] Beck/Geoghegan list several axioms which stipulate the existence of a nonempty set called $\mathbb{Z}$ whose elements are called "integers" which you can "add" and "multiply". Certain algebraic properties such as "$a+b=b+a$" and "$c\cdot(a+b)=(c\cdot a)+(c\cdot a)$" are given as true and so is the existence of an additive neutral unit "0" and a multiplicative neutral unit "1". Besides those algebraic properties the existence of a strict subset $\mathbb{N}$ called "positive integers" is assumed which has, among others, the property that any $z\in\mathbb{Z}$ either satisfies $z\in\mathbb{Z}$ or $-z\in\mathbb{Z}$ or $z=0$. Finally there is the induction axiom which states that if you create the sequence $1,\ 1+1,\ (1+1)+1,\ \dots$ then you capture all of $\mathbb{N}$. This axiom is the basis for the principle of mathematical induction (see thm.**??** on p. **??**). □

Once we have the undefined terms and axioms for a mathematical system, we can begin defining new terms and proving theorems (or lemmas, or corollaries) within the system.

Remark 4.18 (Axioms vs. Definitions)**.**

You can define anything you want but if you are not careful you may have a logical contradiction and the set of all items that satisfy that definition is empty. In contrast, axioms will postulate the

[21] https://en.wikipedia.org/wiki/Euclidean_geometry#Axioms

existence of an item or an entire collection of items which satisfy all axioms. If the axioms contradict each other we have a theory which is inconsistent and the only way to deal with it is to discard it and rework its foundations An example for this was set theory in its early stages. Anything that you could phrase as "Let A be the set which contains ..." was fair game to define a set. We saw in remark **??** (Russell's Antinomy) on p.**??** that this lead to problems so serious that they caused some of the leading mathematicians of the time to revisit the foundations of mathematics. ☐

Example 4.33.

For example you can define an oddandeven integer to be any $z \in \mathbb{Z}$ which satisfies that $z - 212$ is an even number and $z + 48$ is an odd number and you can prove great things for such z. The problem is of course that the set of all oddandeven integers is empty! We have a definition which is useless for all practical purposes, but no mathematical harm is done.

On the other hand, if you add as an additional axiom for $\mathbb{Z}$ in example 4.32**(c)** that $\mathbb{Z}$ must contain one or more oddandeven integers then you are in a conundrum because <u>you postulated the existence</u> of a set $\mathbb{Z}$ which satisfies all axioms and the existence of such a set is logically impossible! ☐

4.6.2 Rules of Inference

Remark 4.19 (Most important rules of inference)**.**

In Notations 4.2 on p.85 we described the term "rule of inference" as "a logical rule that is used to deduce the truth of a statement from the truth of others". The most important rules of inference are those that allow you to draw a conclusion of the form "if A is true then I am allowed to deduce the the truth of C." This basically amounts to having is a list of premises $A_1, A_2, \dots, A_n$ and a conclusion C such that

$$\text{the compound statement } \big[A_1 \wedge A_2 \wedge \cdots \wedge A_n\big] \to C \text{ is a tautology.} \tag{4.76}$$

In other words, the column for the conclusion C in the truth table for this statement must have the value **true** for each combination of truth values which is not logically impossible.

Observe that the order of the premises does not matter because the **and** connective is commutative. ☐

Theorem 4.7.

Let $P_1, P_2, \dots, P_n$ and C be statements. Then the statement $(P_1 \wedge P_2 \wedge \cdots \wedge P_n) \to C$ is a tautology if and only if the following combination of truth values is logically impossible:

$$P_j \textit{ is } \textbf{\textit{true}} \textit{ for each } j = 1, 2, \dots, n \textit{ and } C \textit{ is } \textbf{\textit{false}}. \tag{4.77}$$

PROOF:

Let $P := (P_1 \wedge P_2 \wedge \cdots \wedge P_n)$. Then "$P_j$ is **true** for each $j = 1, 2, \dots, n''$ means according to the definition of the $\wedge$ operator the same as the truth of P. Hence proving the theorem is equivalent to proving that the statement $P \to C$ is a tautology if and only if the combination of truth values

$$P \text{ is } \textbf{true} \text{ and } C \text{ is } \textbf{false} \text{ is logically impossible.} \tag{4.78}$$

In other words, we must prove that $P \to C$ is a tautology if and only if the row with the combination P:T, C:F, i.e., row 3, is logically impossible and can be ignored. This is is obvious as row 3 is the only one for which $P \to C$ evaluates to **false**.

