Math 330 - Additional Material

Student edition with proofs

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5 Relations, Functions and Families

We now give an in depth presentation of the material of ch.**??** (A First Look at Functions, Sequences and Families).

5.1 Cartesian Products and Relations

Definition 5.1 (Cartesian Product of Two Sets)**.**

The **cartesian product** of two sets A and B is

$$A \times B \ := \ \{(a,b) : a \in A, b \in B\},$$

i.e., it consists of all pairs (a,b) with $a \in A$ and $b \in B$.

Let $(a_1,b_1),(a_2,b_2) \in A \times B$. We say they are **equal**, and we write $(a_1,b_1) = (a_2,b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$.

As a shorthand, we abbreviate $A^2 := A \times A$.

It follows from this definition of equality that the pairs (a,b) and (b,a) are different unless $a = b$. In other words, the order of a and b is important. We express this by saying that the cartesian product consists of **ordered pairs**. □

Example 5.1 (Coordinates in the plane)**.** Here is the most important example of a Cartesian Product of Two Sets. Let $A = B = \mathbb{R}$. Then $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$ is the set of pairs of real numbers, i.e., the points in the plane, expressed by their x– and y–coordinates.

Examples of such points are are: $(1,0) \in \mathbb{R}^2$ (a point on the x–axis), $(0,1) \in \mathbb{R}^2$ (a point on the y–axis), $(1.234, -\sqrt{2}) \in \mathbb{R}^2$.

You should understand why we do not allow two pairs to be equal if we flip the coordinates: Of course $(1,0)$ and $(0,1)$ are different points in the xy–plane! □

Remark 5.1 (Function graphs as subsets of cartesian products)**.** We gave the preliminary definition of a function in Definition **??**, p.**??** of ch.**??** (A First Look at Functions, Sequences and Families). [1] A function

$$f : X \to Y; \quad y = f(x)$$

which assigns each $x \in X$ to a unique function value $f(x) \in Y$, e.g., $f(x) = x^2$, is characterized by its graph

$$\Gamma_f \ := \ \{\big(x, f(x)\big) : x \in X\}$$

which is a subset of the cartesian product $X \times Y$. For example, if $X = [-2,3]$ and $Y = [0,10]$ then $\Gamma_f \ := \ \{(x,x^2) : -2 \leq x \leq 3\}$ is a subset of $[-2,3] \times [0,10]$. We will examine the connection between functions and their graphs in detail later in this chapter. □

[1]The precise definition of a function will be given in section 5.2 on p.159.

Remark 5.2 (Empty cartesian product).
Note that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$ or both are empty. □

Definition 5.2 (Relation).

Let X and Y be two sets and $R \subseteq X \times Y$ a subset of their cartesian product $X \times Y$. We call R a **relation** on (X, Y). A relation on (X, X) is simply called a relation on X. If $(x, y) \in R$ we say that x **and** y **are related** and we usually write xRy instead of $(x, y) \in R$.

A relation on X is
- **(a)** **reflexive** if xRx for all $x \in X$,
- **(b)** **symmetric** if x_1Rx_2 implies x_2Rx_1 for all $x_1, x_2 \in X$,
- **(c)** **transitive** if x_1Rx_2 and x_2Rx_3 implies x_1Rx_3 for all $x_1, x_2, x_3 \in X$,
- **(d)** **antisymmetric** if x_1Rx_2 and x_2Rx_1 implies $x_1 = x_2$ for all $x_1, x_2 \in X$. □

Here are some examples of relations.

Example 5.2 (Equality as a relation). Given a set X let $R := \{(x, x) : x \in X\}$ [2]
, i.e., xRy if and only if $x = y$. This defines a relation on X which is reflexive, symmetric, antisymmetric and transitive. □

Example 5.3 (Set inclusion as a relation). Given a set X let $R := \{(A, B) : A, B \subseteq X \text{ and } A \subseteq B\}$, i.e., ARB if and only if $A \subseteq B$. This defines a relation on 2^X which is reflexive, antisymmetric and transitive. □

Remark 5.3. Unless a relation on a set X is symmetric there will be at least one pair $x, y \in X$ such that x is related to y whereas y is related to x is false. This is different from how we think of relatedness in a non–mathematical context.

Consider Example 5.3. If A is a proper subset of B then A is related to B but it is not true that B is related to A. □

Example 5.4 (Function graphs as relations). We saw in rem.5.1 on p.153 that functions $f : X \to Y$ are characterized by their graphs $\Gamma_f := \{\big(x, f(x)\big) : x \in X\}$ which are subsets of $X \times Y$, i.e., Γ_f is a relation on $X \times Y$. □

Example 5.5 (Size of sets as a relation). Let X be a set and

$$R := \{(A, B) : A, B \subseteq X \text{ and } \big|A\big| = \big|B\big| \},$$

i.e., ARB if and only if A and B possess the same number of elements. [3] In particular ARB is true if $|A| = |B| = \infty$. This defines a relation on the power set 2^X of X which is reflexive, symmetric and transitive. □

[2] This set is commonly referred to as the **diagonal** of X^2.
[3] See Definition **??** (preliminary definition of the size of a set) on p.**??**.

Example 5.6 (Empty relation)**.** Given two sets X and Y let $R := \emptyset$. This **empty relation** is the only relation which exists on (X, Y) if X or Y is empty. $\square$

Example 5.7. Let $X := \mathbb{R}^2$ be the xy–plane. For any point $\vec{x} = (x_1, x_2)$ in the plane let

$$\|\vec{x}\|\_2 \ := \ \sqrt{x_1^2 + x_2^2}\,; \qquad R \ := \ \{\, (\vec{x}, \vec{y}) \in \mathbb{R}^2 \times \mathbb{R}^2 : \ \|\vec{x}\|_2 \ = \ \|\vec{y}\|_2 \,\}\,.$$

Let In other words, $\|\vec{x}\|\_2$ is the length of the straight line which extends from the origin of the plane to $\vec{x}$ [4] and two points in the plane are related when they have the same length: they are located on a circle which is centered at the origin and has radius $r = \|\vec{x}\|_2 = \|\vec{y}\|_2$. The relation R is reflexive, symmetric and transitive but not antisymmetric. $\square$

The relations given in examples 5.2, 5.5, 5.6 and 5.7 are reflexive, symmetric and transitive. Such relations are so important that they deserve a special name:

Definition 5.3 (Equivalence relations and equivalence classes)**.**

Let R be a relation on a set X.

(a) If R is • reflexive, • symmetric, • transitive, we call R an **equivalence relation** on X.

(b) For an equivalence relation R it is customary to write $x \sim x'$ rather than xRx' (or $(x, x') \in R$). We say in this case that x and x' are **equivalent**.

(c) Given is an equivalence relation "$\sim$" on a set X. For $x \in X$ let

$$[x]_\sim \ := \ \{x' \in X : x' \sim x\} \ = \ \{ \text{ all items equivalent to } x \ \}. \tag{5.1}$$

We call $[x]_\sim$ the **equivalence class** of x. If it is clear from the context what equivalence relation is referred to then we can write $[x]$ instead of $[x]_\sim$. $\square$

Proposition 5.1 (see [1] B/G prop.6.4 & B/G prop.6.5)**.**

Let "$\sim$" be an equivalence relation on a nonempty set X and $x, y \in X$. Then

(a) $x \in [x]$,

(b) $x \sim y \ \Leftrightarrow [x] = [y]$,

(c) *either* $[x] = [y]$ *or* $[x] \cap [y] = \emptyset$.

PROOF of **(a)**: This follows from the reflexivity of "$\sim$".

PROOF of **(b)**:

We first show that if $x \sim y$ then $[x] = [y]$. Let $z \in [x]$. It follows from the definition of $[x]$ that $z \sim x$, hence $z \sim y$ (transitivity of "$\sim$"), hence $z \in [y]$. This proves $[x] \subseteq [y]$. We switch the roles of x and y and repeat the above to obtain $[y] \subseteq [x]$.

[4] See Definition **??** on p.**??**. of the length or Euclidean norm of a vector in n-dimensional space.

We now prove that if $[x] = [y]$ then $x \sim y$. It follows from $[x] = [y]$ that $x \in [y]$, hence $x \sim y$ by (5.1).
PROOF of **(c)**: This proof is left as exercise 5.4 (see p.194). ■

For the next proposition recall Definition **??** on p.**??** of a partition.

Proposition 5.2 (see [1] B/G prop.6.6 for parts **(a)** and **(b)**).

(a) *Let "$\sim$" be an equivalence relation on a nonempty set X and let $\mathscr{P}_\sim := \{[x] : x \in X\}$ be the set of all its equivalence classes. Then $\mathscr{P}_\sim$ is a partition of X.*
(b) *Conversely, let $\mathscr{P}$ be a partition of X and define a relation "$\sim_{\mathscr{P}}$" on X as follows: $x \sim_{\mathscr{P}} y \Leftrightarrow$ there is $P \in \mathscr{P}$ such that $x, y \in P$. Then $\sim_{\mathscr{P}}$ is an equivalence relation on X.*
(c) *Let "$\sim$" be an equivalence relation on X. Let $\mathscr{P}_\sim$ be the associated partition of its equivalence classes. Let "$\sim_{\mathscr{P}_\sim}$" be the equivalence relation associated with the partition $\mathscr{P}_\sim$. Then "$\sim_{\mathscr{P}_\sim}$" = "$\sim$" (i.e., both equivalence relations are equal as subsets of $X \times X$.*
(d) *Let $\mathscr{P}$ be a partition of X. Let $\sim_{\mathscr{P}}$ be the associated equivalence relation defined in part **(b)**. Let $\mathscr{P}_{\sim_{\mathscr{P}}}$ be the associated partition of its equivalence classes. Then $\mathscr{P}_{\sim_{\mathscr{P}}} = \mathscr{P}$.*

PROOF of **(a)**:

We observe that a set contains no duplicates: If $[x] = [y]$, we do not count $[x]$ and $[y]$ as separate members of $\mathscr{P}_\sim$! It follows from prop.5.1**(c)** that those members are mutually disjoint.

It remains to prove that their union is X. Let $x \in X$. Then $x \in [x]$ (reflexivity of "$\sim$") and $[x] \in \mathscr{P}_\sim$. It follows that $x \in \bigcup \big[P : P \in \mathscr{P}_\sim\big]$ and we obtain that $\bigcup \big[P : P \in \mathscr{P}_\sim\big] = X$. We have proved that $\mathscr{P}_\sim$ is a partition of X.

PROOF of **(b)**: For the following assume that $x, y, z \in X$.

It follows from $\biguplus \big[P : P \in \mathscr{P}\big] = X$ that for each $x \in X$ there exists $P \in \mathscr{P}$ such that $x \in P$. It further follows from the mutual disjointness of the elements of $\mathscr{P}$ that there exists exactly one such P and we are justfied to write P_x for this uniquely defined set P. In other words, the assignment $x \mapsto P_x$ defines a function $P(\cdot) : X \to \mathscr{P}$.

Reflexivity: $x \in P_x$ implies $x \sim_{\mathscr{P}} x$.

Symmetry: Let $x \sim_{\mathscr{P}} y$. This implies $P_x = P_y$, hence $y \in P_x$, hence $y \sim_{\mathscr{P}} x$.

Transitivity: Let $x \sim_{\mathscr{P}} y$ and $y \sim_{\mathscr{P}} z$. $x \sim_{\mathscr{P}} y$ implies $P_x = P_y$ and $y \sim_{\mathscr{P}} z$ implies $P_y = P_z$. It follows that $P_z = P_x$, i.e., $x \sim_{\mathscr{P}} z$.

PROOF of **(c)**: ⋆

Let $x, y \in X$. Then

$$x \sim y \Leftrightarrow [x] = [y] \Leftrightarrow \text{ both } x, y \text{ belong to the same element of } \mathscr{P}_\sim \Leftrightarrow x \sim_{\mathscr{P}_\sim} y.$$

PROOF of **(d)**: ⋆

Let $P \in \mathscr{P}$ and $x, y \in X$. Let $[x]$ and $[y]$ be the equivalence classes of x and y for "$\sim_{\mathscr{P}}$". If $x \in P$ then

$$y \in P \Leftrightarrow x \sim_{\mathscr{P}_\sim} y \Leftrightarrow [x] = [y].$$

It follows that $P = [x]$, hence $P \in \mathscr{P}_{\sim_{\mathscr{P}}}$ This true for any $P \in \mathscr{P}$ and it follows that $\mathscr{P} \subseteq \mathscr{P}_{\sim_{\mathscr{P}}}$.

Now let $x \in X$. If $y \in X$ then

$$y \in [x] \;\Leftrightarrow y \sim_{\mathscr{P}} x \;\Leftrightarrow\; y \in P_x.$$

It follows that any equivalence class $[x]$ with respect to "$\sim_{\mathscr{P}}$" is an element of $\mathscr{P}$, hence $\mathscr{P}_{\sim_{\mathscr{P}}} \subseteq \mathscr{P}$. We have shown that $\mathscr{P}_{\sim_{\mathscr{P}}} = \mathscr{P}$ and this finishes the proof of Proof of **(d)**. ■

Relations which are reflexive, antisymmetric and transitive like the relation of example 5.3 (set inclusion) allow to compare items for "bigger" and "smaller" or "before" and "after". They also deserve a special name:

Definition 5.4 (Partial Order Relation)**.**

Let R be a relation on a set X.

If R is reflexive, antisymmetric and transitive, it is called a **partial ordering** of X aka **partial order relation** on X. It is customary to write "$x \preceq y$" or "$y \succeq x$" rather than "xRy" for a partial ordering R. We say that "x **before** y" or "y **after** x".

We then call $(X, \preceq)$ a **partially ordered set** aka **POset**. □

Remark 5.4.

The properties of a partial ordering can now be phrased as follows:

(5.2)	$x \preceq x$ for all $x \in X$	**reflexivity**
(5.3)	$x \preceq y$ and $y \preceq x \;\Rightarrow\; y = x$	**antisymmetry**
(5.4)	$x \preceq y$ and $y \preceq z \;\Rightarrow\; x \preceq z$	**transitivity** □

Remark 5.5.

⋆ Some authors, Dudley among them, do not include reflexivity into the definition of a partial ordering and then distinguish between **strict partial orders** and **reflexive partial orders**.

Remark 5.6. Note the following:

(A) According to the above definition, the following are partial orderings of X:

1. $X = \mathbb{R}$ and $x \preceq y$ if and only if $x \leq y$.
2. $X = 2^{\Omega}$ for some set Ω and $A \preceq B$ if and only if $A \subseteq B$ (example 5.3).
3. $X = \mathbb{R}$ and $x \preceq y$ if and only if $x \geq y$.

(B) The following relations are **not** partial orderings of X because none of them is reflexive.

4. $X = \mathbb{R}$ and $x \preceq y$ if and only if $x < y$.
5. $X = 2^{\Omega}$ for some set Ω and $A \preceq B$ if and only if $A \subset B$ (i.e., $A \subseteq B$ but $A \neq B$).
6. $X = \mathbb{R}$ and $x \preceq y$ if and only if $x > y$.

Note that each one of those three relations is antisymmetric. For example, let us look at $x < y$. It is indeed true that the premise $\left[x < y \textbf{ and } y < x\right]$ allows us to conclude that $y = x$ as there are no such numbers x and y and a premise that is known never to be true allows us to conclude anything we want!

(C) An equivalence relation $\sim$ is a never a partial ordering of X except in the very uninteresting case where you have $x \sim y$ if and only if $x = y$.

(D) A partial ordering of X, as any relation on X in general, is inherited by any subset $A \subseteq X$ as follows: Let $\preceq$ be a partial ordering on a set X and let $A \subseteq X$. We define a relation $\preceq_A$ on A as follows: Let $x, y \in A$. Then $x \preceq_A y$ if and only if $x \preceq y$. ☐

What makes a partial ordering more general than the "$x \leq y$" order relation on sets of numbers? The answer: You can compare any two numbers: either $x \leq y$ or $y \leq x$ or $x = y$. Set inclusion on 2^Ω on the other hand does not have this property. For example, if $A = [0, 2]$ and $B = [1, 3]$ then neither $A \subseteq B$ nor $B \subseteq A$ nor $A = B$. We call POsets with the property that any two elements can be compared linearly or totally ordered:

Definition 5.5 (Linear orderings)**.**

⋆

(a) Let $(X, \preceq)$ be a nonempty POset, i.e., $\preceq$ is a partial ordering on X (see Definition 5.4 on p.157). We say that $\preceq$ is a **linear ordering** , also called a **total ordering** of X if and only if, for all x and $y \in X$ such that $x \neq y$, either $x \preceq y$ or $y \preceq x$. We call $(X, \preceq)$ a **linearly ordered set** or a **totally ordered set** set.

