Math 330 - Additional Material

Student edition with proofs

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8 More on Sets, Relations, Functions and Families

8.1 More on Set Operations

The material in this chapter is a continuation of Chapter **??** (Arbitrary Unions and Intersections).

Recall that we had defined unions and intersections of arbitrary collections of sets in Definition **??** (Arbitrary unions and intersections) on p.**??**.

It is convenient to allow unions and intersections for the empty index set $J = \emptyset$.

For intersections, the next definition requires the use of a universal set.

Definition 8.1.

We define

(8.1) $$\bigcup_{i \in \emptyset} A_i \ := \ \emptyset, \qquad \text{If there is a universal set } \Omega\text{:} \ \bigcap_{i \in \emptyset} A_i \ := \ \Omega. \ \square$$

Remark 8.1.
Note that this definition is consistent with the fact that

- unions over fewer sets become smaller, so the union over $\emptyset$ should be the smallest set possible, i.e., the empty set,
- intersections over fewer sets become bigger, so the intersection over $\emptyset$ should be the largest set possible, i.e., the universal set. □

We give some more examples of non-finite unions and intersections.

Example 8.1. For any set A we have $A = \bigcup_{a \in A} \{a\}$. According to (8.1) this also is true if $A = \emptyset$. □

The following trivial lemma is useful if you need to prove statements of the form $A \subseteq B$ or $A = B$ for sets A and B. Be sure to understand what it means if you choose $J = \{1, 2\}$ (draw one or two Venn diagrams).

Lemma 8.1 (Inclusion lemma)**.**

Let J be an arbitrary, nonempty index set. Let U, X_j, Y, Z_j, W $(j \in J)$ be sets such that

$$U \ \subseteq \ X_j \ \subseteq \ Y \ \subseteq \ Z_j \ \subseteq \ W$$

for all $j \in J$. Then

(8.2) $$U \ \subseteq \ \bigcap_{j \in J} X_j \ \subseteq \ Y \ \subseteq \bigcup_{j \in J} Z_j \ \subseteq W.$$

PROOF: Note that we need at various places in this proof the existence of some $j_0 \in J$, i.e. the assumption that $J \neq \emptyset$ is essential.

(a) Let $x \in U$. Then $x \in X_j$ for all $j \in J$, hence $x \in \bigcap_{j \in J} X_j$. This proves the first inclusion.

(b) Now let $x \in \bigcap_{j \in J} X_j$ and $j_0 \in J$. Then $x \in X_j$ for all $j \in J$; in particular, $x \in X_{j_0}$. It follows from $X_{j_0} \subseteq Y$ that $x \in Y$ and we have shown the second inclusion.

(c) Let $x \in Y$ and $j_0 \in J$. It follows from $Y \subseteq Z_{j_0}$ that $x \in Z_{j_0}$. But then $x \in \{z : z \in Z_j \text{ for some } j \in J\}$, i.e., $x \in \bigcup_{j \in J} Z_j$. This proves the third inclusion.

(d) Finally, assume $x \in \bigcup_{j \in J} Z_j$ It follows from the definitions of unions that there exists $j_0 \in J$ such that $x \in Z_{j_0}$. But then $x \in W$ as W contains Z_{j_0}. It follows that $\bigcup_{j \in J} Z_j \subseteq W$. This finishes the proof of the rightmost inclusion. ■

Definition 8.2 (Disjoint families)**.**

Let J be a nonempty set. We call a family of sets $(A_i)_{i \in J}$ a **mutually disjoint family** if for any two different indices $i, j \in J$ it is true that $A_i \cap A_j = \emptyset$, i.e., if any two sets in that family with different indices are mutually disjoint. □

We recall from Chapter **??** (Sets and Basic Set Operations), p.**??**, Definition **??** of a partition: For $\mathfrak{A} \subseteq 2^{\Omega}$, $\mathfrak{A}$ is a partition or a partitioning of Ω if

(a) $A \cap B = \emptyset$ for any two $A, B \in \mathfrak{A}$ such that $A \neq B$, **(b)** $\Omega = \biguplus \left[A : A \in \mathfrak{A} \right]$.