	P	C	$P \to C$
1.	F	F	T
2.	F	T	T
3.	T	F	**false**
4.	T	T	T

■

Notation 4.5.

Rules of inference are commonly written in the following form:

Your explanations go into this area	A_1 A_2 $\cdots$ A_n ——— $\therefore C$

Read "$\therefore$" as "therefore". The following, more compact notation can also be found:

$$\frac{A_1, A_2, \dots, A_n}{\therefore C}$$

Theorem 4.8 (The three most important inference rules)**.**

The following lists three inference rules, i.e., those arrow statements are indeed tautoloties:

(4.79) *Modus Ponens (Law of detachment - the mode that affirms the antecedent (the premise))*

$$\begin{array}{l} A \\ A \to C \\ \hline \therefore C \end{array}$$

(4.80) *Modus Tollens (The mode that Denies the consequent (the conclusion))*

$$\begin{array}{l} \neg C \\ A \to C \\ \hline \therefore \neg A \end{array}$$

(4.81) *Hypothetical syllogism*

$$\begin{array}{l} A \to B \\ B \to C \\ \hline \therefore A \to C \end{array}$$

Here is the compact notation:

Modus Ponens	*Modus Tollens*	*Hypothetical syllogism*
$A,\ A \to C$	$\neg C,\ A \to C$	$A \to B,\ B \to C$
————	————	——————
$\therefore C$	$\therefore \neg A$	$\therefore A \to C$

Note that the proof that the hypothetical syllogism is a tautology was given in thm.4.1 on p.65

PROOF:

■

Example 4.34.
Here are five more inference rules.

(4.82) Disjunction Introduction

$$\begin{array}{l} A \\ \hline \therefore A \lor B \end{array}$$

(4.83) Conjunction elimination

$$\begin{array}{l} A \land B \\ \hline \therefore A \end{array}$$

(4.84) Disjunctive syllogism

$$\begin{array}{l} A \lor B \\ \neg A \\ \hline \therefore B \end{array}$$

(4.85) Conjunction introduction

$$\begin{array}{l} A \\ B \\ \hline \therefore A \land B \end{array}$$

(4.86) Constructive dilemma

$$\begin{array}{l} (A \to B) \land (C \to D) \\ A \lor C \\ \hline \therefore B \lor D \end{array}$$

Compact notation:

Disjunction Introduction	Conjunction elimination	Disjunctive syllogism
A ——— $\therefore A \vee B$	$A \wedge B$ ——— $\therefore A$	$A \vee B$, $\neg A$ ——— $\therefore B$
Conjunction introduction	Constructive dilemma	
A, B ——— $\therefore A \wedge B$	$(A \to B) \wedge (C \to D), A \vee C$ ——— - $\therefore B \vee D$	

□

None of the rules of inference that were given in this chapter involve quantifiers. You can find information about that topic in ch.2, section 1.6 (Rules of Inference for Quantifiers) of [2] Bryant, Kirby Course Notes for MAD 2104.

4.6.3 An Example of a Direct Proof

We illustrate in detail a mathematical proof by applying some the tools you have learned so far in this chapter on logic. For an example we will prove the theorem that each polynomial is differentiable. We define a polynomial as a function $f(x) = \sum_{j=0}^{n} c_j x^j$ for some $n = 0, 1, 2, \ldots$, i.e., for some $n \in \mathbb{Z}_{\geq 0}$ and we write $\mathscr{D}$ for the set of all differentiable functions. We now can formulate our theorem.

Theorem 4.9.
Given the statements

$$\textbf{a:}\quad A := \text{“}(n \in \mathbb{Z}_{\geq 0}) \wedge (c_0 \in \mathbb{R}) \wedge (c_1 \in \mathbb{R}) \wedge \cdots \wedge (c_n \in \mathbb{R}) \wedge \ \big(f(x) = \sum_{j=0}^{n} c_j x^j\big)\text{''},$$

$$\textbf{b:}\quad B := \text{“}f(x) \in \mathscr{D}\text{''},$$

the following is valid: $A \Rightarrow B$. [22]

PROOF:
We first collect the necessary ingredients.