(b) Let $(X, \preceq)$ be a nonempty POset and $C \subseteq X$. C is a **chain** in X if $(C, \preceq)$ is linearly ordered (with the same ordering). ☐

Example 5.8.

(a) The real numbers line $(\mathbb{R}, \leq)$ with its usual "$\leq$" ordering is a linearly ordered set. So is $(\mathbb{R}, \geq)$(!)

(b) If X is a set with at least two elements then set inclusion is **not** a linear order on 2^X.

(c) Ordered integral domains $(R, \oplus, \odot, P)$ are totally ordered. ☐

Definition 5.6 (Inverse Relation)**.**

⋆

Let X and Y be two sets and $R \subseteq X \times Y$ a relation on (X, Y). Let

$$R^{-1} := \{ (y, x) : (x, y) \in R \}.$$

Clearly R^{-1} is a subset of $Y \times X$ and hence a relation on (Y, X). We call R^{-1} the **inverse relation** of the relation R. ☐

Example 5.9. Let $R := \{(x, x^3) : x \in \mathbb{R}\}$. Then this relation is the graph Γ_f of the function $y = f(x) = x^3$. We obtain

$$R^{-1} \;=\; \{(x^3, x) : x \in \mathbb{R}\} \;=\; \{(y, y^{1/3}) : y \in \mathbb{R}\}.$$

In other words, R^{-1} is the graph $\Gamma_{f^{-1}}$ of the inverse function $x = f^{-1}(y) = y^{1/3}$. $\square$

5.2 Functions (Mappings) and Families

5.2.1 Some Preliminary Observations about Functions

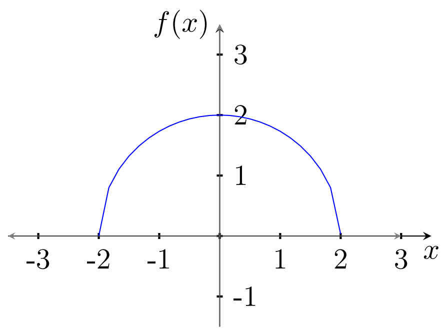
Remark 5.7 (A layman's definition of a function)**.**
We look at the set $\mathbb{R}$ of all real numbers [5] and the function $y = f(x) = \sqrt{4 - x^2}$ which associates with certain real numbers x (the "argument" or "independent variable") another real number $y = \sqrt{4 - x^2}$ (the "function value" or "dependent variable"):

$f(0) = \sqrt{(4-0} = 2,\ f(2) = f(-2) = \sqrt{(4-4} = 0,\ f(2/3) = f(-2/3) = \sqrt{(36-4)/9} = \sqrt{30}/3,\ \dots$

You can think of this function as a rule or law which specifies what item y is obtained as the output or result if the item x is provided as input.

Let us take a closer look at the function $y = f(x) = \sqrt{4 - x^2}$ and its properties:

- **(a)** For some real numbers x there is no function value: For example, if $x = 10$ then $4 - x^2 = -96$ is negative and the square root cannot be taken.
- **(b)** For some other x, e.g., $x = 0$ or $x = 2/3$, there is a function value $f(x)$. A moment's reflection shows that the biggest possible set of potential arguments for our function, called by some authors the **natural domain** of the function (e.g., [2] Brewster/Geoghegan), is the interval $[-2, 2]$. It is customary to write D_f for the natural domain of a function $y = f(x)$.
- **(c)** For a given x there is never more than one function value $f(x)$. This property allows us to think of a function as an assignment rule: It assigns to certain arguments x a unique function value $f(x)$. We observed in **(b)** that $f(x)$ exists if and only if $x \in [-2, 2]$.
- **(d)** Not every $y \in \mathbb{R}$ is suitable as a function value: A square root cannot be negative, hence no x exists such that $f(x) = -1$ or $f(x) = -\pi$.
- **(e)** On the other hand, there are numbers y such as $y = 0$, which are "hit" more than once by the function: $f(2) = f(-2) = 0$. [6]
- **(f)** Graphs as drawings: We are used to look at the graphs of functions. Here is a picture of the graph of $f(x) = \sqrt{4 - x^2}$.



[5]Real numbers were defined informally in ch.**??** (Numbers)
[6]Matter of fact, only for $y = 2$ there exists a single argument x such that $y = f(x)$ ($x = 0$). All other y-values in

(g) Graphs as sets: Drawings as the one above have limited precision (the software should have drawn a perfect half circle with radius 2 about the origin but there seem to be wedges at $x \approx \pm 1.8$). Also, how would you draw a picture of a function which assigns a 3–dimensional vector [7] (x, y, z) to its distance $w = F(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ from the zero vector $(0, 0, 0)$? You would need four dimensions, one each for x, y, z, w, to draw the graph! To express the graph of a function without a picture, let us look at a verbal description: The graph of a function $f(x)$ is the collection of the pairs $(x, f(x))$ for all points x which belong to the set $[-2, 2]$ of potential arguments (see **(a)**). In mathematical parlance: The graph of the function $f(x)$ is the set

$$\Gamma_f \ := \ \{\big(x, f(x)\big) : x \in D_f\}$$

(see remark 5.1 on p. 153). □

We now make adjustments to some of those properties which will get us closer to the definition of a function as it is used in abstract mathematics.

Remark 5.8 (A better definition of a function)**.**

We make the following alterations to remark 5.7.

- ▶ We require an upfront specification of the set A of items that will be allowed as input (arguments) for the function and we require that $y = f(x)$ makes sense for each $x \in A$. Given the function $y = f(x) = \sqrt{4 - x^2}$ from above this means that A must be a subset of $[-2, 2]$.
- ▶ We require an upfront specification of the set B of items that will be allowed as output (function values) for the function. This set must be so big that each $x \in A$ has a function value $y \in B$. We do not mind if B contains redundant y values. For $y = f(x) = \sqrt{4 - x^2}$ any superset of the closed interval $[0, 2]$ will do. We may choose, e.g., $B := [0, 2]$ or $B := [-2, 2\pi]$ or $B := [0, 4]$ or $B := \mathbb{R} \cup \{$ all inhabitants of Chicago $\}$.

Doing so gives us the following: A function consists of three items: a set A of inputs, a set B of outputs and an assignment rule $x \mapsto f(x)$ with the following properties:

(1) For **all** inputs $x \in A$ there is a function value $f(x) \in B$.

(2) For any input $x \in A$ there is never more than one function value $f(x) \in B$. It follows from property 1 that each $x \in A$ uniquely determines its function value $y = f(x)$. This property is what allows us to think of a function as an assignment rule: It assigns to each $x \in A$ a unique function value $f(x) \in B$.

(3) Not every $y \in B$ needs to be a function value $f(x)$ for some $x \in A$, i.e., the set $\{x \in A : f(x) = y\}$ can be empty.

(4) On the other hand there may be numbers y which are "hit" more than once by f.
Example: Let $A := \mathbb{N}$, $B := \mathbb{R}$, $f(x) := (-1)^x$. Then both -1 and 1 are mapped to infinitely often by f.

the interval $[0, 2]$ are "mapped to" by two different arguments $x = \pm\sqrt{4 - y^2}$.

[7] Skip this example on first reading if you do not know about functions of several variables. You will find information about this in chapter **??** ("Vectors and vector spaces") on p.**??**.

(5) The graph Γ_f of a function $f(x)$ is the collection of the pairs $(x, f(x))$ for all points x which belong to the set A, i.e.,

$$\Gamma_f \;:=\; \{\big(x, f(x)\big) : x \in A\}. \tag{5.5}$$

Γ_f has the following properties:

(5a) $\Gamma_f \subseteq A \times B$, i.e., Γ_f is a relation on (A, B) (see Definition 5.2 on p.154).

(5b) For each $x \in A$ there exists a unique $y \in B$ such that $(x, y) \in \Gamma_f$

(5c) If $x \mapsto g(x)$ is another function with inputs A and outputs B which is different from $x \mapsto f(x)$ (i.e., there is at least one $a \in A$ such that $f(a) \neq g(a)$) then the graphs Γ_f and Γ_g do not coincide

(6) Conversely, if A and B are two nonempty sets, then any relation Γ on (A, B) which satifies **5a** and **5b** uniquely determines a function $x \mapsto f(x)$ with inputs A and outputs B as follows: For $a \in A$ we define $f(a)$ to be the element $b \in B$ for which $(a, b) \in \Gamma$. We know from **5b** that such b exists and is uniquely determined. $\square$

Here is a complicated way of looking at the example above: Let $X = [-2, 2]$ and $Y = \mathbb{R}$. Then $y = f(x) = \sqrt{4 - x^2}$ is a rule which "maps" each element $x \in X$ to a uniquely determined number $y \in Y$ which depends on x as follows: Subtract the square of x from 4, then take the square root of that difference.

Mathematicians are very lazy as far as writing is concerned and they figured out long ago that writing "depends on xyz" all the time not only takes too long, but also is aesthetically very unpleasing and makes statements and their proofs hard to understand. They decided to write "(xyz)" instead of "depends on xyz" and the modern notion of a function or mapping $y = f(x)$ was born.

Here is another example: if you say $f(x) = x^2 - \sqrt{2}$, it's just a short for "I have a rule which maps a number x to a value $f(x)$ which depends on x in the following way: compute $x^2 - \sqrt{2}$." It is crucial to understand from which set X you are allowed to pick the "arguments" x and it is often helpful to state what kinds of objects $f(x)$ the x–arguments are associated with, i.e., what set Y they will belong to.

We now are ready to give the precise definition of a function.

5.2.2 Definition of a Function and Some Basic Properties

Introduction 5.1.

Remark 5.8 on p.160 made it plausible that a function can be thought of equivalently as an assignment rule $x \mapsto f(x)$ or as a graph $\Gamma_f \;:=\; \{\big(x, f(x)\big) : x \in A\}$, i.e., as a relation on (X, Y) (see example 5.4 on p.154). Mathematicians prefer the latter approach because "assignment rule" is a rather vague term (an undefined term in the sense of ch. **??** (Building blocks of mathematical theories) on p.**??**) whereas "relation" is entirely defined in the language of sets.

Not every relation Γ on $X \times Y$ is can serve as the graph of a function with domain X and codomain Y since we decided that the following is important:

(a) For each $x \in X$ there must be a function value $f(x)$, i.e., some $y \in Y$ such that $(x, y) \in \Gamma$,

(b) There cannot be more than one such function value $f(x)$, i.e., for each $x \in X$ there must be exactly one $y \in Y$ such that $(x, y) \in \Gamma$. $\square$

The above now leads us to the official definition of a function as a relation which satisfies those properties **(a)** and **(b)**.

Definition 5.7 (Mappings (functions))**.**

Given are two arbitrary nonempty sets X and Y and a relation Γ on (X, Y) (see 5.2 on p.154) which satisfies the following:

$$\text{for each } x \in X \text{ there exists exactly one } y \in Y \text{such that } (x, y) \in \Gamma. \tag{5.6}$$

We call the triplet $f(\cdot) := (X, Y, \Gamma)$ a **function** or **mapping** from X to Y. The set X is called the **domain** or **source** and Y is called the **codomain** or **target** of the mapping $f(\cdot)$. We will usually use the words "domain" and "codomain" in this document.

Usually mathematicians simply write f instead of $f(\cdot)$ We mostly follow that convention, but sometimes include the "$(\cdot)$" part to emphasize that a function rather than an "ordinary"' element of a set is involved. We write Γ_f or $\Gamma(f)$ if we want to stress that Γ is the relation associated with the function $f = (X, Y, \Gamma)$. Let $x \in X$. We write $f(x)$ for the uniquely determined $y \in Y$ such that $(x, y) \in \Gamma$. It is customary to write

$$f : X \to Y, \qquad x \mapsto f(x) \tag{5.7}$$

instead of $f = (X, Y, \Gamma)$ and we henceforth follow that convention. We abbreviate that to $f : X \to Y$ if it is clear or irrelevant how to compute $f(x)$ from x. We read the expression "$a \mapsto b$" as "a is assigned to b" or "a maps to b".

We call Γ the **graph** of the function f. Clearly

$$\Gamma \ = \ \Gamma_f \ = \ \Gamma(f) \ = \ \{(x, f(x)) : x \in X\}. \tag{5.8}$$

We refer to $\mapsto$ as the **maps to operator** or **assignment operator**.
Domain elements $x \in X$ are called **independent variables** or **arguments** and $f(x) \in Y$ is called the **function value** of x. The subset

$$f(X) \ := \{y \in Y : y = f(x) \text{ for some } x \in X\} \ = \ \{f(x) : x \in X\} \tag{5.9}$$

of Y is called the **range** or **image** of the function $f(\cdot)$.
We say "f maps X into Y" and "f maps the domain value x to the function value $f(x)$".
We say that two functions $f = (X, Y, \Gamma)$ and $f' = (X', Y', \Gamma')$ are **equal** if $X = X'$, $Y = Y'$, and $\Gamma = \Gamma'$. Note that $X = X'$ follows from $\Gamma = \Gamma'$ because

$$x \in X \ \Leftrightarrow \ (x, y) \in \Gamma \text{ for some (unique) } y \in Y \ \Leftrightarrow \ (x, y) \in \Gamma' \text{ for some } y \in Y \ \Leftrightarrow \ x \in X'. \ \square$$

Note that the codomain) Y of f and its range $f(X)$ can be vastly different. For example, if $f : \mathbb{R} \to \mathbb{R}$ is given by the assignment $f(x) = \sin(x)$ then $f(\mathbb{R}) = [-1, 1]$ is a very small part of the codomain!

Figure 5.1 on p.163 illustrates the graph of a function as a subset of $X \times Y$.

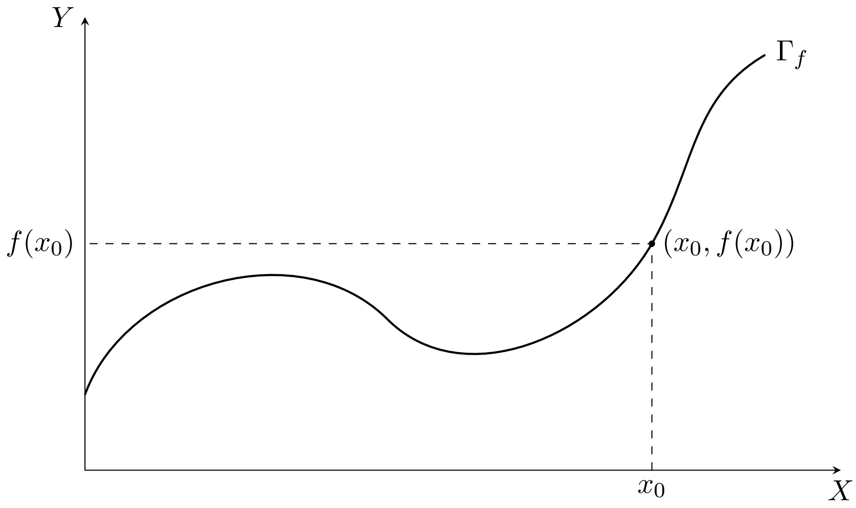


Figure 5.1: Graph of a function.

Remark 5.9. Note that if $Y \subsetneq Y'$ and $f = (X, Y, \Gamma)$ is a function then $f' = (X, Y', \Gamma)$ also is a function: Γ) is a subset of $X \times Y'$ and (5.6) remains valid for Y' in place of Y. But note that the domain X of f is determined by the graph Γ as follows:

$$X \;=\; \{x : (x, y) \in \Gamma \text{ for some } y\}. \;\square$$

Remark 5.10 (Mappings vs. functions)**.** Mathematicians do not always agree 100% on their definitions. The issue of what is called a function and what is called a mapping is subject to debate. Some mathematicians call a mapping a function only if its codomain is a subset of the real numbers, [8] but the majority does what this document tries to adhere to: We use “mapping” and “function” interchangeably and we talk about **real–valued functions** rather than just functions if the codomain is a subset of $\mathbb{R}$ (see (5.16) on p.183). $\square$

Remark 5.11.

The symbol x chosen for the argument of the function is a **dummy variable** in the sense that it does not matter what symbol you use.