We extend this to arbitrary families and hence finite collections and sequences of subsets of Ω:

Definition 8.3 (Partition)**.**

Let J be an arbitrary nonempty set, let $(A_j)_{j \in J}$ be a family of subsets of Ω. We call $(A_j)_{j \in J}$ a **partition** or a **partitioning** of Ω if it is a mutually disjoint family which satisfies $\Omega = \biguplus \left[A_j : j \in J \right]$.

In other words,

- $(A_j)_{j \in J}$ is a partition of Ω if and only if $\mathfrak{A} := \{A_j : j \in J\}$ is a partition of Ω. □

Note that duplicate nonempty sets cannot occur in a disjoint family of sets because otherwise the condition of mutual disjointness does not hold.

Example 8.2. Here are some examples of partitions.

(a) For any set Ω the collection $\{ \{\omega\} : \omega \in \Omega\}$ is a partition of Ω.

(b) The empty set is a partition of the empty set and it is its only partition. Do you see that this is a special case of **(a)**?

(c) The set of half open intervals $\{ \,]k, k+1] : k \in \mathbb{Z}\}$ is a partitioning of $\mathbb{R}$.

(d) Given is a strictly increasing sequence $n_0 = 0 < n_1 < n_2 < \dots$ of nonnegative integers. For $k \in \mathbb{N}$ let $A_k := \{j \in \mathbb{N} : n_{k-1} < j \le n_k\}$. Then the set $\{A_k : k \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ (**not** of $\mathbb{Z}_{\ge 0}$!) ☐

Theorem 8.1 (De Morgan's Law)**.**

*Let there be a universal set Ω (see (**??**) on p.**??**). Then the following "duality principle" holds for any indexed family $(A_\alpha)_{\alpha \in I}$ of sets:*

(8.3) $$\textbf{(a)}\ \Big(\bigcup_\alpha A_\alpha\Big)^{\complement} = \bigcap_\alpha A_\alpha^{\complement} \qquad \textbf{(b)}\ \Big(\bigcap_\alpha A_\alpha\Big)^{\complement} = \bigcup_\alpha A_\alpha^{\complement}$$

We can state the above without the use of mathematical symbols as follows.

- *The complement of an arbitrary union is the intersection of the complements*
- *The complement of an arbitrary intersection is the union of the complements.*

Generally speaking the formulas are a consequence of the duality principle for set operations which states that any true statement involving a family of subsets of a universal sets can be converted into its "dual" true statement by replacing all unions with intersections and all intersections with unions.

PROOF of De Morgan's law, formula **(a)**:

1) First we prove that $(\bigcup_\alpha A_\alpha)^{\complement} \subseteq \bigcap_\alpha A_\alpha^{\complement}$:

Assume that $x \in (\bigcup_\alpha A_\alpha)^{\complement}$. Then $x \notin \bigcup_\alpha A_\alpha$ which is the same as saying that x does not belong to any of the A_α. That means that x belongs to each $A_\alpha^{\complement}$ and hence also to the intersection $\bigcap_\alpha A_\alpha^{\complement}$.

2) Now we prove that $(\bigcup_\alpha A_\alpha)^{\complement} \supseteq \bigcap_\alpha A_\alpha^{\complement}$:

Let $x \in \bigcap A_\alpha^{\complement}$. Then x belongs to each of the $A_\alpha^{\complement}$ and hence to none of the A_α. Then it also does not belong to the union of all the A_α and must therefore belong to the complement $(\bigcup_\alpha A_\alpha)^{\complement}$. This completes the proof of formula **(a)**.

The proof of formula **(b)** is very similar and given as exercise 8.3 on p.322. ■

You should draw the Venn diagrams involving just two sets A_1 and A_2 for both formulas a and b so that you understand the visual representation of De Morgan's law.

Proposition 8.1 (Distributivity of unions and intersections)**.**

Let $(A_i)_{i\in I}$ be an arbitrary family of sets and let B be a set. Then

$$\bigcup_{i\in I}(B\cap A_i) = B\cap\bigcup_{i\in I}A_i, \tag{8.4}$$

$$\bigcap_{i\in I}(B\cup A_i) = B\cup\bigcap_{i\in I}A_i. \tag{8.5}$$

PROOF: We only prove (8.4). The proof of (8.5) is left as exercise 8.5.