We define the following statements which serve as abbreviations so that the formulas we will build are reasonably compact.

[22] Note here and for the other theorems the use of $A_2 \Rightarrow B_2$ instead of $A_2 \to B_2$: We assume that Thm-2 has been proved, i.e., $A_2 \to B_2$ is a tautology.

a: $Z_j := \text{“}j \in \mathbb{Z}_{\geq 0}\text{''}$,
b: $C_j := Z_j \wedge \text{“}c_j \in \mathbb{R}\text{''}$,
c: [23] $X_j := Z_j \wedge \text{“}x^j \in \mathscr{D}\text{''}$,
d: $D_j := Z_j \wedge \text{“}c_j x^j \in \mathscr{D}\text{''}$,
e: $E := Z_n \wedge \text{“}f(x) = \sum_{j=0}^{n} c_j x^j\text{''}$,
f: $B := \text{“}f(x) \in \mathscr{D}\text{''}$ (repeated for convenient reference)

We now can write our theorem as

$$(Z_n \wedge C_0 \wedge C_1 \wedge \cdots \wedge C_n \wedge E) \to B. \tag{4.87}$$

We assume that the following three theorems were proved previously, hence we may use them without giving a proof.

Theorem Thm-1: If $p(x)$ is a power of x, i.e., $p(x) = x^n$ for some $n = 0, 1, 2, \ldots$, then is $p(x)$ differentiable.

We rewrite Thm-1 as an implication which uses the statements above. Let

$$A_1 := Z_n \;\wedge\; \text{“}p(x) = x^n\text{''}, \quad B_1 := X_n.$$

Then Thm-1 states that $A_1 \Rightarrow B_1$. [24]

Theorem Thm-2: The product of a constant (real number) and a differentiable function is differentiable.

We rewrite Thm-2 as an implication. Let
$A_2 := \text{“}c \in \mathbb{R}\text{''} \wedge \text{“}h(x) \in \mathscr{D}\text{''} \wedge \text{“}g(x) = c \cdot h(x)\text{''}$,
$B_2 := \text{“}h(x) \in \mathscr{D}\text{''}$,
Then Thm-2 states that $A_2 \Rightarrow B_2$.

Theorem Thm-3: The sum of differentiable functions is differentiable

We rewrite Thm-3 as an implication. Let

$$A_3 := \text{“}Z_n \wedge \text{“}h_1(x) \in \mathscr{D}\text{''} \wedge \text{“}h_2(x) \in \mathscr{D}\text{''} \wedge \cdots \wedge \text{“}h_n(x) \in \mathscr{D}\text{''} \wedge \text{“}g(x) = \sum_{j=0}^{n} h_j(x)\text{''},$$

$B_3 := \text{“}g(x) \in \mathscr{D}\text{''}$,
Then Thm-3 states that $A_3 \Rightarrow B_3$.

	Assertion	Reason
a:	$Z_0, Z_1, \ldots Z_n$	evident from $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$
b:	$C_0, C_1, \ldots C_n$	part of the premise of $A \to B$ (see (4.87))
c:	$Z_j \to X_j \;\; (j = 0, 1, \ldots n)$	Thm-1 with $n := j$
d:	$X_j \;\; (j = 0, 1, \ldots n)$	**(c)** and modus ponens
e:	$(Z_j \wedge C_j \wedge X_j) \to D_j \;\; (j = 0, 1, \ldots n)$	Thm-2 with $c := c_j$ and $h(x) := x^j$
f:	$D_j \;\; (j = 0, 1, \ldots n)$	**(e)** and modus ponens
g:	E	part of the premise of $A \to B$
h:	$(Z_n \wedge D_0 \wedge D_1 \wedge \cdots \wedge D_n \wedge E) \to B$	**(g)** and Thm-3 with $h_j(x) := c_j x^j$ and $g(x) := f(x)$
i:	B	**(h)** and modus ponens

[23] The expression x^j in **(c)** and **(d)** denotes the function $x \mapsto x^j$.