The following each define the same function with domain $[0, \infty[$ and codomain $\mathbb{R}$ which assigns to any nonnegative real number its (positive) square root:

$$\begin{aligned} f : [0, \infty[\to \mathbb{R}, &\qquad x \mapsto \sqrt{x}, \\ f : [0, \infty[\to \mathbb{R}, &\qquad y \mapsto \sqrt{y}, \\ f : [0, \infty[\to \mathbb{R}, &\qquad f(\gamma) \;=\; \sqrt{\gamma}. \end{aligned}$$

[8] or if the codomain is a subset of the complex numbers, but we won’t discuss complex numbers in this document.

Matter of fact, not even the symbol you choose for the function matters as long as the operation (here: assign a number to its square root) is unchanged. In other words, the following still describe the same function as above:

$$\begin{aligned} &\varphi : [0,\infty[\to \mathbb{R}, && t \mapsto \sqrt{t},\\ &A : [0,\infty[\to \mathbb{R}, && x \mapsto \sqrt{x},\\ &g : [0,\infty[\to \mathbb{R}, && g(A) \ = \ \sqrt{A}. \end{aligned}$$

In contrast, the following three functions all are different from each other and none of them equals f because domain and/or codomain do not match:

$$\begin{aligned} &\psi :]0,\infty[\ \to \ \mathbb{R}, && x \mapsto \sqrt{x} \quad \text{(different domain)}.\\ &B : [0,\infty[\ \to]0,\infty[, && x \mapsto \sqrt{x} \quad \text{(different codomain)},\\ &h : [0,1[\ \to \ [0,1[, && x \mapsto \sqrt{x} \quad \text{(different domain and codomain)}. \ \square \end{aligned}$$

The next topic is function composition. We have already dealt with its associativity in ch.**??**. See prop.**??** on p.**??**.

Definition 5.8 (Function composition)**.**

Given are three nonempty sets X, Y and Z and two functions $f : X \to Y$ and $g : Y \to Z$. Given $x \in X$ we know the meaning of the expression $g\big(f(x)\big)$:

$y := f(x)$ is the function value of x for the function f, i.e., the unique $y \in Y$ such that $(x, y) \in \Gamma_f$.

$z := g(y) = g\big(f(x)\big)$ is the function value of $f(x)$ for the function g, i.e., the unique $z \in Z$ such that $\big(f(x), z\big) = \big(f(x), g(f(x))\big) \in \Gamma_g$.

The set $\Gamma := \{\big(x, g(f(x)) : x \in X\big)\}$ is a relation on (X, Z) such that

(5.10)

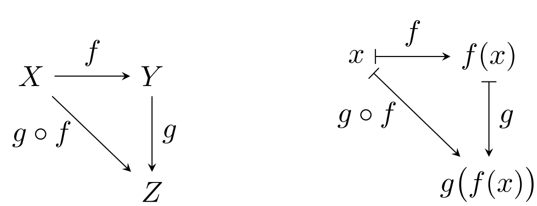
$$\text{for each } x \in X \text{ there exists exactly one } z \in Z, \text{namely, } \ z = g\big(f(x)\big), \text{such that } (x, z) \in \Gamma.$$

It follows that Γ is the graph of a function $h = (X, Z, \Gamma)$ with function values $h(x) = g\big(f(x)\big)$ for each $x \in X$. We call h the **composition** of f and g and we write $h = g \circ f$ (“g after f”).

As far as notation is concerned it is OK to write either of $g \circ f(x)$ or $(g \circ f)(x)$. The additional parentheses may give a clearer presentation if f and/or g are defined by fairly complex formulas. $\square$

The following shows how you diagram the composition of two functions. The left picture shows the domains and codomains for each mapping and the right one shows the element assignments.

(5.11) Function composition



The simplest functions are those that map every domain value to one and the same function value.

Definition 5.9 (Constant functions)**.**

Let Y be a nonempty set and $y_0 \in Y$. You can think of y_0 as a function from any nonempty set X to Y as follows:

$$y_0(\cdot) : X \to Y; \qquad x \mapsto y_0.$$

In other words, the function $y_0(\cdot)$ assigns to each $x \in X$ one and the same value y_0. We call such a function which only takes a single value a **constant function**.

The most important constant function is the **zero function** $0(\cdot)$ which maps any $x \in X$ to the number zero. We usually just write 0 for this function unless doing so would confuse the reader. $\square$

We have a special name for the "do nothing function" which assigns each argument to itself:

Definition 5.10 (identity mapping)**.**

Given any nonempty set X, we use the symbol id_X for the **identity** mapping defined as

$$id_X \,:\, X \to X; \qquad x \mapsto x.$$

We drop the subscript if it is clear what set is referred to. $\square$

5.2.3 Examples of Functions

We now give some examples of functions. You might find some of them rather difficult to understand at first reading.

Example 5.10.
Let $\Gamma := \{(x, x^3) : x \in \mathbb{R}\} \subseteq \mathbb{R} \times \mathbb{R}$. Then $f = (\mathbb{R}, \mathbb{R}, \Gamma)$ is the function

$$f : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^3. \ \square$$

Example 5.11.
Let $\Gamma := \{(x, x^2 + 1) : x \in \mathbb{R}\}$. Then $g = (\mathbb{R}, \mathbb{R}, \Gamma)$ is the function

$$g : \mathbb{R} \to \mathbb{R}, \qquad x \mapsto x^2 + 1. \ \square$$

Example 5.12.
Let $\Gamma := \{(a, \ln(a)) : a \in]0, \infty[\}$. Here $\ln(a)$ denotes the natural logarithm of a. Then $h = (]0, \infty[, \mathbb{R}, \Gamma)$ is the function

$$h :]0, \infty[\to \mathbb{R}, \qquad x \mapsto \ln(x). \ \square$$

Example 5.13.
Let $\Gamma := \{(x, \sqrt{x}) : x \in [0, \infty[\ \}$. Then $\varphi = ([0, \infty[, \mathbb{R}, \Gamma)$ is the function

$$\varphi : [0, \infty[\to \mathbb{R}, \qquad x \mapsto \sqrt{x}.\ \square$$

Example 5.14.
Let $\Gamma := \{(x, \sqrt{x}) : x \in [0, \infty[\ \}$. We can consider Γ as a subset of $[0, \infty[\times\mathbb{R}$ but also as a subset of $[0, \infty[\times[0, \infty[$. In the first case we obtain a function $\varphi = ([0, \infty[, \mathbb{R}, \Gamma)$, i.e., the function

$$\varphi : [0, \infty[\to \mathbb{R}, \qquad x \mapsto \sqrt{x}.$$

In the second case we obtain a different(!) function $\psi = ([0, \infty[, [0, \infty[, \Gamma)$, i.e., the function

$$\psi : [0, \infty[\to [0, \infty[, \qquad x \mapsto \sqrt{x}.\ \square$$

If you have taken multivariable calculus or linear algebra then you know that functions need not necessarily map numbers to numbers but they can also map vectors to numbers, numbers to vectors (curves) or vectors to vectors.

Example 5.15.
We define a function which maps two-dimensional vectors to numbers. Let

$$A := \{\big((x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\ \}, \qquad \Gamma := \{\big((x, y), \sqrt{1 - x^2 - y^2}\big) : (x, y) \in A\}.$$

Then $F = (A, \mathbb{R}, \Gamma)$ is the function

$$F : A \to \mathbb{R}, \qquad (x, y) \mapsto \sqrt{1 - x^2 - y^2}.$$

Note that the domain is not a set of real numbers but of points in the plane and that the graph of F is a set of points (x, y, z) in 3–dimensional space. (It is the upper half of the surface of the three dimensional ball centered at the origin and with radius 1). $\square$

Example 5.16.
We define a function which maps numbers to two-dimensional vectors (a curve in the plane). Let $\Gamma := \{\big(t, (\sin t, \cos t)\big) : t \in \mathbb{R}\ \}$. Then $G = (\mathbb{R}, \mathbb{R}^2, \Gamma)$ is the function

$$G : \mathbb{R} \to \mathbb{R}^2, \qquad t \mapsto (\sin t, \cos t).$$

whose image $G(\mathbb{R}$ is the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ Note that the codomain is not a set of real numbers but the Euclidean plane. $\square$

Example 5.17.
Let $\Gamma := \{\big((x, y), (2x - y/3,\ x/6 + 4y)\big) : x, y \in \mathbb{R}\ \}$. Then $H = (\mathbb{R}^2, \mathbb{R}^2, \Gamma)$ is the function

$$H : \mathbb{R}^2 \to \mathbb{R}^2, \qquad (x, y) \mapsto (2x - y/3,\ x/6 + 4y).$$

Note that both domain and codomain are the Euclidean plane. $\square$

We now reformulate the last example in the framework of linear algebra. Skip this next example if you do not know about matrix multiplication.

Example 5.18.

As is customary in linear algebra we now think of $\mathbb{R}^2$ as the collection of column vectors $\{\begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \mathbb{R}\}$ rather than the cartesian product $\mathbb{R} \times \mathbb{R}$ which is the collection of row vectors $\{(x, y) : x, y \in \mathbb{R}\}$. Let A be the 2×2 matrix

$$A := \begin{pmatrix} 2 & -1/3 \\ 1/6 & 4 \end{pmatrix}.$$

We then obtain for any pair of numbers $\vec{x} = (x, y)^\top$ [9] that

$$A\vec{x} = \begin{pmatrix} 2 & -1/3 \\ 1/6 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x - y/3 \\ x/6 + 4y \end{pmatrix}$$

Let $\Gamma := \{ (\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 2x - y/3 \\ x/6 + 4y \end{pmatrix}) : x, y \in \mathbb{R} \}$. Then $H = (\mathbb{R}^2, \mathbb{R}^2, \Gamma)$ is the function

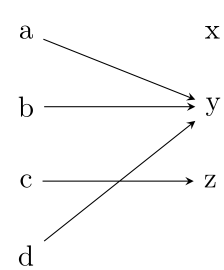
$$H : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Note that both domain and codomain are the Euclidean plane. ☐

If you want to construct a counterexample to a mathematical statement concerning functions it often is best to construct functions with small domain and codomain so that you can draw a picture that completely describes the assignments. The next example will illustrate this.

Example 5.19.

Let $X := \{a, b, c, d\}$, $Y := \{x, y, z\}$, $\Gamma := \{ (a, y), (b, y), (c, z), (d, y)\}$. Then $I = (X, Y, \Gamma)$ is the function which maps the elements of X to Y according to the diagram on the right. Note that nothing was said about the nature of the elements of X and Y. One need not know about it to make observations like the following: Examine items **(3)** and **(4)** of remark 5.8 (A better definition of a function) on p.160. Convince yourself that $x \in Y$ is an example for **(3)**: Not every element of Y needs to be a function value and that $y \in Y$ is an example for **(4)**: There may be elements of Y which are “hit” more than once by the function. ☐



Example 5.20.

[9] Here $(x, y)^\top = \begin{pmatrix} x \\ y \end{pmatrix}$ is the **transpose** of $(x.y)$, i.e., the operation that switches rows and columns of any matrix. In particular it transforms a row vector into a column vector and vice versa.

This example represents a mathematical model for computing probabilities of the outcomes of rolling a fair die and demonstrates that probability can be thought of as a function that maps sets to numbers.

If we roll a die then the outcome will be an integer between 1 and 6, i.e., the "state space" for this random action will be $X := \{1,2,3,4,5,6\}$. For $A \subseteq X$ let $\text{Prob}(A)$ denote the probability that rolling the die results in an outcome $x \in A$.

For example Prob(an even number occurs) $= \text{Prob}\big(\{2,4,6\}\big) = 50\% = 1/2$. Clearly we have for singletons consisting of a single outcome that

$$\text{Prob}\big(\{1\}\big) \;=\; \text{Prob}\big(\{2\}\big) \;=\; \cdots \;=\; \text{Prob}\big(\{6\}\big) \;=\; 1/6 \;=\; 16.\bar{6}\%.$$

Your everyday experience tells you that if $A = \{x_1, x_2, \dots, x_k\}$ where $x_j \in X$ for each index j (and hence $k \leq 6$ because a set does not contain duplicates) then

$$\text{Prob}(A) \;=\; \text{Prob}\big(\{x_1\}\big) + \text{Prob}\big(\{x_2\}\big) + \cdots + \text{Prob}\big(\{x_k\}\big) \;=\; \sum_{j=1}^{k} \text{Prob}\big(\{x_j\}\big).$$

What if A is the event that the roll of the die does not result in any outcome, i.e., $A = \emptyset$? We do not worry about the die getting stuck in mid-air or the dog snatching it before we get a chance to see the outcome and consider this event impossible, i.e., $\text{Prob}(\emptyset) = 0$.

We now have a probability associated with every $A \subseteq X$, i.e., with every $A \in 2^X$ and can finally write this probability as a function. Let $\Gamma \;:=\; \{(A, \text{Prob}(A)) : A \subseteq X\}$. Then $P = (2^X, [0,1], \Gamma)$ is the function

$$P : 2^X \to [0,1], \qquad A \mapsto \text{Prob}(A).$$

Why do we use $[0,1]$ and not $\mathbb{R}$ as the codomain? The answer is that we could have done so but no event has a probablity that exceeds 100% or is negative, so $[0,1]$ is big enough and by choosing this set as the codomain we do not deviate from standard presentation of mathematical probability theory. $\square$

Example 5.21.

In this example we will define a function $I(\cdot)$ for which the domain $\mathscr{F}$ is a set of functions, and the codomain $\mathscr{G}$ is a set of equivalence classes of functions. For the necessary background on antiderivatives see rem.**??** on p.**??**.

Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ and let $X :=]a,b[$ be the open (end points a, b are excluded) interval of all real numbers between a and b. Let $x_0 \in]a,b[$ be "fixed but arbitrary". Let

$$\begin{aligned} \mathscr{F} &:= \{f :]a,b[\to \mathbb{R} \text{ such that } f \text{ is continuous on }]a,b[\ \}, \\ \mathscr{G} &:= \{[g]_\sim : g \text{ is differentiable on }]a,b[\ \ \}, \text{ where } g \sim g' \;\Leftrightarrow\; g - g' = \text{ const.} \end{aligned}$$

We have seen in rem.**??** on p.**??** that for each $f \in \mathscr{F}$ there exists a differentiable function g, unique up to a constant, such that $g' = f$, i.e., g is an antiderivative of f.

Using "$\sim$" and writing $[g]$ for $[g]_\sim$ this can be rephrased as follows: For each $f \in \mathscr{F}$ there exists a unique $[g]_\sim \in \mathscr{G}$, such that g is an antiderivative of f, i.e., $g' = f$.

We now define a function $I : \mathscr{F} \to \mathscr{G}$ by specifying its graph as the set

$$\Gamma \;:=\; \{\big(f, [g]_\sim\big) : f \in \mathscr{F}, [g] \in \mathscr{G}, g' = f \;\}. \;\square$$

Example 5.22.

Compare the following to example 5.21.

Let $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ and let $X :=]a,b[$ be the open (end points a, b are excluded) interval of all real numbers between a and b. Let $x_0 \in]a,b[$ be "fixed but arbitrary". For example, we could choose $x_0 := \frac{a+b}{2}$. Let

$$\begin{aligned} \mathscr{F} &:= \{f : f \text{is a real–valued function with domain }]a,b[\ \}, \\ \mathscr{C} &:= \{f :]a,b[\to \mathbb{R} \text{ such that } f \text{ is continuous on }]a,b[\ \}, \\ \mathscr{D} &:= \{f :]a,b[\to \mathbb{R} \text{ such that } g \text{ is differentiable on }]a,b[\ \}. \end{aligned}$$

Note that $\mathscr{D} \subseteq \mathscr{C} \subseteq \mathscr{F}$ because differentiable functions are continuous. We define the following equivalence relation on $\mathscr{D}$: $f \sim g \ \Leftrightarrow\ f - g =$ const. [10] Let

$$\mathscr{A} := \{[f] : f \in \mathscr{D}\ \}$$

be the set of all equivalence classes of differentiable functions on $]a,b[$. Then

$$I : \mathscr{C} \to \mathscr{A}; \quad f \mapsto [I(f)] \text{ where } I(f) :]a,b[\to \mathbb{R} \text{ is the function } x \mapsto I(f)(x) := \int_{x_0}^{x} f(u)du,$$

is a function whose domain $\mathscr{C}$ is a set of functions and whose codomain $\mathscr{A}$ is a set of equivalence classes (i.e., sets(!)) of functions. ☐

5.2.4 A First Look at Direct Images and Preimages of a Function

Introduction 5.2. We continue with yet another example. It leads to the very important definition of the direct images of subsets of the domain, and of the preimages of subsets of the codomain of a function. ☐

Example 5.23. Let X and Y be nonempty sets and $f : X \to Y$. We define two functions $f_\star$ and $f^\star$ which are associated with f and for which both arguments and function values are sets(!) as follows.