PROOF of "$\subseteq$": It follows from $B\cap A_i\subseteq A_i$ for all i that $\bigcup_i(B\cap A_i)\subseteq\bigcup_i A_i$. Moreover, $B\cap A_i\subseteq B$ for all i implies $\bigcup_i(B\cap A_i)\subseteq\bigcup_i B$ which equals B. It follows that $\bigcup_i(B\cap A_i)$ is contained in the intersection $\left(\bigcup_i A_i\right)\cap B$.

PROOF of "$\supseteq$": Let $x\in B\cap\bigcup_i A_i$. Then $x\in B$ and $x\in A_{i^\star}$ for some $i^\star\in I$, hence $x\in B\cap A_{i^\star}$, hence $x\in\bigcup_i(B\cap A_i)$. ■

Note that the next proposition is about finite unions and can be formulated and proven with what has been taught in chapter **??** (Preliminaries about Sets, Numbers and Functions) on p.**??**.

Proposition 8.2 (Rewrite unions as disjoint unions)**.**

Let $(A_j)_{j\in\mathbb{N}}$ be a sequence of sets which all are contained within the universal set Ω. Let

(a) $B_n := \bigcup_{j=1}^{n} A_j = A_1\cup A_2\cup\cdots\cup A_n\ (n\in\mathbb{N})$,

(b) $C_1 := A_1 = B_1,\quad C_{n+1} := A_{n+1}\setminus B_n\ (n\in\mathbb{N})$.

Then,

(c) *The sequence $(B_j)_j$ is increasing: $m<n\Rightarrow B_m\subseteq B_n$.*

(d) *For each $n\in\mathbb{N}$, $\bigcup_{j=1}^{n}A_j = \bigcup_{j=1}^{n}B_j$.*

(e) *The sets C_j are mutually disjoint and $\bigcup_{j=1}^{n}A_j = \biguplus_{j=1}^{n}C_j$.*

(f) *The sets C_j $(j\in\mathbb{N})$ form a partition of the set $\bigcup_{j=1}^{\infty}A_j$.*

PROOF of **(c)** and of **(d)**: Left as exercise 8.1 (p.322).

PROOF of **e:** Let $1\le j\le n$. We note that $C_j\subseteq A_j\subseteq B_j\subseteq B_n$ and obtain

$$C_j\cap C_{n+1}\subseteq B_n\cap C_{n+1} = B_n\cap(A_{n+1}\setminus B_n) = B_n\cap(A_{n+1}\cap B_n^{\complement}) = A_{n+1}\cap(B_n\cap B_n^{\complement}) = \emptyset.$$

We have proved that for any $j,k\in\mathbb{N}$ such that $j<k$ the sets C_j and C_k have empty intersection (we replaced $n+1$ with k) and it follows that the entire sequence of sets C_j is disjoint.

Next, we show that $\bigcup_{j=1}^{n}A_j = \bigcup_{j=1}^{n}B_j = \biguplus_{j=1}^{n}C_j$. The first equation follows from **(d)**. For the second equation, $\biguplus_{j=1}^{n}C_j\subseteq\bigcup_{j=1}^{n}A_j$ is obvious, since $C_n\subseteq A_n$ for all $n\in\mathbb{N}$.

We finally prove that $\bigcup_{j=1}^{n} A_j \subseteq \biguplus_{j=1}^{n} C_j$. Let $x \in \bigcup_{j=1}^{n} A_j$. Then $x \in A_j$ for at least one $1 \leq j \leq n$. Let j_0 be the smallest such j. If $j_0 = 1$ then $x \in C_1$ because $C_1 = A_1$, hence $x \in \biguplus_{j=1}^{n} C_j$ and we are done. Otherwise $x \notin A_j$ for all $1 \leq j < j_0$, hence $x \notin \bigcup_{j=1}^{j_0-1} A_j = B_{j_0-1}$, hence $x \in A_{j_0} \setminus B_{j_0-1}$, i.e., $x \in C_{j_0}$. It follows that $x \in \biguplus_{j=1}^{n} C_j$.

PROOF of **f:** This is a trivial consequence of **(e)**. ■

8.2 Rings and Algebras of Sets ⋆

Note that this entire section is starred, hence optional.