[24] As is the case for the theorem we want to prove, note here and for Thm-2 and Thm-3 below the use of $A_1 \Rightarrow B_1$ instead of $A_1 \to B_1$: Thm-1 has been proved already, i.e., we know that $A_1 \to B_1$ is a tautology.

We have demonstrated that the truth of the premise A of our theorem implies that of its conclusion B and this proves the theorem. ■

Remark 4.20.

Let us reflect on the steps involved in the proof above.

a: Break down all statements involved – not only those in the theorem you want to prove but also in all theorems, axioms and definitions you reference – into reusable components and name those components with a symbol so that it is easier to understand what assertions you employ and how they lead to the truth of other assertions. Example: D_j references the component $Z_j \;\wedge\;$ "$c_j x^j \in \mathscr{D}$" (which itself references the component $Z_j =$ "$j \in \mathbb{Z}_{\geq 0}$").

b: Rewrite the theorem to be proved as an implication $A \Rightarrow B$.

c: Do the same for the three other theorems that we assumed as already having been proved.
The following is specific to our example but can be modified to other problems.

d: Start by using the premise A and the definition $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$ to get the first two rows. Show that what you have implies the truth of the premise of Thm-1 and then use the modus ponens inference rule to deduce the truth of its conclusion X_j. This allows X_j to become an additional assertion.

e: Use that new assertion to obtain the truth of the premise of Thm-2 and then use again modus ponens to deduce the truth of its conclusion D_j. Now D_j becomes an additional assertion.

f: Use that new assertion to obtain the truth of the premise of Thm-3 and then use again modus ponens to deduce the truth of D_j. Now D_j becomes an additional assertion. □

4.6.4 Invalid Proofs Due to Faulty Arguments

Remark 4.21 (Fallacies in logical arguments)**.**

People who are not very analytical often commit the following errors in their argumentation:

(4.88)	Affirming the Consequent (proving the wrong direction)	$P \to Q$ Q ――― $\therefore P$
(4.89)	Denying the Antecedent (indirect proof in the wrong direction)	$P \to Q$ $\neg P$ ――― $\therefore \neg Q$
(4.90)	Circular Reasoning	The argument incorporates use of the (not yet proven) conclusion

□

The reason that the above are fallacies stems from the fact that the above "rules of inferences" are not tautologies.

Example 4.35 (Fallacies in reasoning)**.**

(a) Affirming the Consequent:
"If you are a great mathematician then you can add $2+2$". It is true that you can add $2+2$. You conclude that you are a great mathematician.

(b) Denying the Antecedent:
"If this animal is a cat then it can run quickly". This is not a cat. You conclude that this animal cannot run quickly.

(c) Circular Reasoning: [25]
"If xy is divisible by 5 then x is divisible by 5 or y is divisible by 5".

The following incorrect proof uses the yet to be proven fact that the factors can be divided evenly by 5.

PROOF:
If xy is divisible by 5 then $xy = 5k$ for some $k \in \mathbb{Z}$. But then $x = 5m$ or $y = 5n$ for some $m, n \in \mathbb{Z}$ (this is the spot where the conclusion was used). Hence x is divisible by 5 or y is divisible by 5. ■

4.7 Categorization of Proofs (Understand this!)

There are different methods by which you can attempt to prove an "**if** ...**then**" statement $P \Rightarrow Q$. They are:

(a) Trivial proof
(b) Vacuous proof
(c) Direct proof
(d) Proof by contrapositive
(e) Indirect proof (proof by contradiction)
(f) Proof by cases

4.7.1 Trivial Proofs

The underlying principle of a trivial proof is the following: If we know that the conclusion Q is true then any implication $P \Rightarrow Q$ is valid, regardless of the hypothesis P.

Example 4.36 (Trivial proof)**.**

Prove that if it rains at least 60 days per year in Miami then $25 + 35 = 60$.

PROOF: There is nothing to prove as it is known that $25 + 35 = 60$. It is irrelevant whether or not in rains (or snows, if you prefer) 60 days per year in Miami. ■

[25] This is example 1.8.3 in ch.3 (Methods of Proofs) of [2] Bryant, Kirby Course Notes for MAD 2104.