(a) $f_\star : 2^X \to 2^Y; \qquad A \mapsto f_\star(A) := \{f(a) : a \in A\}$,
(b) $f^\star : 2^Y \to 2^X; \qquad B \mapsto f^\star(B) := \{x \in X : f(x) \in B\}$.

You should convince yourself that indeed $f_\star$ maps any subset of X to a subset of Y, and that $f^\star$ maps any subset of Y to a subset of X. ☐

The sets $f_\star(A)$ and $f^\star(B)$ are used pervasively in abstract mathematics, but it is much more common nowadays to write $f(A)$ rather than $f_\star(A)$ and $f^{-1}(B)$ rather than $f^\star(B)$. We will follow this convention.

Definition 5.11.

[10] Note that $f \sim g \ \Leftrightarrow\ f - g =$ const also defines equivalence relations on the supersets $\mathscr{C}$ and $\mathscr{F}$.

Let X, Y be two nonempty sets and $f : X \to Y$. We associate with f the functions

$$f : 2^X \to 2^Y; \qquad A \mapsto f(A) := \{f(a) : a \in A\}, \tag{5.12}$$

$$f^{-1} : 2^Y \to 2^X; \qquad B \mapsto f^{-1}(B) := \{x \in X : f(x) \in B\}. \tag{5.13}$$

We call $f : 2^X \to 2^Y$ the **direct image function** and $f^{-1} : 2^Y \to 2^X$ the **indirect image function** or **preimage function** associated with $f : X \to Y$.
For each $A \subseteq X$ we call $f(A)$ the **direct image** of A under f, and for each $B \subseteq Y$ we call $f^{-1}(B)$ the **indirect image** or **preimage** of B under f. ☐

Note that the range $f(X)$ of f (see (5.9) on p.162 is a special case of a direct image.

Notational conveniences I:

If we have a set that is written as $\{\dots\}$ then we may write $f\{\dots\}$ instead of $f(\{\dots\})$ and $f^{-1}\{\dots\}$ instead of $f^{-1}(\{\dots\})$. Specifically for singletons $\{x\} \subseteq X$ and $\{y\} \subseteq Y$ we obtain $f\{x\}$ and $f^{-1}\{y\}$.
Many mathematicians will write $f^{-1}(y)$ instead of $f^{-1}\{y\}$ but this author sees no advantages doing so whatsoever. There seemingly are no savings with respect to time or space for writing that alternate form but we are confounding two entirely separate items: a <u>subset</u> $f^{-1}\{y\}$ of X v.s. the function value $f^{-1}(y)$ of $y \in Y$ which is an <u>element</u> of X. We are allowed to talk about the latter only in case that the inverse function f^{-1} of f exists.

The same symbol f is used for the original function $f : X \to Y$ and the direct image function $f : 2^X \to 2^Y$, and the symbol f^{-1} which is used here for the indirect image function $f^{-1} : 2^Y \to 2^X$ will be used at the start of ch.5.2.5 to define the inverse function $f^{-1} : Y \to X$ of f in case this can be done. [11] Be careful not to let this confuse you! ☐

Example 5.24 (Direct images)**.** Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$.

(a) $f(]-4,-2[) = \{\, x^2 : x \in]-4,-2[\,\} = \{\, x^2 : -4 < x < -2 \,\} =]4,16[$.
(b) $f([1,2]) = \{\, x^2 : x \in [1,2] \,\} = \{\, x^2 : 1 \le x \le 2 \,\} = [1,4]$.
(c) $f([5,6]) = \{\, x^2 : x \in [5,6] \,\} = \{\, x^2 : 5 \le x \le 6 \,\} = [25,36]$.
(d) $f(]-4,-2[\cup [1,2] \cup [5,6]) = \{\, x^2 : x \in]-4,-2[\text{ or } x \in [1,2] \text{ or } x \in [5,6] \,\}$
$=]4,16[\cup [1,4] \cup [25,36] = [1,16[\cup [25,36]$. ☐

Example 5.25 (Direct images)**.** Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$.

(a) $f(]-4,2[) = \{\, x^2 : x \in]-4,2[\,\} = \{\, x^2 : -4 < x < 2 \,\} =]4,16[$.
(b) $f([1,3]) = \{\, x^2 : x \in [1,3] \,\} = \{\, x^2 : 1 \le x \le 3 \,\} = [1,9]$.
(c) $f(]-4,2[\cap [1,3]) = \{\, x^2 : x \in]-4,2[\text{ and } x \in [1,3] \,\} = \{\, x^2 : 1 \le x < 2 \,\} = [1,4[$.
☐

And here are the results for the preimages of the same sets with respect to the same function $x \mapsto x^2$.

Example 5.26 (Preimages)**.** Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$. Determine
a. $f^{-1}(]-4,-2[)$, **b.** $f^{-1}([1,2])$, **c.** $f^{-1}([5,6])$, **d.** $\{-4 < f < -2$ **or** $1 \leq f \leq 2$ **or** $5 \leq f < 6]\}$.

Solution:
a. $f^{-1}(]-4,-2[) = \{ x \in \mathbb{R} : x^2 \in]-4,-2[\} = \{ -4 < f < -2 \} = \emptyset$.
b. $f^{-1}([1,2]) = \{ x \in \mathbb{R} : x^2 \in [1,2] \} = \{ 1 \leq f \leq 2 \} = [-\sqrt{2},-1] \cup [1,\sqrt{2}]$.
c. $f^{-1}([5,6]) = \{ x \in \mathbb{R} : x^2 \in [5,6] \} = \{ 5 \leq f \leq 6 \} = [-\sqrt{6},-\sqrt{5}] \cup [\sqrt{5},\sqrt{6}]$.
d. $\{-4 < f < -2$ **or** $1 \leq f \leq 2$ **or** $5 \leq f < 6]\} = f^{-1}(]-4,-2[\cup [1,2] \cup [5,6])$
$= \{ x \in \mathbb{R} : x^2 \in]-4,-2[$ or $x^2 \in [1,2]$ **or** $x^2 \in [5,6] \}$
$= [-\sqrt{2},-1] \cup [1,\sqrt{2}] \cup [-\sqrt{6},-\sqrt{5}] \cup [\sqrt{5},\sqrt{6}]$. □

Example 5.27 (Preimages)**.** Let $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$. Determine
a. $f^{-1}(]-4,2[)$, **b.** $f^{-1}([1,3])$, **c.** $\{-4 < f < 2$ **and** $1 \leq f \leq 3\}$.

Solution:
a. $f^{-1}(]-4,2[) = \{ x \in \mathbb{R} : x^2 \in]-4,2[\} = \{ x \in \mathbb{R} : -4 < x^2 < 2 \} =]-\sqrt{2},\sqrt{2}[$.
b. $f^{-1}([1,3]) = \{ x \in \mathbb{R} : x^2 \in [1,3] \} = \{ x \in \mathbb{R} : 1 \leq x^2 \leq 3 \} = [-\sqrt{3},-1] \cup [1,\sqrt{3}]$.
c. $\{-4 < f < 2$ **and** $1 \leq f \leq 3\} = f^{-1}(]-4,2[\cap [1,3])$
$= \{ x \in \mathbb{R} : x^2 \in]-4,2[$ **and** $x^2 \in [1,3] \}$
$= \{ x \in \mathbb{R} : 1 \leq x^2 < 2 \} =]-\sqrt{2},-1] \cup [1,\sqrt{2}[$. □

Remark 5.12 (Notational conveniences II:)**.**

In probability theory the following notation is also very common:
$\{f \in B\} := f^{-1}(B)$, $\{f = y\} := f^{-1}\{y\}$.
Let $\mathscr{R}$ be either of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. Assume that the codomain of f is considered a subset of $\mathscr{R}$. Let $a, b \in \mathscr{R}$ such that $a < b$. We write $\{a \leq f \leq b\} := f^{-1}([a,b]_{\mathscr{R}})$, $\{a < f < b\} := f^{-1}(]a,b[_{\mathscr{R}})$, $\{a \leq f < b\} := f^{-1}([a,b[_{\mathscr{R}})$, $\{a < f \leq b\} := f^{-1}(]a,b]_{\mathscr{R}})$, $\{f \leq b\} := f^{-1}(]-\infty,b]_{\mathscr{R}})$, etc. □

Proposition 5.3.

Some simple properties:

(5.14) $f(\emptyset) = f^{-1}(\emptyset) = \emptyset$

(5.15) $A_1 \subseteq A_2 \subseteq X \Rightarrow f(A_1) \subseteq f(A_2)$ (***monotonicity of*** $f\{\dots\}$)

(5.16) $B_1 \subseteq B_2 \subseteq Y \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$ (***monotonicity of*** $f^{-1}\{\dots\}$)

(5.17) $x \in X \Rightarrow f(\{x\}) = \{f(x)\}$

(5.18) $f(X) = Y \Leftrightarrow f$ *is "surjective"* (*see def.*5.12 *on p.*172)

(5.19) $f^{-1}(Y) = X$ *always!*

PROOF: Left as exercise **??** on p.**??**. ■

5.2.5 Injective, Surjective and Bijective functions

Introduction 5.3.

Given two nonempty sets X and Y we did not find every relation $\Gamma \subseteq X \times Y$ suitable to serve as the graph of a function $X \to Y$: We demanded that for each $x \in X$ there should be one and only one $y \in Y$ suitable as a function value, i.e., there should be one and only one $y \in Y$ such that $y \in \Gamma$. The example $f : \mathbb{R} \to \mathbb{R};\ \ x \mapsto x^2$ demonstrates that this relationship between domain elements $x \in X$ and codomain elements $y \in Y$ is not symmetric: One can find $y \in \mathbb{R}$ for which zero elements $x \in \mathbb{R}$ can be found such that $(x, y) \in \Gamma_f$: that would be all negative numbers y. Moreover there also are many $y \in \mathbb{R}$ for which more than one $x \in \mathbb{R}$ exists which is mapped to y: If $y > 0$ then both $\sqrt{y}$ and $-\sqrt{y}$ have y as function value.

Let X, Y be two nonempty sets and $f : X \to Y$ an arbitrary function with domain X and codomain Y. Restricting the domain of f to a small enough subset $A \subseteq X$ may have the effect that the resulting function f' possesses at most one $(x, y) \in \Gamma_{f'}$ whenever $x \in A$. We will call such functions injective. Also, restricting the codomain of f to a small enough subset $B \subseteq Y$ may result in a function f'' which satisfies the following: For each $y \in B$ there exists at least one $x \in X$ such that $(x, y) \in \Gamma_{f''}$. We will call such functions surjective.

We demonstrate this by using again the function $f : \mathbb{R} \to \mathbb{R};\ \ x \mapsto x^2$ as an example. If we restrict its domain $\mathbb{R}$ to $[0, \infty[$ or any nonempty subset thereof then the resulting function will be injective, and if we restrict its codomain $\mathbb{R}$ to $f(\mathbb{R}) = [0, \infty[$ (the range of f) then we wills say of the resulting function that it is surjective. □

The above leads to the following definition.

Definition 5.12 (Surjective, injective, bijective)**.**

Let $f : X \to Y$, with graph Γ_f.

a. Surjectivity: It need not be true that $f(X) = \{f(x) : x \in X\}$ equals the entire codomain Y, i.e., that

(5.20) for each $y \in Y$ there exists at least one $x \in X$ such that $(x, y) \in \Gamma_f$.

But if $f(X) = Y$, i.e., if (5.20) holds, we call f **surjective** aka **surjection**. aka **onto function**. We also say that f maps X **onto** Y.

b. Injectivity: It need not be true that if $y \in f(X)$, then $y = f(x)$ for a unique x, i.e., that if there is another $x_1 \in X$ such that also $y = f(x_1)$ then it follows that $x_1 = x$. But if this is the case, i.e., if

(5.21) for each $y \in Y$ there exists at most one $x \in X$such that $(x, y) \in \Gamma_f$.

then we call f **injective** aka **injection** aka **one to one** function.

We can express (5.21) also as follows: If $x, x_1 \in X$ and $y \in Y$ are such that $(x, y) \in \Gamma_f$ and $(x_1, y) \in \Gamma_f$ then it follows that $x_1 = x$.

c. Bijectivity: Let $f : X \to Y$ be both injective and surjective. Such a function is called **bijective**, aka **bijection**. We often write $f : X \xrightarrow{\sim} Y$ for a bijective function f.

It follows from (5.20) and (5.21) that f is bijective if and only if

$$\text{for each } y \in Y \text{ there exists exactly one } x \in X \text{such that } (x, y) \in \Gamma_f. \tag{5.22}$$

We rewrite (5.22) by employing Γ_f's inverse relation $\Gamma_f^{-1} = \{(y, x) : (x.y) \in \Gamma\}$ (see def. 5.6 on p.158) and obtain

$$\text{for each } y \in Y \text{ there exists exactly one } x \in X \text{such that } (y, x) \in \Gamma_f^{-1}. \tag{5.23}$$

But this implies, according to (5.6), that Γ_f^{-1} is the graph of a function $g := (Y, X, \Gamma_f^{-1})$ with domain Y and codomain X where, for a given $y \in Y$, $g(y)$ stands for the uniquely determined $x \in X$ such that $(y, x) \in \Gamma_f^{-1}$. Note that

$$\Gamma_f^{-1} \ = \ \Gamma_g. \tag{5.24}$$

We call g the **inverse mapping** or **inverse function** of f and write f^{-1} instead of g. ☐

Notation 5.1.

We will occasionally use special arrow symbols to give a visual clue about injectivity, surjectivity and bijectivity of a function.

- **a)** $f : X \twoheadrightarrow Y$ and $X \overset{f}{\twoheadrightarrow} Y$ indicate that the function f is surjective,
- **b)** $f : X \rightarrowtail Y$ and $X \overset{f}{\rightarrowtail} Y$ indicate that the function f is injective,
- **c)** $f : X \xrightarrow{\sim} Y$ and $f : X \xrightarrow{\cong} Y$ indicate that the function f is bijective. ☐

Moreover, $X \cong Y$ implies that there exists a bijection between the sets X and Y.

Remark 5.13.

(a) It follows from (5.24) that

$$\Gamma_f^{-1} \ = \ \Gamma_{f^{-1}}. \tag{5.25}$$

(b) Each $x \in X$ is mapped to $y = f(x)$ which is the only element of Y such that $f^{-1}(y) = x$,

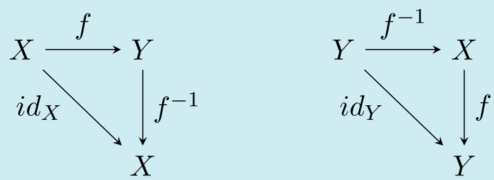
(c) Each $y \in Y$ is mapped to $x = f^{-1}(y)$ which is the only element of X such that $f(x) = y$.

(d) It follows from **(b)** and **(c)** that

(5.26) $$\text{if } x \in X, y \in Y \text{ then } f(x) = y \;\Leftrightarrow\; x = f^{-1}(y).$$

(e) It also follows from **(b)** and **(c)** that $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$.
In other words, $f^{-1} \circ f = id_X$ and $f \circ f^{-1} = id_Y$. Here is the picture:

(5.27) Inverse function:



□

Theorem 5.1 (Characterization of inverse functions)**.**

Let X and Y be nonempty sets and $f : X \to Y$. The following are equivalent:
- ***(a)*** *f is bijective.*
- ***(b)*** *There exists $g : Y \to X$ such that both $g \circ f = id_X$ and $f \circ g = id_Y$.*

PROOF of **(a)** ⇒ **(b)**: We have seen in part **(e)** of remark 5.13 that $g := f^{-1}$ satisfies **(b)**.
PROOF of **(b)** ⇒ **(a)**: We must show that f is both surjective and injective. First we show that f is surjective. Let $y \in Y$. we must find some $x \in X$ such that $f(x) = y$. Let $x := g(y)$. Then

$$f(x) \;=\; f\big(g(y)\big) \;=\; f \circ g(y) \;=\; id_Y(y) \;=\; y.$$

We have $f(x) = y$ and this proves surjectivity. Now we show that f is injective. Let $x_1, x_2 \in X$ and $y \in Y$ such that $f(x_1) = f(x_2) = y$. We are done if we can prove that $x_1 = x_2$. We have

$$x_1 = id_X(x_1) = g \circ f(x_1) = g\big(f(x_1)\big) = g(y) = g\big(f(x_2)\big) = g \circ f(x_2) = id_X(x_2) = x_2,$$

i.e., $x_1 = x_2$. This proves injectivity of f. ■

Example 5.28 (Bijective functions)**.**

- **(a)** Let $R = (R, \oplus, \odot)$ be an integral domain and let $a \in R$. Then the function $\varphi : R \to R;\ x \mapsto x \oplus a$ is bijective since it has the function $\varphi^{-1} : R \to R;\ y \mapsto y \ominus a$ as an inverse.
- **(b)** Let $R = (R, \oplus, \odot)$ be an integral domain. Then the function $\psi : R \to R;\ x \mapsto \ominus x$ is bijective since it has the function $\psi^{-1} : R \to R;\ y \mapsto \ominus y$ as an inverse. Note that $\psi^{-1} = \psi$!