Definition 8.4 (Rings, algebras, and σ–Algebras of Sets)**.**

A subset $\mathscr{R}$ of 2^Ω (a set of sets!) is called a **ring of sets** if it is closed with respect to the operations "$\cup$" and "$\setminus$", i.e.,

$$R_1 \cup R_2 \in \mathscr{R} \text{ and } R_1 \setminus R_2 \in \mathscr{R} \quad \text{whenever } R_1, R_2 \in \mathscr{R}. \tag{8.6}$$

A subset $\mathscr{A}$ of 2^Ω is called an **algebra of sets** if $\Omega \in \mathscr{A}$ and $\mathscr{A}$ is a ring of sets.

A subset $\mathscr{F}$ of 2^Ω is called a **σ–algebra** if $\mathscr{F}$ is an algebra of sets which satisfies

$$(A_n)_{n\in\mathbb{N}} \in \mathfrak{F} \qquad \Rightarrow \qquad \bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{F} \ \square$$

It is worthwhile mentioning that σ–algebras are fundamental objects in measure theory and graduate level probability theory.

Parts **2a** through **2h** of the next proposition have already been encountered in prop.**??** on p.**??** of ch.**??** (Sets and Basic Set Operations). They have now been tagged with names such as "associativity of $\triangle$" which emphasize the connection to the rings we studied in ch.**??** (The Axiomatic Method).

Proposition 8.3.

(1) *Let $\mathscr{R}$ be a ring of sets and $A, B \in \mathscr{R}$. Then $\emptyset \in \mathscr{R}$, $A \triangle B \in \mathscr{R}$, and $A \cap B \in \mathscr{R}$.*

(2) *Let A, B, C, Ω be sets such that $A, B, C \subseteq \Omega$. Then*

- ***(a)*** $(A \triangle B) \triangle C = A \triangle (B \triangle C)$ *(associativity of $\triangle$)*
- ***(b)*** $A \triangle \emptyset = \emptyset \triangle A = A$ *(neutral element $\emptyset$ for $\triangle$)*
- ***(c)*** $A \triangle A = \emptyset$ *(inverse element $A^{-1} = A$ for $\triangle$)*
- ***(d)*** $A \triangle B = B \triangle A$ *(commutativity of $\triangle$)*

Further, we have the following for the intersection operation:

- ***(e)*** $(A \cap B) \cap C = A \cap (B \cap C)$ *(associativity of $\cap$)*
- ***(f)*** $A \cap \Omega = \Omega \cap A = A$ *(neutral element Ω for $\cap$)*
- ***(g)*** $A \cap B = B \cap A$ *(commutativity of $\cap$)*

Also, we have the following interrelationship between $\triangle$ and $\cap$:
(h) $A \cap (B \triangle C) \;=\; (A \cap B) \triangle (A \cap C)$ *(distributivity)*

PROOF:
For the proof of **2.a** see the one of prop.**??**. The proofs of the other properties are left as an exercise. ■

Remark 8.2 (Algebras of Sets as Rings)**.**

(1) Prop.8.3**(1)** states that the assignments $(A, B) \mapsto A \triangle B$ and $(A, B) \mapsto A \cap B$ are binary operations on $\mathscr{R}$.
(2) Items **(a)** – **(d)** of prop.8.3**(2)** assert that $(\mathscr{R}, \triangle)$ is an abelian group with neutral element $\emptyset$ and inverse $A^{-1} = A$.
(3) If $\Omega \in \mathscr{R}$, i.e., $\mathscr{R}$ is an algebra of sets, Items **(e)** – **(g)** of prop.8.3**(2)** assert that $(\mathscr{R}, \cap)$ is a commutative monoid with unit Ω.
(4) Assume that Ω is not empty. Then the "additive" neutral elenent $\emptyset$ is different from Ω, the "multiplicative" neutral elenent.
(5) **(1)** – **(4)** plus Proposition 8.3**(2)**.**(h)** imply that, if $\Omega \neq \emptyset$, then $(\mathscr{R}, \triangle, \cap)$ satisfies Definition **??** on p.**??**, i.e., $(\mathscr{R}, \triangle, \cap)$ is a commutative ring with unit.

The above justifies calling $\mathscr{R} = (\mathscr{R}, \triangle, \cap)$ a ring of sets. The name "algebra of sets" for a ring of sets which contains Ω stems from the fact that such systems of subsets of Ω are "boolean algebras". One can view this "algebra of sets = commutative ring with unit" mismatch in names from a different perspective: $(\mathscr{R}, \triangle, \cap)$ is a non–trivial example of what one generally calls a commutative ring in mathematics: an algebraic structure which satisfies all properties of a commutative ring with unit (Definition **??** on p.**??**), except that there need not be a multiplicative unit.