4.7.2 Vacuous Proofs

The underlying principle of a vacuous proof is that a wrong premise allows you to conclude anything you want: Both P:F, Q:F and P:F, Q:T yield **true** for $P \to Q$.

For example, it was mentioned in remark **??** (Elements of the empty set and their properties) on p.**??** that you can state anything you like about the elements of the empty set as there are none. The underlying principle of proving this kind of assertion is that of a vacuous proof. We prove here assertion **(d)** of that remark.

Theorem 4.10.
Let A be any set. Then $\emptyset \subseteq A$.

PROOF:
According to the definition of $\subseteq$ we must prove that if $x \in \emptyset$ then $x \in A$.

So let $x \in \emptyset$. We stop right here: "$x \in \emptyset$" is a false statement regardless of the nature of x because the empty set, by definition, does not contain any elements. It follows that $x \in A$. ■

Remark 4.22.
You may ask: But is it not equally true that if $x \in \emptyset$ then $x \notin A$? The answer to that is YES, it is equally true that $x \in A$? and $x \notin A$?, but so what? First you'll find me an x that belongs to the empty set and **only then** am I required to show you that it both does and does not belong to A! □

4.7.3 Direct Proofs

In a direct proof of $P \Rightarrow Q$ we assume the truth of the hypothesis P and then employ logical equivalences, including the rules of inference, to show the truth of Q.

We proved in chapter 4.6.3 (An example of a direct proof) on p.91 that each polynomial is differentiable (theorem 4.9). That was an example of a direct proof.

4.7.4 Proof by Contrapositive

A proof by contrapositive makes use of the logical equivalence $(P \Rightarrow Q) \Leftrightarrow (\neg Q \Rightarrow \neg P)$ (see the contrapositive law (4.38) on p.74). We give a direct proof of $\neg Q \Rightarrow \neg P$, i.e., we assume the falseness of Q and prove that then P must also be false. Here is an example.

Theorem 4.11.
Let A, B be two subsets of some universal set Ω such that $A \cap B^{\complement} = \emptyset$. Then $A \subseteq B$.

PROOF: We prove the contrapositive instead: If $A \nsubseteq B$ then $A \cap B^{\complement} \neq \emptyset$.

So let us assume $A \nsubseteq B$. This means that not every element of A also belongs to B. In other words, there exists some $x \in A$ such that $x \notin B$. But then $x \in A \setminus B = A \cap B^{\complement}$, i.e., $A \cap B^{\complement} \neq \emptyset$.

We have proved from the negated conclusion $A \nsubseteq B$ the negated premise $A \cap B^{\complement} \neq \emptyset$. ■

4.7.5 Proof by Contradiction (Indirect Proof)

A proofs by contradiction are a generalization of proofs by contrapositive. We assume that it is possible for the implication $P \Rightarrow Q$ that the premise P can be true and Q can be false at the same time and construct the assumption of the truth of $P \cap \neg Q$ a statement R such that both R and $\neg R$ must be true. Here is an example.

Theorem 4.12.
Let $A \subseteq \mathbb{Z}$ with the following properties:

$$m, n \in A \;\Rightarrow\; m + n \in A, \tag{4.91}$$
$$m, n \in A \;\Rightarrow\; mn \in A, \tag{4.92}$$
$$0 \notin A, \tag{4.93}$$
$$\text{if } n \in \mathbb{Z} \text{ then either } n \in A \text{ or } -n \in A \text{ or } n = 0. \tag{4.94}$$

Then $1 \in A$.

Proof by contradiction: Assume that A is a set of integers with properties (4.91) – (4.94) but that $1 \notin A$. We will show that then $1 \in A$ must be true. This finishes the proof because it is impossible that both $1 \notin A$ and $1 \in A$ are true.
(a) It follows from $1 \notin A$ and (4.94) and $1 \neq 0$ that $-1 \in A$.
(b) It now follows from (4.92) that $(-1) \cdot (-1) \in A$, i.e., $1 \in A$.
We have reached our contradiction. ■

Remark 4.23.
In this simple proof the statement R for which both R and $\neg R$ were shown to be true happens to be the conclusion $1 \in A$. This generally does not need to be the case. □