(c) Let $G := \{f : \mathbb{R} \to \mathbb{R} : f(x) = ax + b$ for some $a, b \in \mathbb{R}$ where $a \neq 0\}$ of all polynomials of degree 1. We computed in prop.**??** on p.**??** for each element $f : \mathbb{R} \to \mathbb{R};\ f(x) = ax + b$ of G its inverse f^{-1} as the function $y \mapsto \frac{1}{a}y - \frac{b}{a}$. Thus each element $f \in G$ is a bijection $\mathbb{R} \xrightarrow{\sim} \mathbb{R}$.

(d) Let X be a nonempty set and let

$$\mathfrak{E} := \{\sim : \sim \text{ is an equivalence relation on } X\}, \quad \mathfrak{P} := \{\mathscr{P} : \mathscr{P} \text{ is a partition of } X\}.$$

In prop.5.2 on p.156 we associated with an equivalence relation $\sim$ the partition $\mathscr{P}_\sim = \{[x]_\sim : x \in X$ of its equivalence classes, and we associated with a partition $\mathscr{P}$ of X the equivalence relation $\sim_\mathscr{P}$ on X defined as $x \sim_\mathscr{P} y \Leftrightarrow x, y$ belong to the same element of $\mathscr{P}$.

With those notations let $\varphi : \mathfrak{E} \to \mathfrak{P}$ be defined as $\varphi(\sim) := \mathscr{P}_\sim$, and let $\psi : \mathfrak{P} \to \mathfrak{E}$ be defined as $\psi(\mathscr{P}) := \sim_\mathscr{P}$. We saw in prop.5.2**(c)** that $\sim_{\mathscr{P}_\sim} = \sim$, i.e., that $\psi(\varphi(\sim)) = \sim$ for any $\sim \in \mathfrak{E}$. We further saw in prop.5.2**(d)** that $\mathscr{P}_{\sim_\mathscr{P}} = \mathscr{P}$, i.e., that $\varphi(\psi(\mathscr{P})) = \mathscr{P}$ for any $\mathscr{P} \in \mathfrak{P}$. This allows us to restate parts **(c)** and **(d)** of prop.5.2 as follows: The function φ defines a bijection $\mathfrak{E} \xrightarrow{\sim} \mathfrak{P}$, and ψ is the inverse of φ. ☐

Remark 5.14. [Horizontal and vertical line tests] Let X and Y be nonempty sets and $f : X \to Y$. The following needs to be taken with a grain of salt because X and Y need not be sets of real numbers. Let $R \subseteq X \times Y$.

(a) (5.6) on p.162 states that R is the graph of a function with domain X and codomain Y if and only if it passes the "vertical line test": Any "vertical line", i.e., any subset of $X \times Y$ of the form $V(x_0) := \{(x_0, y) : y \in Y\}$ for a fixed $x_0 \in X$ intersects R in **exactly one** point.

(b) (5.20) on p.172 states that R is the graph of a surjective function with domain X and codomain Y if and only if it passes in addition to the vertical line test the following "horizontal line test": any "horizontal line", i.e., any subset of $X \times Y$ of the form $H(y_0) := \{(x, y_0) : x \in X\}$ for a fixed $y_0 \in Y$ intersects R in **at least one** point.

(c) (5.21) on p.172 states that R is the graph of an injective function with domain X and codomain Y if and only if it passes in addition to the vertical line test the following horizontal line test: any "horizontal line", i.e., any subset of $X \times Y$ of the form $H(y_0) := \{(x, y_0) : x \in X\}$ for a fixed $y_0 \in Y$ intersects R in **at most one** point.

(d) It follows from (5.22) on p.173 but also from the above that that R is the graph of a bijective function with domain X and codomain Y if and only if it passes in addition to the vertical line test the following horizontal line test: any "horizontal line", i.e., any subset of $X \times Y$ of the form $H(y_0) := \{(x, y_0) : x \in X\}$ for a fixed $y_0 \in Y$ intersects R in **exactly one** "point". Note the symmetry between this test and the one for vertical lines. The above is another indication that the inverse graph R^{-1} of a bijective function is a graph of a function (the inverse function f^{-1}). ☐

Proposition 5.4. *Let $(R, \oplus, \odot, P)$ be an ordered integral domain*

(A) *Let $b \in R$. Then the function*

$$T : R \to R; \quad x \mapsto x \oplus b,$$

is a bijection.

(B) *Let $a \in R, a \neq 0$. Then the function*

$$D : R \to a \odot R; \quad x \mapsto a \odot x,$$

is a bijection. (As usual, $a \odot R = aR = \{a \odot r : r \in R\}$.)

The proof is left as exercise 5.11 (see p.195). ■

Remark 5.15.

Abstract math is about proving theorems and propositions. Functions are very important tools for many proofs, and in many instances it is very important to know or to show that a certain function is injective or surjective or both. But these properties depend on the choice of domain and codomain, and for this reason domain and codomain are very important for the complete specification of a function.

Here is a simple example.

Let $f : A \to B$ be the function $f(x) := x^2$.

$A = \mathbb{R}, B = \mathbb{R}$:	f is neither injective nor surjective
$A =]-2, 3[, B = [0, 9[$:	f is surjective but not injective
$A =]0, 3[, B = [0, 9]$:	f is injective but not surjective
$A =]0, 3[, B =]0, 9[$:	f is bijective □

Proposition 5.5.

Let $X, Y, Z \neq \emptyset$. Let $f : X \to Y$ and $g : Y \to Z$.

(a) *If both f, g are injective then $g \circ f$ is injective.*

(b) *If both f, g are surjective then $g \circ f$ is surjective.*

(c) *If both f, g are bijective then $g \circ f$ is bijective.*

The proof of **(a)** and **(b)** is left as exercise 5.9 on p.195.

PROOF of **(c)**: Follows from **(a)** and **(b)** because bijective = injective + surjective. ■

Corollary 5.1.

Let $X, Y, Z \neq \emptyset$. Let $f : X \to Y$ and $g : Y \to Z$.

(a) *If f is bijective and g is injective then both $g \circ f$ and $f \circ g$ are injective.*

(b) *If f is bijective and g is surjective then both $g \circ f$ and $f \circ g$ are surjective.*

(c) *If f is bijective and g is bijective then both $g \circ f$ and $f \circ g$ are bijective.*

PROOF:

(a) follows from prop.5.5**(a)** because bijective functions are injective.

(b) follows from prop.5.5**(b)** because bijective functions are surjective.

(c) follows from prop.5.5**(c)**. ■

The following proposition is easy to prove and will be used when we compare the sizes of sets later on.

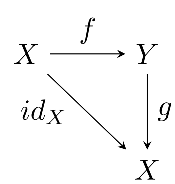
Proposition 5.6.

> ★ *Let X be an arbitrary set and let A be a nonempty proper subset of X. so that $X = A \uplus A^{\complement}$ is a partitioning of X into two nonempty subsets A and $A^{\complement}$. Let $a \in A$, $a_0 \in A^{\complement}$ and $A' := (A \setminus \{a\}) \uplus \{a_0\}$. Then the function*
>
> $$\varphi : A' \xrightarrow{\sim} A; \qquad x \mapsto \begin{cases} x & if x \neq a_0, \\ a & if x = a_0 \end{cases}$$
>
> *is a bijection.*

PROOF: The proof is left as exercise 5.10. ■

We now examine conditions under which there are functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$, i.e.,

(5.28) $$g\big(f(x)\big) = x \text{ for all } x \in X :$$



Proposition 5.7.

> *Let $X, Y \neq \emptyset$. Let $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$. Then*
> **(a)** *f is injective,*
> **(b)** *g is surjective.*

PROOF of **(a)**: Let $x_1, x_2 \in X$. If $f(x_1) = f(x_2)$ then

$$x_1 = id_X(x_1) = g(f(x_1)) = g(f(x_2)) = id_X(x_2) = x_2.$$

This proves injectivity of f.

PROOF of **(b)**: Let $x_0 \in X$. Let $y := f(x_0)$. Then $g(y) = g(f(x_0)) = g \circ f(x_0) = x_0$. We found for an arbitrary x_0 in the codomain of g some y which maps to x_0. This proves surjectivity of g. ■

Proposition 5.8.

Let $X, Y \neq \emptyset$.
- ***(a)*** *Let $f : X \to Y$. If f is injective then there exists $g : Y \to X$ such that $g \circ f = id_X$ and any such function g is necessarily surjective.*
- ***(b)*** *Let $g : Y \to X$. If g is surjective then there exists $f : X \to Y$ such that $g \circ f = id_X$ and any such function f is necessarily injective.*

PROOF of **(a)**: Let $Y' := f(X)$ and

$$f' : X \to Y', \qquad x \mapsto f(x),$$

i.e., $f(x) = f'(x)$ for all $x \in X$. The only difference between f and f' is that we shrunk the codomain from Y to $f(X)$, thus making f' not only injective but also surjective, hence bijective. It follows that the inverse $(f')^{-1} : Y' \to X$ exists.

Let x_0 be an arbitrary, but fixed, element of X. We define $g : Y \to X$ as follows.

$$g(y) \ := \begin{cases} (f')^{-1}(y) & \text{if } y \in Y', \\ x_0 & \text{if } y \notin Y'. \end{cases}$$

Let $x \in X$. Then $f(x) \in Y'$, hence $g \circ f(x) = g \circ f'(x) = (f')^{-1}\big(f'(x)\big) = x$. As x was an arbitrary element of x, this proves $g \circ f = id_X$. We observe that g is surjective according to prop.5.7**(b)**.

PROOF of **(b)**: If $x \in X$ then the surjectivity of g implies that $g^{-1}\{x\} \neq \emptyset$. We thus can associate with each $x \in X$ some $y_x \in g^{-1}\{x\}$. [12]

Let $f : X \to Y$ be the function $x \mapsto y_x$ described by the above association. If $x \in X$ then

$$g \circ f(x) \ = \ g(y_x) \ = \ x.$$

The first equality follows from the definition of f and the second one is true because $y_x \in g^{-1}\{x\}$. It follows from prop.5.7**(a)** that f is injective. ■

There are special names for functions f and g which are related by (5.28).

Definition 5.13 (Left inverses and right inverses)**.**

Let $X, Y \neq \emptyset$. Let $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = id_X$. We say that
- **(a)** f possesses a **left inverse**,
- **(b)** g is a **left inverse** of f,
- **(c)** g possesses a **right inverse**,
- **(d)** f is a **right inverse** of g. □

[12] The ability to do such selections $y_x \in g^{-1}\{x\}$ regardless of the nature of X, Y and of the surjective function $g : Y \to X$ is not something one can prove. It requires acceptance of the **Axiom of Choice**. See Chapter 5.3 (optional) in which a complete proof is given that the Axiom of Choice is equivalent to the existence of $f : X \to Y$ such that $g \circ f = id_X$ for any surjective $g : Y \to X$. See also Remark **??** on p.**??** in ch.**??** (Applications of Zorn's Lemma).

Remark 5.16. There is no good way to remember which function in the composition of f and g is/has a left inverse and which one is/has a right inverse since the order of f and g in the expression $g \circ f$ is reversed in the expression $X \xrightarrow{f} Y \xrightarrow{g} Z$. The author's suggestion:

- f has (g as) a left inverse since g is to the left of f in the expression $g \circ f$,
- g has (f as) a right inverse since f is to the right of g in the expression $g \circ f$. $\square$

We combine the definition of left/right inverses with the preceding two proposition and obtain

Theorem 5.2.

Let $X, Y \neq \emptyset$.

(a) *Let $f : X \to Y$. Then f is injective $\Leftrightarrow$ f has a left inverse (which is necessarily surjective).*

(b) *Let $g : Y \to X$. Then g is surjective $\Leftrightarrow$ g has a right inverse (which is necessarily injective).*

(c) *An injection $X \to Y$ exists $\Leftrightarrow$ a surjection $Y \to X$ exists.*

PROOF of **(a)** $\Rightarrow$): prop.5.8**(a)**.
PROOF of **(a)** $\Leftarrow$): prop.5.7**(a)**.

PROOF of **(b)** $\Rightarrow$): prop.5.8**(b)**.
PROOF of **(b)** $\Leftarrow$): prop.5.7**(b)**.

PROOF of **(c)** $\Rightarrow$): Let $f : X \to Y$ be injective. According to part **(a)** there exists a left inverse $g : Y \to X$ and this function is surjective
PROOF of **(c)** $\Leftarrow$): Let $g : Y \to X$ be surjective. According to part **(b)** there exists a right inverse $f : X \to Y$ and this function is injective $\blacksquare$

Remark 5.17. Let X and Y be two nonempty sets. No assumptions are made concerning how X and Y might be related.

(a) Let $y_0 \in Y$. Then the function

$$f : X \xrightarrow{\sim} \{y_0\} \times X; \quad x \mapsto (y_0, x) \tag{5.29}$$

is bijective because f has the function $(y_0, x) \mapsto x$ as an inverse.

(b) Let u, v elements of some set. An injection/surjection/bijection $X \to Y$ exists if and only if an injection/surjection/bijection $\{u\} \times X \to \{v\} \times Y$ exists.

(c) Let u, v elements of some set such that $u \neq v$. Then the sets $\{u\} \times X$ and $\{v\} \times Y$ are disjoint. $\square$

5.2.6 Binary Operations and Restrictions and Extensions of Functions

Introduction 5.4.

When we defined groups, integral domains and other algebraic structures in ch.?? (The Axiomatic Method) we made use of binary operations such as "$\odot$" which assign to any two elements x and y of such an algebraic structure $\mathfrak{A}$ a third element $z \in \mathfrak{A}$, and also of the "unary operations $\ominus$ and $\cdot^{-1}$" which assign to $x \in \mathfrak{A}$ its inverse $\ominus x \in \mathfrak{A}$ or $x^{-1} \in \mathfrak{A}$ if it exists.

Beside formalizing these notions we will also define restrictions of functions to subsets of their domain and extensions of functions to supersets of their domain. We have previously discussed in the introduction to ch.5.2.5 (Injective, Surjective and Bijective functions) that confining a function to a smaller domain may make that restriction injective. $\square$

We start with the formal definition of unary and binary operations as functions.

Definition 5.14 (Binary and unary operations)**.**

⋆ Let X be a nonempty set.

A **binary operation** on X is a function

$$\diamond : X \times X \longrightarrow X; \quad (x, y) \mapsto x \diamond y := \diamond(x, y). \tag{5.30}$$

A **unary operation**, on X is a function

$$\bullet : X \longrightarrow X; \quad x \mapsto \bullet(x). \ \square \tag{5.31}$$

One often writes $x^\bullet$ or $\bullet x$ instead of $\bullet(x)$. For example, $-x$ instead of $-(x)$ and x^{-1} rather than $^{-1}(x)$.

Example 5.29. The following are examples of binary operations.

(a) Addition on $X = \mathbb{N}$ or $X = \mathbb{Z}$ or $X = \mathbb{Q}$ or $X = \mathbb{R}$ is a binary operation

$$+ : X \times X \longrightarrow X; \quad (x, y) \mapsto x + y. \tag{5.32}$$

(b) Multiplication on $X = \mathbb{N}$ or $X = \mathbb{Z}$ or $X = \mathbb{Q}$ or $X = \mathbb{R}$ is a binary operation

$$\cdot : X \times X \longrightarrow X; \quad (x, y) \mapsto x \cdot y. \tag{5.33}$$

(c) Let X be a nonempty set and $\mathscr{F} := \{$ functions $f : X \to X\}$. Function composition

$$\circ : \mathscr{F} \times \mathscr{F} \longrightarrow \mathscr{F}; \quad (f, g) \mapsto g \circ f \tag{5.34}$$

where $g \circ f : X \to X$ is the function defined by $x \mapsto g \circ f(x) := g\big(f(x)\big)$ $(x \in X)$.

Here are some examples of unary operations.