Note that we do not have an integral domain if Ω contains at least two elements ω and ω': Let $A \subseteq \Omega$ such that $\omega \in A$ and $\omega' \in A^{\complement}$. Then $A \neq \emptyset$ and $A^{\complement} \neq \emptyset$ but $A \cap A^{\complement} = \emptyset$, i.e., A and $A^{\complement}$ are a pair of zero divisors in $(\mathscr{R}, \triangle, \cap)$. □

8.3 Cartesian Products of More Than Two Sets

In this chapter we will extend the notion of a Cartesian product to more than two factors. Matter of fact, we will not stop at a finite number of factors and extend that concept to the product of factors X_i where the indices i are the members of an arbitrary index set.

Remark 8.3 (Associativity of cartesian products)**.** Assume we have three sets A, B and C. We can then look at

$$\begin{aligned} (A \times B) \times C &= \{((a, b), c) : a \in A, b \in B, c \in C\} \\ A \times (B \times C) &= \{(a, (b, c)) : a \in A, b \in B, c \in C\} \end{aligned}$$

The mapping

$$F : (A \times B) \times C \to A \times (B \times C), \qquad \big((a,b),c\big) \mapsto \big(a,(b,c)\big)$$

is bijective because it has the mapping

$$G : A \times (B \times C) \to (A \times B) \times C, \qquad \big(a,(b,c)\big) \mapsto \big((a,b),c\big)$$

as an inverse. For both $(A \times B) \times C$ and $A \times (B \times C)$ there are bijections to the set $\{(a,b,c) : a \in A, b \in B, c \in C\}$ of all triplets (a,b,c): the obvious bijections would be $(a,b,c) \mapsto \big((a,b),c\big)$ and $(a,b,c) \mapsto \big((a,b),c\big)$. $\square$

The last remark leads us to the following definition.

Definition 8.5 (Cartesian Product of three or more sets)**.** The **cartesian product** of three sets A, B and C is defined as

$$A \times B \times C \ := \ \{(a,b,c) : a \in A, b \in B, c \in C\}$$

i.e., it consists of all pairs (a,b,c) with $a \in A$, $b \in B$ and $c \in C$.
More generally, for N sets $X_1, X_2, X_3, \dots, X_N$ ($N \in \mathbb{N}$), we define the **cartesian product** as

$$X_1 \times X_2 \times X_3 \times \ldots \times X_N \ := \ \{(x_1, x_2, \dots, x_N) : x_j \in X_j \text{ for all } 1 \leq j \leq N\}$$

Note that the elements of this set are finite sequences in the sense of Definition **??** (finite sequences) on p.**??**.

Two elements $(x_1, x_2, \dots, x_N)$ and $(y_1, y_2, \dots, y_N)$ of $X_1 \times X_2 \times X_3 \times \ldots \times X_N$ are called **equal** if and only if $x_j = y_j$ for all j such that $1 \leq j \leq N$. In this case we write $(x_1, x_2, \dots, x_N) = (y_1, y_2, \dots, y_N)$.

As a shorthand, we abbreviate $X^N := \underbrace{X \times X \times + \cdots \times X}_{N \text{ times}}$. $\square$

Example 8.3 (N–dimensional coordinates)**.**
Here is the most important example of a cartesian product of N sets. Let $X_1 = X_2 = \ldots = X_N = \mathbb{R}$. Then

$$\mathbb{R}^N \ = \ \{(x_1, x_2, \dots, x_N) : x_j \in \mathbb{R} \text{ for } 1 \leq j \leq N\}$$

is the set of points in N–dimensional space. You may not be familiar with what those are unless $N = 2$ (see example **??** above) or $N = 3$.

In the 3–dimensional case it is customary to write (x,y,z) rather than (x_1, x_2, x_3) . Each such triplet of real numbers represents a point in (ordinary 3–dimensional) space and we speak of its x–coordinate, y–coordinate and z–coordinate. $\square$

Remark 8.4.

Here are some points concerning Definition 8.5 (Cartesian Product of three or more sets).

(a) For the sake of completeness: If $N = 1$, the item $(x) \in \mathbb{R}^1$ (where $x \in \mathbb{R}$; observe the parentheses around x) is considered the same as the real number x. In other words, we "identify" $\mathbb{R}^1$ with $\mathbb{R}$. Such a "one–dimensional point" is simply a point on the x–axis.