4.7.6 Proof by Cases

Sometimes an assumption P is too messy to take on in its entirety and it is easier to break it down into two or more cases $P_1, P_2, \dots, P_n$ each of which only covers part of P but such that $P_1 \vee P_2 \vee \dots \vee P_n$ covers all of it, i.e., we assume

$$P_1 \vee P_2 \vee \dots \vee P_n \;\Leftrightarrow\; P. \tag{4.95}$$

Proof by cases then rests on the following theorem:

Theorem 4.13. *Let $P, Q, P_1 \vee P_2 \vee \dots \vee P_n$ be statements such that (4.95) is true. Then*

$$\big(P \Rightarrow Q\big) \;\Leftrightarrow\; \big[\, (P_1 \Rightarrow Q) \;\vee\; (P_2 \Rightarrow Q) \;\vee\; \dots\; (P_n \Rightarrow Q) \,\big]. \tag{4.96}$$

Proof (outline): You would do the proof by induction. Prove (4.96) first for $n = 2$ by expressing $A \to B$ as $\neg A \vee B$ and then building a truth table that compares $\big(\neg(P_1 \vee P_2)\big) \vee Q$ with $\neg P_1 \vee Q \vee \neg P_2 \vee Q$. Then do the induction step in which (4.95) becomes $P_1 \vee P_2 \vee \dots \vee P_{n+1} \;\Leftrightarrow\; P$ by setting $A := P_1 \vee P_2 \vee \dots \vee P_n$ and this way reducing the proof of (4.96) for $n + 1$ to that of 2 components. You make the validity of $\big(A \Rightarrow Q\big) \;\Leftrightarrow\; \big[\, (P_1 \Rightarrow Q) \;\vee\; (P_2 \Rightarrow Q) \;\vee\; \dots\; (P_n \Rightarrow Q) \,\big]$ the induction assumption. ■

Theorem 4.14.
Prove that for any $x \in \mathbb{R}$ *such that* $x \neq 5$ *we have*

$$\frac{x}{x-5} > 0 \ \Rightarrow\ \big[\ (x < 0)\ \textit{or}\ (x > 5)\ \big]. \tag{4.97}$$

PROOF: There are two cases for which $x/(x-5) > 0$:
either both $x > 0$ and $x - 5 > 0$ or both $x < 0$ and $x - 5 < 0$. We write
$P :=$ "$x/(x-5) > 0$", [26] $P_1 := x > 0$ **and** $x - 5 > 0$, $P_2 := x < 0$ **and** $x - 5 < 0$. Then $P = P_1 \vee P_2$.
case 1. P_1:
Obviously $x > 0$ and $x - 5 > 0$ if and only if $x > 5$, so we have proved $P_1 \Rightarrow (x > 5)$.
case 2. P_2:
Obviously $x < 0$ and $x - 5 < 0$ if and only if $x < 0$, so we have proved $P_2 \Rightarrow (x < 0)$.
We now conclude from $P = P_1 \vee P_2$ and theorem 4.13 the validity of (4.97). ■

[26] $P :=$ "$x/(x-5) > 0$ **and** $x \neq 5$" if you want to be a stickler for precision

4.8 Blank Page after Ch.4

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References

[1] Matthias Beck and Ross Geoghegan. The Art of Proof. Springer, 1st edition, 2010.

[2] John Bryant and Penelope Kirby. Course Notes for MAD 2104 Discrete Mathematics I. Florida State University.

List of Symbols

F_0 – contradiction stmt , 61
T_0 – tautology stmt , 61
$\Leftrightarrow$ – logical equivalence , 63
$\Rightarrow$ – implication , 65
$\mathscr{U}$ universe of discourse , 54
$\exists$ – exists , 79
$\exists!$ – exists unique , 79
$\forall$ – for all , 79
$\leftrightarrow$ – double arrow logic op. , 62
$\neg$ – negation , 58
$\rightarrow$ – arrow operator , 65
$\vee$ – disjunction , 63
$\wedge$ – conjunction , 58
xor – exclusive or , 64

$\therefore$ – therefore , 89
false , 53
true , 53
L/I – logically impossible , 59

F – false , 53

T – true , 53

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