(d) Negative number: Let $X = \mathbb{N}$ or $X = \mathbb{Z}$ or $X = \mathbb{Q}$ or $X = \mathbb{R}$. Then

$$- : \; X \to X; \quad x \;\mapsto\; -x. \tag{5.35}$$

is a unary operation.

(e) Reciprocal: Let $X = \mathbb{Q}_{\neq 0}$ or $X = \mathbb{R}_{\neq 0}$

$$\cdot^{-1} : \; X \to X; \quad x \;\mapsto\; x^{-1} \;=\; 1/x. \tag{5.36}$$

is a unary operation.

(f) Let X be a nonempty set and $\mathscr{B} \;:=\; \{$ bijective functions $f : X \to X\}$. Let

$$\cdot^{-1} : \; \mathscr{B} \to \mathscr{B}; f(\cdot) \;\mapsto\; f^{-1}(\cdot) \tag{5.37}$$

be the function which assigns to the function $x \mapsto f(x)$ its (uniquely determined) inverse function $y \mapsto f^{-1}(y)$). Then this assignment is a unary operation on $\mathscr{B}$.

Note that assignment of the reciprocal number and assignment of the inverse function both are denoted by the symbol "$\cdot^{-1}$". There is no danger of confusing the two unary operations because one of them operates on a set of numbers and the other one on a set of functions. □

Definition 5.15 (Restriction/Extension of a function)**.**

Given are three nonempty sets A, X and Y such that $A \subseteq X$ and a function $f : X \to Y$ with domain X.

(a) We define the **restriction of** f **to** A as the function

$$f\big|_A : A \to Y \quad \text{defined as} \quad f\big|_A(x) := f(x) \text{ for all } x \in A. \tag{5.38}$$

(b) Conversely, let $f : A \to Y$ and $\varphi : X \to Y$ be functions such that $f = \varphi\big|_A$. We then call φ an **extension** of f to X. □

Example 5.30. For an example let $X := \mathbb{R}$, $A := [0, 1]$ and $f(x) := 3x^2 (x \in [0, 1])$. For any $\alpha \in \mathbb{R}$ the function $\varphi_\alpha : \mathbb{R} \to \mathbb{R}$ defined as $\varphi_\alpha(x) := 3x^2$ if $0 \leq x \leq 1$ and αx otherwise defines a different extension of f to $\mathbb{R}$. □

Notation 5.2. As the only difference between f and $f\big|_A$ is the domain, it is customary to write f instead of $f\big|_A$ to make formulas look simpler if doing so does not give rise to confusions. □

Remark 5.18.

The restriction $f\big|_A$ is always uniquely determined by f. Such is not the case for extensions if A is a strict subset of X unless some conditions are imposed on the nature of the extension.

For example, if we had asked for continuity [13] of the extension φ_α of f in example 5.30 above, only $\varphi_1(x) = 3x^2$ if $0 \leq x \leq 1$ and x otherwise would qualify. □

[13] Continuity of functions $y = f(x)$ with real numbers x and y will be defined in ch.**??** (Convergence and Continuity in $\mathbb{R}$). See Definition **??** on p.**??**. Until then use your knowledge from calculus.

Proposition 5.9.

Let X, Y be nonempty sets. Let $f : X \xrightarrow{\sim} Y$ be bijective

(a) *Let $\emptyset \neq A \subseteq X$, $B := f\big|_A(A) = \{f(a) : a \in A\}$.* [14] *Let $f' : A \to B$; $x \mapsto f(x)$, i.e., $f' = f\big|_A$, except that we have shrunken the codomain Y to B. Then f' is bijective.*

(b) *Let $\emptyset \neq V \subseteq Y$. Let $U := \{x \in X : f(x) \in V\}$.* [15] *Let $f'' : U \to V$; $x \mapsto f(x)$, i.e., $f'' = f\big|_U$, except that we have shrunken the domain X to U. Then f'' is bijective.*

The proof of **(a)** is left as exercise 5.17. See p.197.

PROOF of **(b)**:

We first prove injectivity. Let $u, u' \in U$ such that $u \neq u'$. Then $f(u) \neq f(u')$ because f is injective. But then $f'(u) = f(u) \neq f(u') = f'(u')$. It follows that f' is injective.

Let $b \in V$. Since $U = \{x \in X : f(x) \in V\}$, the set U contains all items with function values in V, hence there exists $u \in U$ such that $f(u) = v$. We have proven surjectivity. ■

Example 5.31.

For example let $f : [0, \infty[\to [0, \infty[;\ x \to x^2$. Then f is bijective.

(a) Let $A := [0, 2]$. Then $f(A) = f\big|_A(A) = [0, 4]$, and $f\big|_{[0,2]}[0, 2] \xrightarrow{\sim} [0, 4]$ is bijective.

(b) Let $V := [1, 9]$. Then $f^{-1}(V) = [1, 3]$, and $f\big|_{[1,3]}[1, 3] \xrightarrow{\sim} [1, 9]$ is bijective. □

5.2.7 Real–Valued Functions and Polynomials

Introduction 5.5. If we deal with functions such as $f(x) = sin(2x) - 3x^3$ or $g(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ or $h(x, y, z) = (x^y)^z$ for which the codomain is (a subset of) $\mathbb{R}$, i.e., each function value is a real number, then we can add those function values or multiply them or do anything else one can do with real numbers. In particular we can define for two functions $f_1, f_2 : X \to \mathbb{R}$ with matching domain X their sum $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ and their product $(f_1 \cdot f_2)(x) = f_1(x)f_2(x)$. Thus we have $g + h(x, y, z) = \sqrt{x^2 + y^2 + z^2} + (x^y)^z$ and $gh(x, y, z) = \sqrt{x^2 + y^2 + z^2}(x^y)^z$.

Does it matter at all what kind of structure if any the domain has been endowed with? Not for the subject matter that will be discussed here. For example let $C :=$ { all inhabitants of Chicago }, let $a : X \to [0, \infty[$ be the function which assigns to each person x who lives in Chicago her/his age in days $a(x)$, and let $s : C \to [0, \infty[$ be the function which assigns to x the number of days $s(x)$ s/he has been severely ill so far. Then we can build the function $s/a : C \to [0, \infty[\,;\ \ x \mapsto 100\big(s(x)/a(x)\big)$ which assigns to each inhabitant of Chicago the percentage of time they have been sick so far.

We just mention in passing that we can apply this principle to codomains which carry any kind of structure. For example if $(G, \diamond)$ is a group and X is a nonempty set then we can associate with $f, gX \to G$ the functions $f \diamond g : x \mapsto f(x) \diamond g(x)$ and $f^{-1} : x \mapsto \big(f(x)\big)^{-1}$.

Here is an example where each function value is a set. Let C again denote the inhabitants of Chicago. We assign to each $x \in C$ the set $P(x) \subseteq C$ of all people who live in Chicago and whom x knows professionally, and the set $F(x)$ of all friends that x has in Chicago. We thus have defined two functions $P, F : C \to 2^C$. We cannot build the sum $P + F$ or the quotient P/F, but we can construct functions such as $P \cap F : x \mapsto P(x) \cap F(x)$, the set of all friends who live in Chicago whom x knows professionally and $P^{\complement} : x \mapsto P(x)^{\complement}$, the set of all Chicagoans with whom x does not have a professional relationship. □

We start with the definition of a real–valued function.

Definition 5.16 (Real–Valued Function)**.**

Let X be an arbitrary, nonempty set. If the codomain Y of a mapping

$$f : X \to Y; \qquad x \mapsto f(x)$$

is a subset of $\mathbb{R}$, then we call $f(\cdot)$ a **real function** or **real–valued function**. □

Note that the above definition does not exclude the case $Y = \mathbb{R}$ because $Y \subseteq \mathbb{R}$ is in particular true if both sets are equal.

As we mentioned in the introduction to this section real–valued functions are a pleasure to work with because, given any fixed argument x_0, the object $f(x_0)$ is just an ordinary number. In particular you can add, subtract, multiply and divide real–valued functions. Of course, division by zero is not allowed:

Definition 5.17 (Operations on real–valued functions)**.**

⋆

Let X be an arbitrary nonempty set.
Given are two real–valued functions $f(\cdot), g(\cdot) : X \to \mathbb{R}$ and a real number α. The **sum** $f+g$, **difference** $f-g$, **product** fg or $f \cdot g$, **quotient** f/g, and **scalar product** αf are defined by doing the operation in question with the numbers $f(x)$ and $g(x)$ for each $x \in X$. In other words these items are defined by the following equations:

$$\begin{aligned}(f+g)(x) &:= f(x)+g(x),\\ (f-g)(x) &:= f(x)-g(x),\\ (fg)(x) &:= f(x)g(x),\\ (f/g)(x) &:= f(x)/g(x) \qquad \text{for all } x \in X \text{ where } g(x) \neq 0,\\ (\alpha f)(x) &:= \alpha \cdot g(x). \quad \square\end{aligned} \tag{5.39}$$

Remark 5.19.

Note that scalar multiplication $(\alpha f)(x) = \alpha \cdot f(x)$ is a special case of multiplying two functions $(gf)(x) = g(x)f(x)$, namely the case where $g(x) = \alpha$ for all $x \in X$ (constant function α). □

Definition 5.18 (Negative function). ⋆

Let X be an arbitrary, nonempty set and let $f : X \to \mathbb{R}$. The function

$$-f(\cdot) : X \to \mathbb{R}; \qquad x \mapsto -f(x).$$

is called **negative** f or **minus** f. We usually write $-f$ for $-f(\cdot)$. □

Note that this definition does not exclude the case $Y = \mathbb{R}$ because $Y \subseteq \mathbb{R}$ is in particular true if both sets are equal.

All those last definitions about sums, products, scalar products, ... of real–valued functions are very easy to understand if you remember that, for any fixed $x \in X$, you just deal with ordinary numbers!

Example 5.32 (Arithmetic operations on real–valued functions).
For simplicity, let $X := \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Let

$$\begin{aligned} f &: \mathbb{R}_{\geq 0} \to \mathbb{R}; && x \mapsto (x-1)(x+1) \\ g &: \mathbb{R}_{\geq 0} \to \mathbb{R}; && x \mapsto (x-1) \\ h &: \mathbb{R}_{\geq 0} \to \mathbb{R}; && x \mapsto (x+1) \end{aligned}$$

Then

$$\begin{aligned} (f+h)(x) &= (x-1)(x+1) + x + 1 \ = x^2 - 1 + x + 1 \ = \ x(x+1) \ \forall x \in \mathbb{R}_{\geq 0}, \\ (f-g)(x) &= (x-1)(x+1) - (x-1) \ = x^2 - 1 - x + 1 \ = \ x(x-1) \ \forall x \in \mathbb{R}_{\geq 0}, \\ (gh)(x) &= (x-1)(x+1) \ = f(x) \ \ \forall x \in \mathbb{R}_{\geq 0}, \\ (f/h)(x) &= (x-1)(x+1)/(x+1) \ = x - 1 \ = \ g(x) \ \forall x \in \mathbb{R}_{\geq 0}, \\ (f/g)(x) &= (x-1)(x+1)/(x-1) \ = x + 1 \ = \ h(x) \ \forall x \in \mathbb{R}_{\geq 0} \setminus \{1\} \end{aligned}$$

It is really, really important for you to understand that f/g and h are **not the same functions**. Here is the reason. f/g is not defined for $x = 1$ because $\frac{(1-1)(1+1)}{1-1} =$ "0/0". The domain of f/g is $\mathbb{R}_{\geq 0} \setminus \{1\}$. It is different from $\mathbb{R}_{\geq 0}$, the domain of h. It follows that both functions are different. □

Definition 5.19 (Polynomials).

Let A be subset of the real numbers and let $p(\cdot) : A \to \mathbb{R}$ be a real–valued function on A. $p(\cdot)$ is called a **polynomial**. if there is an integer $n \geq 0$ and real numbers $a_1, a_2, \dots, a_n$ which are constant (they do not depend on x) so that $p(\cdot)$ can be written as a sum

$$p(x) \ = \ a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \ = \ \sum_{j=0}^{n} a_j x^j. \tag{5.40}$$

In other words, polynomials are linear combinations of the **monomials** $x \to x^k \quad (k \in (Z)_{\geq 0}$. If $a_n \neq 0$ then we call n the **degree** of p. The zero function $x \mapsto 0 = 0 \cdot x^0$ is a polynomial which we call the **zero polynomial**. Note that it has no degree because we cannot represent it in the form (5.40) with a non–zero coefficient a_n. We call $z \in A$ a **root** of the polynomial p if $p(z) = 0$.

If we talk about polynomials without explicitly specifying the domain then it is implied that the domain is $\mathbb{R}$. □

Proposition 5.10.

If p_1 and p_2 are polynomials and if $\lambda \in \mathbb{R}$ then

(a) *The sum $x \mapsto p_1(x) + p_2(x)$ is a polynomial.*
(b) *The "scalar product" $x \mapsto \lambda p_1(x)$ is a polynomial.*

The proof is left as exercise 5.19 (see p.197). ■

Example 5.33. Polnomials may not always be given in their "normalized form" (5.40) on p.184. For $x \in \mathbb{R}$ let

$$\begin{aligned} p(x) &:= b_0 x^0 (1-x)^n + b_1 x^1 (1-x)^{n-1} + \ldots + b_{n-1} x^{n-1} (1-x)^1 + b_n x^n \\ &= \sum_{k=0}^{n} b_k x^k (1-x)^{n-k} \end{aligned} \tag{5.41}$$

We have $b_k x^k (1-x)^{n-k} = b_k x^k + (-b_k) x^n$. Let $b := \sum_{k=0}^{n} b_k$. Then $p(x) = \sum_{k=0}^{n} b_k x^k - b x^n$ is of the form (5.40) (define $a_k := b_k$ for $0 \leq k < n$ and $a_n := b_n - b$) and hence is a polynomial.

The so called Bernstein polynomials which we will examine in ch.**??** are of the form (5.41). □

Many more properties of functions will be discussed later. Now we look at families, sequences and some additional properties of sets.

5.2.8 Families, Sequences, and Functions as Families

Introduction 5.6. In Chapter **??** (A First Look at Functions, Sequences and Families) We were introduced to the notion of a family as collection $(x_i)_{i \in I}$ of items x_i which are subscripted or indexed by the elements i of an arbitrary index set I. We saw that any such family can be thought of as a function with index set I, and that sequences are families with index sets of integers that contain a smallest element, the start index. We also noticed that families (in particular, sequences) are functions in disguise shich associate with each index $i \in I$ the corresponding indexed item x_i.

We never gave a reason why one would introduce families if, whatever one can do with them, also can be done with functions. It's really just convenience: The expression

$$\Big(\bigcup_{j \in J} (A_i \cap B_j)\Big)_{i \in I}$$

is significantly shorter than the expression

$$\varphi : I \to 2^{\Omega}; \;\; \varphi(i) = \bigcup_{j \in J} \big(A(i) \cap B(j)\big) \qquad \text{where} \quad A : I \to 2^{\Omega} \;\text{ and }\; B : J \to 2^{\Omega}.$$

Again, we define sequences as special families, those with index set $J = J = [k_0, \infty[_{\mathbb{Z}}$ for some initial index $k_0 \in \mathbb{Z}$. We also give in this chapter the definition of an infinite subsequence which is consistent with the one given in Chapter **??**,

Even though finite sequences and their subsequences were already defined in Chapter **??**, you will not find the exact definitions here. That must wait until Chapter **??** (Cardinality I: Finite and Countable Sets) when the precise definition of finiteness is available. □

We now are ready to give the definition of a family.

Definition 5.20 (Indexed families)**.**

Let J and X be nonempty sets and assume that

for each $\imath \in J$ there exists exactly one indexed item $x_\imath \in X$.

(a) We write $(x_\imath)_{\imath \in J}$ for this collection of indexed elements and call it an **indexed family** or simply a **family** in X.
(b) J is called the **index set** of the family.
(c) For each $\jmath \in J$, $x_\jmath$ is called a **member of the family** $(x_\imath)_{\imath \in J}$. □

Remark 5.20.

(a) The index $\imath$ is a dummy variable: $(x_\imath)_{\imath \in J}$ and $(x_k)_{k \in J}$ describe the same family as long as $\imath \mapsto x_\imath$ and $k \mapsto x_k$ describe the same function $x(\cdot) : J \to X$. This should not surprise you if you recall remark 5.11 on p.163.

(b) Let $R := \{(\imath, x_\imath) : \imath \in J\}$. Then R is a relation on (J, X) which satisfies (5.6) of the definition of a function

$$x(\cdot) : \; J \longrightarrow X, \qquad \imath \mapsto x(\imath) := x_\imath$$

(see Definition 5.7 on p.162), whose graph $\Gamma_{x(\cdot)}$ equals R. In other words,

- Every family $(x_\imath)_{\imath \in J}$ in X can be interpreted as a function

$$x(\cdot) : J \longrightarrow X; \quad \imath \mapsto x_\imath \,. \;\square$$

Remark 5.21.

The codomain X does not occur in the notation $(x_\imath)_{\imath \in J}$. This is not a problem because we do not care about surjectivity or injectivity of families. The only thing that matters about the set X is that it is big enough to contain each indexed item. Here are two natural choices for a codomain.

(a) If there is a universal set X which contains all tagged items of the family then selecting X as codomain makes perfect sense.

(b) If there is no universal set then you can think of

$$X = \bigcup [x_\imath : \imath \in J] := \{x : x = x_{\imath_0} \text{ for some } \imath_0 \in I\}$$

as the codomain. [16] □

Definition 5.21 (Equality of families)**.**

Two families $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are equal if
(a) $I = J$,
(b) $x_i = y_i$ for all $i \in I$. □

Remark 5.22.

Equality of families and equality of functions are not identical concepts, since no demand is made in the latter that both families are families in the same set, say, X. But of course, if $(x_i)_{i \in I}$ is a family in X and $(y_j)_{j \in J}$ is a family in Y and those two families are equal then this necessitates

$$\{x_i : i \in I\} = \{y_j : j \in J\} \subseteq X \cap Y. \quad \square \tag{5.42}$$

Note 5.1 (Simplified notation for families)**.**

If there is no confusion about the index set then it can be dropped from the specification of a family and we simply write $(x_\imath)_\imath$ instead of $(x_\imath)_{\imath \in J}$. We even may shorten this to $(x_\imath)$ if doing so does not lead do confusion.

For example, a proposition may start as follows: Let (A_α) and (B_α) be two families of subsets of Ω indexed by the same set. Then .....

It is clear from the formulation that we deal in fact with two families $(A_\alpha)_{\alpha \in J}$ and $(B_\alpha)_{\alpha \in J}$. Nothing is said about the index set, probably because the proposition is valid for any index set or because this set was fixed once and for all earlier on for the entire section. □

Example 5.34. Here is an example of a family of subsets of $\mathbb{R}$ which are indexed by real numbers: Let $J = [0, 1]$ and $X := 2^{\mathbb{R}}$. For $0 \leq x \leq 1$ let $A_x := [x, 2x]$ be the set of all real numbers between x and $2x$. Then $(A_x)_{x \in [0,1]}$ is such a family. □

[16] General unions and intersections will be defined in ch.**??** (More on set operations). See Definition **??** on p.**??**.

Remark 5.23.

If a family is just some kind of function, why bother with yet another definition? We already gave an answer in the introduction to this section: There we saw an example where writing something as a collection of indexed items rather than as a function is a notational convenience. Here is another example. Take a peek at theorem **??** (De Morgan's Law) on p.**??**. One of the formulas there states that for any indexed family $(A_\alpha)_{\alpha \in I}$ of subsets of a universal set Ω it is true that

$$\Big(\bigcup_\alpha A_\alpha\Big)^{\complement} = \bigcap_\alpha A_\alpha^{\complement}.$$

Without the notion of a family you might have to say something like this: Let $A : I \to 2^\Omega$ be a function which assigns its arguments to subsets of Ω . Then

$$\Big(\bigcup_\alpha A(\alpha)\Big)^{\complement} = \bigcap_\alpha A(\alpha)^{\complement}.$$

The additional parentheses around the index α just add complexity to the formula. □

Example 5.35 (Sequences as families)**.** We have worked with special families before: those where the index set is $J = \mathbb{N} = [1, \infty[_{\mathbb{Z}}$ or $J = [0, \infty[_{\mathbb{Z}}$ or, more generally $J = [k_0, \infty[_{\mathbb{Z}}$ for some "start index" $k_0 \in \mathbb{Z}$, and where X is a subset of the real numbers. Example: $x_n := 1/n$. Here

$$(x_n)_{n\in\mathbb{N}} \text{ corresponds to the indexed collection } 1,\ \frac{1}{2},\ \frac{1}{3},\ \dots,\ \frac{1}{n},\ \dots \quad \square$$

The families from the last example will be called sequences. Preliminary definitions for sequences, subsequences, finite sequences and finite subsequences were given in Definition **??** on p.**??** and Definition **??** on p.**??**. We will now give precise definitions for sequences and subsequences. Those for finite sequences and finite subsequences will have to wait until ch.**??** (Finite Sequences and Subsequences and Eventually True Properties).

Definition 5.22 (Sequences and subsequences)**.**

Let $n_\star \in \mathbb{Z}$, let

$$J := [n_\star, \infty[_{\mathbb{Z}} = \{k \in \mathbb{Z} : k \geq n_\star\}.$$

Let X be an arbitrary nonempty set. An indexed family $(x_n)_{n\in J}$ in X with index set J is called a **sequence** in X with **start index** $n_\star$. We will also write

$$(x_n)_{n\geq n_\star} \quad\text{or}\quad (x_n)_{n=n_\star}^\infty \quad\text{or}\quad x_{n_\star}, x_{n_\star+1}, x_{n_\star+2}, \dots$$

for this sequence. As for families, the name of the index variable of a sequence is unimportant as long as it is applied consistently. It does not matter whether one writes, e.g.,

$$(x_n)_{n\geq n_\star} \quad\text{or}\quad (x_j)_{j\geq n_\star} \quad\text{or}\quad (x_\beta)_{\beta\geq n_\star} \quad\text{or}\quad (x_A)_{A=n_\star}^\infty.$$ [17]

Let $(n_j)_{j=1}^\infty$ be a sequence of integers n_j such that

1) $n_j \in J$ (i.e., a sequence of indices for the above sequence $(x_j)_{j=n_\star}^\infty$)
2) $i < j \Rightarrow n_i < n_j$ for all $i, j \in \mathbb{N}$.

Note that $n_j \in J$ for all $j \in \mathbb{N}$ implies $n_\star \leq n_1 < n_2 < \dots$. If we write $I := \{n_j : j \in \mathbb{N}\}$ then we see that $(x_n)_{n \in I} = (x_{n_j})_{j \in \mathbb{N}}$, thus this object is an indexed family whose index set I is a subset of the original index set J. We call $(x_{n_j})_{j \in \mathbb{N}} = (x_{n_j})_{j=1}^\infty$ a **subsequence** of the sequence $(x_j)_{j=n_\star}^\infty$. This is an appropriate name since we obtain $(x_{n_j})_{j=1}^\infty$ from $(x_j)_{j \in J}$ by removing all members x_n such that none of the n_j equals n. Be sure to understand that, according to this definition, the sequence $(n_j)_{j \in \mathbb{N}}$ is a subsequence of the full sequence of indices $(n)_{n=n_\star}^\infty$. We will also write

$$(x_{n_j})_{j \in \mathbb{N}} \quad \text{or} \quad (x_{n_j})_{j \geq 1} \quad \text{or} \quad (x_{n_j})_{j=1}^\infty \quad \text{or} \quad x_{n_1}, x_{n_2}, x_{n_3}, \dots$$

for this subsequence. □

Note 5.2 (Simplified notation for sequences).

(a) It is customary to choose either of i, j, k, l, m, n as the symbol of the index variable of a sequence and to stay away from other symbols whenever possible.
(b) By default the index set for a sequence is $\mathbb{N} = \{1, 2, 3, 4, \dots\}$.
(c) We are allowed to write $(x_n)_n$ or just (x_n) if there is no confusion about the value of $n_\star$ or if this value is irrelevant for the statement at hand.
(d) Customary simplified notation for subsequences is either of $(x_{n_j})_{j \in \mathbb{N}}$, $(x_{n_j})_{j \geq 1}$, $(x_{n_j})_j$ or simply (x_{n_j}).

Compare this to note 5.1 about simplified notation for families. □

Part **(b)** of the above note deserves repeating:

Assumption 5.1 (indices of sequences).

Unless explicitly stated otherwise, sequences are always indexed $1, 2, 3, \dots$, i.e., the first index is 1, there is no largest index and, given any index, you obtain the next one by adding 1 to it. □

Example 5.36. For $j \in \mathbb{N}$ let $x_j := (-1)^j$. Then $\big((-1)^n\big)_{n=1}^\infty$ is the sequence

$$x_1 = -1, \quad x_2 = 1, \quad x_3 = -1, \quad x_4 = 1, \quad x_5 = -1, \ \dots$$

With the notations of Definitions **??** and **??** we have $X = \mathbb{Z}$ and $n_\star = 1$ (i.e., $J = \mathbb{N}$). If we choose $n_j := 2j$, then the corresponding index set $\{2, 4, 6, \dots\}$ is the set of all even indices, and we obtain the subsequence

$$\big(x_{n_j}\big)_{j=1}^\infty \ = \ \big((-1)^{2j}\big)_{j=1}^\infty \ = \ 1, 1, 1, 1, \dots.$$

If we choose $n_j := 2j - 1$ then we obtain as index set the subset $\{1, 3, 5, \dots\}$ of all odd indices, and thus the subsequence

$$\left(x_{n_j}\right)_{j=1}^{\infty} \;=\; \left((-1)^{2j-1}\right)_{j=1}^{\infty} \;=\; -1, -1, -1, -1, \dots \;\; \square$$

Here is another example of a sequence.

Example 5.37 (Series (summation sequence)). Let $s_k := 1 + 2^{-1} + 2^{-2} + \dots + 2^{-k}$ $(k = 1, 2, 3, \dots)$:

$$\begin{aligned} s_1 =& 1, \quad s_2 = 1 + 1/2 = 2 - 1/2, \quad s_3 = 1 + 1/2 + 1/4 = 2 - 1/4, \quad \dots, \\ s_k =& 1 + 1/2 + \dots + 2^{k-1} = 2 - 2^{k-1}; \quad s = 1 + 1/2 + 1/4 + 1/8 + \dots \quad \text{“infinite sum”}. \end{aligned}$$

You obtain s_{k+1} from $s_k = 2 - 2^{k-1}$ by cutting the difference 2^{k-1} to the number 2 in half (that would be 2^k) and adding that to s_k. It is intuitively obvious from $s_k = 2 - 2^{k-1}$ that the infinite sum s adds up to 2. Such an infinite sum is called a **series**. [18] $\square$

Remark 5.24. Having defined the family $(x_\imath)_{\imath \in J}$ as the function which maps $\imath \in J$ to $x_\imath$ means that a family distinguishes any two of its members $x_\imath$ and $x_\jmath$ by remembering what their indices are, even if they represent one and the same element of X: Think of "$(x_\imath)_{\imath \in J}$" as an abbreviation for

$$\text{“}\big((\imath, x_\imath)\big)_{\imath \in J}\text{''}. \tag{5.43}$$

Doing so should also make it much easier to see the equivalence of functions and families: (5.43) looks at its core very much like the graph $\{(\imath, x_\imath) : \imath \in J\}$ of the function $\imath \mapsto x_\imath$. $\square$

Remark 5.25 (Families and sequences can contain duplicates). One of the important properties of sets is that they do not contain any duplicates (see Definition **??** (sets) on p.**??**). On the other hand, remark 5.24 casually mentions that families, and hence sequences as special kinds of families, can contain duplicates. Let us look at this more closely.

The two sets $A := \{31, 20, 20, 20, 31\}$ and $B := \{20, 31\}$ are equal. On the other hand let $J := \{\alpha, \beta, \pi, \star, Q\}$ and define the family $(w_\imath)_{\imath \in J}$ in B by its associated graph as follows:

$$\Gamma \;:=\; \{(\alpha, 31), (\beta, 20), (\pi, 20), (\star, 20), (Q, 31)\}, \;\; \text{i.e.,,} \;\; w_\alpha = 31, w_\beta = 20, , w_\pi = 20, w_\star = 20, w_Q = 31.$$

The three occurrences of 20 cannot be distinguished as elements of the set A. In contrast to this the items $(\beta, 20), (\pi, 20), (\star, 20)$ as elements of $\Gamma \subseteq J \times A = J \times B$ [19] are different from each other because two pairs (a, b) and (x, y) are equal only if $x = a$ and $y = b$. $\square$

> In contrast to sets, families and sequences allow us to incorporate duplicates.

[18] The precise definition of a series will be given in ch.**??** (Function Sequences and Infinite Series) on p.**??**.

[19] Be sure to understand that $J \times A = J \times B$!

A family $(x_\imath)_{\imath \in J}$ in X is specified by the function $F : J \to X$ which maps $\imath \in J$ to $F(z) = x_z$. Conversely, let X, Y be nonempty sets and let $f : X \to Y$ be a function with domain X and codomain Y. For $x \in X$ let $f_x := f(x)$. Then f can be written as $(f_x)_{x \in X}$, i.e., as a family in Y with index set X. Thus we have

Proposition 5.11 (Functions are families and families are functions)**.**

The following two ways of specifying a function $f : X \to Y, \ x \mapsto f(x)$ are equivalent:
(a) *f is defined by its graph* $\{(x, f(x)) : x \in X\}$.
(b) *f is defined by the following family in Y:* $(f(x))_{x \in X}$

PROOF: This follows from the material leading to the above proposition. ■

Note that the formulation of the last proposition involved a family that needed explicit mention of its codomain, Y.

There will be a lot more on sequences and series (sequences of sums) in later chapters, but we need to develop more concepts, such as convergence, to continue with this subject.

5.3 Right Inverses and the Axiom of Choice ⋆

The following is a greatly expanded version of the online article http://planetmath.org/surjectionandaxiomofchoice about the equivalence of the Axiom of Choice and the existence of right inverses for arbitrary, surjective functions.

Definition 5.23 (Choice function)**.**

Let $\mathscr{A}$ be a collection of nonempty sets and let Ω be a set such that $\bigcup [A : A \in \mathscr{A}] \subseteq \Omega$. Let the function

$$c : \mathscr{A} \longrightarrow \Omega \qquad \text{satisfy} \qquad c(A) \in A \ \text{ for all } A \in \mathscr{A}$$

Then we call c a **choice function** [20] on $\mathscr{A}$. □

The following is a repeat of Proposition 5.8**(b)** and its proof, with emphasis on the use of a choice function. It shows that the acceptance of the Axiom of Choice implies that right inverses exist for any choice of nonempty Y and X and surjective $g : Y \to X$.

Proposition 5.12.

Let $X, Y \neq \emptyset$.
Let $g : Y \to X$. If g is surjective then there exists $f : X \to Y$ such that $g \circ f = id_X$.

PROOF: Let $\mathscr{A} := \{g^{-1}\{x\} : x \in X\}$. The surjectivity of g implies that $g^{-1}\{x\} \neq \emptyset$ for all $x \in X$. According to the Axiom of Choice there exists a choice function $c : \mathscr{A} \to Y$.

Let $f : X \to Y$ be the function $x \mapsto y_x := c(g^{-1}\{x\})$ and let $x \in X$.

Since c is a choice function, $y_x \in g^{-1}\{x\}$ and thus $g(y_x) \in \{x\}$, i.e., $g(y_x) = x$. Thus

$$g \circ f(x) \;=\; g \circ c(g^{-1}\{x\}) \;=\; g(y_x) \;=\; x.$$

The first equality follows from the definition of f and the second one from that of y_x. ■

It is not as easy to show the other direction: Accepting that surjective functions $g : Y \to X$ have right inverses for any choice of nonempty X, Y and $y \mapsto g(y)$ implies the existence of choice functions $\mathscr{A} \to \Omega$ for arbitrary, nonempty Ω and $\mathscr{A} \subseteq 2^\Omega \setminus \emptyset$. This will be shown in the next lemma and subsequent proposition.

Lemma 5.1.

Assume that each surjective function possesses a right inverse, i.e., if Y and X are nonempty and $g : Y \to X$ is surjective then there exists $f : X \to Y$ (necessarily injective) such that $g \circ f = id_X$. See Definition 5.13 (Left inverses and right inverses) on p.178 and the subsequent material. Assume further that $\mathscr{A}$ is a collection of nonempty and disjoint sets.

Then there exists a choice function on $\mathscr{A}$.

PROOF: Let $\Omega := \biguplus[A : A \in \mathscr{A}]$. Since the elements of $\mathscr{A}$ are disjoint there exists for each $\omega \in \Omega$ a unique A_ω such that $\omega \in A_\omega$. Thus the association

$$\omega \mapsto A_\omega \text{ defines a function } g : \Omega \to \mathscr{A} \text{ such that } \omega \in g(\omega) = A_\omega \,. \tag{5.44}$$

(A): We show that g is surjective and thus possesses a right inverse.

Let $A \in \mathscr{A}$ and $\omega \in A$. Such ω exists because $A \neq \emptyset$. Let

$$A_\omega := g(\omega)\,. \tag{5.45}$$

Then $\omega \in A_\omega$ by the definitions of g and A_ω. Since we assumed $\omega \in A$, it follows that $\omega \in A \cap g(\omega)$. Since the elements of $\mathscr{A}$ are disjoint, $A = g(\omega)$. We have found for an arbitrary $A \in \mathscr{A}$ an $\omega \in \Omega$ such that $\omega \in g(\omega)$, thus g is surjective.

(B): By assumption g possesses a right inverse, i.e., there is

$$c : \mathscr{A} \to \Omega \quad \text{such that } c \text{ is injective and} \quad g \circ c \;=\; id_{\mathscr{A}}\,. \tag{5.46}$$

Let $A \in \mathscr{A}$ and $\omega := c(A)$. Then

$$g(\omega) \;=\; g\big(c(A)\big) \;=\; id_{\mathscr{A}}(A) \;=\; A\,. \tag{5.47}$$

Since $c(A) = \omega$, $\omega \in g(\omega)$ by (5.44) and $g(\omega) = A$ by (5.47), it follows that $c(A) \in A$. This holds for arbitrary $A \in \mathscr{A}$, thus c is a choice function on $\mathscr{A}$. ■

We now remove the restrictive assumption that the members of $\mathscr{A}$ must be mutually disjoint.

Proposition 5.13.

Assume that each surjective function possesses a right inverse. Assume further that $\mathscr{A}$ is a collection of nonempty sets. Then there exists a choice function on $\mathscr{A}$.

PROOF:
Let $\Omega := \bigcup[A : A \in \mathscr{A}]$. Let

$$\jmath : \mathscr{A} \longrightarrow \Omega \times \mathscr{A}\,; \qquad A \mapsto \{(\omega', A) : \omega' \in A\}\,.$$

Let $A, A' \in \mathscr{A}$ such that $A \neq A'$. Then $\jmath(A) \cap \jmath(A') = \emptyset$ since all elements $(\omega, A) \in \jmath(A)$ have different second coordinate from the elements $(\omega', A') \in \jmath(A')$, and two elements (x, y) and (x', y') of a cartesian product $X \times Y$ are different unless both $x = x'$ and $y = y'$.
In particular, $A \neq A' \Rightarrow \jmath(A) \neq \jmath(A')$. It follows that the function

$$\imath : \mathscr{A} \xrightarrow{\sim} \jmath(\mathscr{A})\,; \qquad A \mapsto \imath(A) \tag{5.48}$$

bijects $\mathscr{A}$ to a collection of disjoint subsets of $\Omega \times \mathscr{A}$.
We infer from Lemma 5.1 the existence of a choice function $c : \imath(\mathscr{A}) \to \Omega \times \mathscr{A}$.
Let $A \in \mathscr{A}$. Since c is a choice function, $c \circ \imath(A) \in \imath(A)$, i.e.,

$$c \circ \imath(A) \;=\; c \circ \jmath(A) \;\in\; \{(\omega', A) : \omega' \in A\}$$

according to the definition of $\jmath(A)$. Thus

$$\text{there exists } \omega \in A \text{ such that } c \circ \imath(A) \;=\; (\omega, A)\,. \tag{5.49}$$

$$\text{Let } \pi_\Omega : \imath(\mathscr{A}) \longrightarrow \Omega\,; \qquad (\omega', A') \mapsto \omega'\,,$$

be the projection to the first coordinate. Then

$$\pi_\Omega \circ c \circ \imath(A) \;=\; \pi_\Omega\big((\omega, A)\big) \;=\; \omega$$

where ω satisfies $\omega \in A$ according to (5.49). We have shown that the function

$$c^* \;:=\; \pi_\Omega \circ c \circ \imath : \mathscr{A} \longrightarrow \Omega \qquad A \mapsto \omega := \pi_\Omega \circ c \circ \imath : \mathscr{A}(A)$$

maps any $A \in \mathscr{A}$ to an element $\omega \in A$. We conclude that c^* is a choice function on $\mathscr{A}$. ■

We state the content of Proposition 5.12 and Proposition 5.13 as follows.

Theorem 5.3.

The following are equivalent.

(a) *For any sets $X, Y \neq \emptyset$ and surjective $g : Y \to X$ there exists a right inverse for g, i.e., a function $f : X \to Y$ such that $g \circ f = id_X$.*

(b) *The Axiom of Choice holds: For any collection $\mathscr{A}$ of nonempty sets there exists a choice function on $\mathscr{A}$, i.e., a function $c : \mathscr{A} \to \bigcup[A : A \in \mathscr{A}]$ such that $c(A) \in A$ for all $A \in \mathscr{A}$.*

PROOF: See Proposition 5.12 and Proposition 5.13. ■

5.4 Exercises for Ch.5

5.4.1 Exercises for Functions and Relations

Exercise 5.1.

Prove that $A \times B = \emptyset \;\Leftrightarrow\; A = \emptyset$ or $B = \emptyset$ or both are empty. $\square$

Exercise 5.2.

(a) Which of the following is an equivalence relation? a partial ordering? on $\mathbb{R}$?
a1. $xRy \Leftrightarrow x < y$, **a2.** $xRy \Leftrightarrow x \leq y$, **a3.** $xRy \Leftrightarrow x = y$, **a4.** $xRy \Leftrightarrow x \neq y$.

(b) Define $xRy \Leftrightarrow xy > 0$. Is this an equivalence relation on $\mathbb{R}$? on $\mathbb{R}_{\neq 0}$? on $\mathbb{R}_{>0}$? on $\mathbb{R}_{<0}$? $\square$

Exercise 5.3.

It was stated in example 5.8 on p.158 that $(\mathbb{R}, \geq)$ is a linearly ordered set. Prove it. (Prove first that this is a POset.) $\square$

Exercise 5.4.

Prove prop.5.1(c) on p.155 of this document: If "$\sim$" is an equivalence relation on a nonempty set X and $x, y \in X$ then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$. $\square$

Exercise 5.5.

Injectivity and Surjectivity:

- Let $f : \mathbb{R} \to [0, \infty[$; $\quad x \mapsto x^2$.
- Let $g : [0, \infty[\to [0, \infty[$; $\quad x \mapsto x^2$.
  In other words, g is same function as f as far as assigning function values is concerned, but its domain is downsized to $[0, \infty[$.

Answer the following with **true** or **false**.

(a) f is surjective **(c)** g is surjective
(b) f is injective **(d)** g is injective

If your answer is **false** then give a specific counterexample. $\square$

Exercise 5.6 (Excercise 5.5 continued)**.** Let $A \subseteq \mathbb{R}$.

Part 1.

- Let $F_1 : A \to [-2, 20[$; $\quad x \mapsto x^2$.
- Let $F_2 : A \to [2, 20[$; $\quad x \mapsto x^2$.

What choice of A makes

(a) F_1 surjective? **(c)** F_2 surjective?
(b) F_1 injective? **(d)** F_2 injective?

Part 2.

- Let $G_1 : A \to [-2, 20[$; $\quad x \mapsto \sqrt{x}$.
- Let $G_2 : A \to [2, 20[$; $\quad x \mapsto \sqrt{x}$.

What choice of A makes

(e) G_1 surjective? **(g)** G_2 surjective?
(f) G_1 injective? **(h)** G_2 injective?

Part 3.

- Let $G_3 : A \to [-20, 2[; \quad x \mapsto \sqrt{x}$.
- Let $G_4 : A \to [-20, -2[; \quad x \mapsto \sqrt{x}$.

What choice of A makes

(i) G_3 surjective? **k.** G_4 surjective?
(j) G_3 injective? **l.** G_4 injective?

For the questions above

- Write **impossible** if no choice of $A \subseteq \mathbb{R}$ exists.
- Write **NAF** for any of $F_1, F_2, G_1, G_2, G_3, G_4$ which does **not define a function**. $\square$

Exercise 5.7.
Find $f : X \to Y$ and $A \subseteq X$ such that $f(A^{\complement}) \neq f(A)^{\complement}$. Hint: use $f(x) = x^2$ and choose Y as a **one element only** set (which does not leave you a whole lot of choices for X). See example 5.19 on p.167. $\square$

Exercise 5.8.

(a) Prove prop.5.5**(a)**: The composition of two injective functions is injective.
(b) Prove prop.5.5**(b)**: The composition of two surjective functions is surjective. $\square$

Exercise 5.9.
You proved in the previous exercise that
injective $\circ$ injective $=$ injective,
surjective $\circ$ surjective $=$ surjective.
This exercise illustrates that the reverse is not necessarily true.

Find functions $f : \{a\} \to \{b_1, b_2\}$ and $g : \{b_1, b_2\} \to \{a\}$ such that $h := g \circ f : \{a\}$ is bijective but such that it is **not true** that both f, g are injective and it is also **not true** that both f, g are surjective.

Hint: There are not a whole lot of possibilities. Draw possible candidates for f and g in arrow notation as on p.118. You should easily be able to figure out some examples. Think simple and look at example 5.19 on p.167. $\square$

Exercise 5.10. Prove prop.5.6 on p.177: Let X be an arbitrary set and let A be a nonempty proper subset of X. so that $X = A \uplus A^{\complement}$ is a partitioning of X into two nonempty subsets A and $A^{\complement}$. Let $a \in A$, $a_0 \in A^{\complement}$ and $A' := (A \setminus \{a\}) \uplus \{a_0\}$. Then the function $\varphi : A' \xrightarrow{\sim} A$; $\varphi(x) = a$ if $x = a_0$ and $\varphi(x) = x$ else is a bijection. $\square$

Exercise 5.11.
Prove prop.5.4 on p.175 of this document: Let $(R, \oplus, \odot, P)$ be an ordered integral domain

(A) Let $b \in R$. Then the function

$$T : R \to R; \quad x \mapsto x \oplus b,$$

is a bijection.

(B) Let $a \in R, a \neq 0$. Then the function

$$D : R \to a \odot R; \quad x \mapsto a \odot x,$$

is a bijection. (As usual, $a \odot R = aR = \{a \odot r : r \in R\}$.)
Hint: Find the inverses of T (obvious) and D (tricky: you cannot write $a^{-1}y$ since the inverse of a may not exist). ☐

Only the group structure of R was used in part **A** of the previous exercise:

Exercise 5.12.
If $(G, \diamond)$ is a group and and $h \in G$ then the function

$$T : G \to G; \quad g \mapsto g \diamond h,$$

is a bijection. ☐

Exercise 5.13.
Prove **(c)** of remark 5.17 on p.179: Let X and Y be two nonempty sets and $u \neq v$ arbitrary items. Then the sets $\{u\} \times X$ and $\{v\} \times Y$ are disjoint. ☐

Exercise 5.14.
Prove **(b)** of remark 5.17 on p.179: Let X and Y be two nonempty sets and u, v arbitrary.
Then an injection/surjection/bijection $X \to Y$ exists if and only if an injection/surjection/bijection $\{u\} \times X \ \to \ \{v\} \times Y$ exists. ☐

Exercise 5.15.
Let $f : \mathbb{R} \to \mathbb{R}$ be the function $x \mapsto 2x - 4$. Let the relation Γ_f be defined as the graph of f.
- **(a)** Compute the inverse relation $(\Gamma_f)^{-1}$.
- **(b)** Is $(\Gamma_f)^{-1}$ the graph of a function? If yes, what function? Don't forget to include domain and codomain. ☐

Exercise 5.16.
B/G Project 6.9.:
On $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ we define the relation $\sim$ as follows.

$$(m_1, n_1) \ \sim \ (m_2, n_2) \ \Leftrightarrow m_1 \cdot n_2 = n_1 \cdot m_2. \tag{5.50}$$

(a) Prove that $\sim$ defines an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$.
Let

$$\mathfrak{Q} \ := \{[(m, n)] \ : \ m, n \in \mathbb{Z} \text{ and } n \neq 0\} \tag{5.51}$$

be the set of all equivalence classes of $\sim$. We define two binary operations $\oplus$ and $\otimes$ on $\mathfrak{Q}$ as follows;

$$[(m_1, n_1)] \oplus [(m_2, n_2)] \ := [(m_1 n_2 \ + \ m_2 n_1, n_1 n_2)], \tag{5.52}$$
$$[(m_1, n_1)] \otimes [(m_2, n_2)] \ := [(m_1 m_2, n_1 n_2)] \tag{5.53}$$

(b) Prove that these binary operations are defined consistently: the right–hand sides of (5.52) and (5.53) do not depend on the particular choice of elements picked from the sets $[(m_1, n_1)]$ and $[(m_2, n_2)]$. In other words, prove the following:

Let $(p_1, q_1) \sim (m_1, n_1)$ and $(p_2, q_2) \sim (m_2, n_2)$. Then

$$[(m_1n_2 \ + \ m_2n_1, n_1n_2)] \ = [(p_1q_2 \ + \ p_2q_1, q_1q_2)], \tag{5.54}$$
$$[(m_1m_2, n_1n_2)] \ = [(p_1p_2, q_1q_2)]. \tag{5.55}$$

or, equivalently, then

$$(m_1n_2 \ + \ m_2n_1, n_1n_2) \ \sim (p_1q_2 \ + \ p_2q_1, q_1q_2), \tag{5.56}$$
$$(m_1m_2, n_1n_2) \ \sim (p_1p_2, q_1q_2). \ \square \tag{5.57}$$

Exercise 5.17.

Prove prop.5.9**(a)** on p.182: Let A, X, Y be nonempty sets and $A \subseteq X$. Let $f : X \xrightarrow{\sim} Y$ be bijective. Let $B := \{f(a) : a \in A\}$. Let $f' : A \to B;\ \ x \mapsto f(x)$. Then f' is bijective.

Hint: Study the proof of prop.5.9**(b)** Your proof is very similar. $\square$

Exercise 5.18.

What are the graphs $\Gamma_{f_\star}$ and $\Gamma_{f^\star}$ of the functions $f_\star$ and $f^\star$ of example 5.23 on p.169? Do not use the symbols $f_\star$ and $f^\star$ when you write the formulas!

Exercise 5.19.

Prove prop.5.10 on p.185 of this document: If p_1 and p_2 are polynomials and if $\lambda \in \mathbb{R}$ then

(a) The sum $x \mapsto p_1(x) + p_2(x)$ is a polynomial.
(b) The "scalar product" $x \mapsto \lambda p_1(x)$ is a polynomial. $\square$

Exercise 5.20.

Prove (5.42) of rem.5.22 on p.187 of this document: If $(x_i)_{i \in I}$ is a family in X, $(y_j)_{j \in J}$ is a family in Y, and those two families are equal then

$$\{x_i : i \in I\} \ = \{y_j : j \in J\} \ \subseteq \ X \cap Y. \ \square$$

References

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List of Symbols

$\mapsto$ – maps to , 162
$f(\cdot) = (X, Y, \Gamma)$ – function , 162
$f(\cdot)$ – function , 162
$g \circ f$ – function composition , 164
$A \times B$ – cartesian product of 2 sets , 153
$x \sim x'$ – equivalent items , 155

$(x_\imath)_{\imath \in J}$ – family , 186
$f(A)$ – direct image , 170
$f^{-1}(B)$ – indirect image, preimage , 170
$g \circ f(x)$ – function composition , 164
$x \diamond y$ – binary operation , 180
$x^\bullet$ – unary operation, 180
(x_n) – sequence , 189
(x_{n_j}) subsequence , 189
$-f(\cdot), -f$ – negative function , 184
$0(\cdot)$ – zero function , 165
$[x]_\sim, [x]$ – (equivalence class , 155
αf – scalar product of functions , 183
$\Gamma_f, \Gamma(f)$ – graph of f , 162
$\preceq_A$ – partial order on subset , 158
D_f – natural domain of f , 159
$f : X \xrightarrow{\sim} Y$ – bijective function , 173
$f\big|_A$ – restriction of f , 181
$f + g$ – sum of functions , 183
$f - g$ – difference of functions , 183
f/g – quotient of functions , 183
$f^{-1}(\cdot)$ – inverse function , 173
$fg,\ f \cdot g$ – product of functions), 183
$x \preceq y$ – precedes , 157
$x \succeq y$ – succeeds , 157
xRy – equivalent items , 154

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