(b) A short note on vectors and coordinates: For $N \leq 3$ you can visualize the following: Given a point x on the x–axis or in the plane or in 3–dimensional space, there is a unique arrow that starts at the point whose coordinates are all zero (the **origin**) and ends at the location marked by the point x. Such an arrow is customarily called a vector.

(c) We showed for $n = 3$ that grouping factors with parentheses is benign since we can biject

$$(A \times B) \times C \xrightarrow{\sim} A \times (B \times C)\,.$$

For $n > 3$ one has to adopt the methods of Section **??** (Recursive Definitions of Sums, Products and Powers in Integral Domains). Given a sequence of sets $(A_j)_j$, one recursively defines

$$\text{(8.7)} \qquad \text{(i)} \quad \prod_{j=k}^{k} A_j \;=\; A_k, \qquad\qquad \text{(ii)} \quad \prod_{j=k}^{n+1} A_j \;=\; \prod_{j=k}^{n} A_j \;\times\; A_{n+1}.$$

One then proves formulas such as the analogue of **??(a)** on p.**??**:

If $m, n, p \in \mathbb{Z}$ such that $m \leq n < p$, then

$$\prod_{j=m}^{p} A_j \;=\; \prod_{j=m}^{n} A_j \;\times\; \prod_{j=n+1}^{p} A_j\,; \quad m \leq n < p$$

to prove by (strong) induction that no matter how grouping with parentheses is done, all resulting cartesian products of $A_1, \dots, A_n$ biject to $\prod\limits_{j=1}^{n} A_j$.

(d) Because it makes sense in dimensions $1, 2, 3$, an N–**tuple** $(x_1, x_2, \dots, x_N)$ of numbers is called a vector of dimension N. [1] You will read more about this in ch.**??** about vectors and vector spaces on page **??**.

You may recall from high school physics that each $x \in \mathbb{R}^N$ is uniquely identified with the corresponding vector: an arrow that starts in $(\underbrace{0, 0, \dots, 0}_{N\text{times}})$ and ends in x.

More will be said about n-dimensional space in section **??**, p.**??** on vectors and vector spaces. □

Example 8.4 (Parallelepipeds)**.** Let $a_1 < b_1, a_2 < b_2, a_3 < b_3$ be real numbers. Then

$$[a_1, b_1] \times [a_2, b_2] \times [a_3, b_3] \;=\; \{(x, y, z) : a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3\}$$

is the **parallelepiped** (box or quad parallel to the coordinate axes) with sides $[a_1, b_1], [a_2, b_2]$ and $[a_3, b_3]$. This generalizes in an obvious manner to N dimensions:

Let $N \in \mathbb{N}$ and $a_j < b_j$ ($j \in \mathbb{N}, j \leq N, a_j, b_j \in \mathbb{R}$). Then

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \;=\; \{(x_1, x_2, \dots, x_N) : a_j \leq x_j \leq b_j, j \in \mathbb{N}, j \leq N\}$$

is the parallelepiped with sides $[a_1, b_1], \dots, [a_N, b_N]$. □

We now introduce cartesian products of an entire family of sets $(X_i)_{i \in I}$.

[1] See Definition **??** (finite sequences) on p.**??**.

Definition 8.6 (Cartesian Product of a family of sets). ⋆

Let I be an arbitrary, nonempty set (the index set). Let $(X_i)_{i\in I}$ be a family of nonempty sets X_i.

The **cartesian product** of the family $(X_i)_{i\in I}$ is the set

$$\prod_{i\in I} X_i := \Big(\prod X_i\Big)_{i\in I} := \{(x_i)_{i\in I} : x_k \in X_k\ \forall k \in I\} \tag{8.8}$$

of all familes $(x_i)_{i\in I}$ each of whose members x_j belongs to the corresponding set X_j.

$(x_i)_{i\in I}$, $(y_k)_{k\in I} \in \prod\limits_{i\in I} X_i$ are called **equal** (we write $(x_i)_{i\in I} = (y_k)_{k\in I}$), if $x_j = y_j$ for all $j \in I$.

If all sets X_i are equal to one and the same set X, we also write

$$X^I := \prod_{i\in I} X := \prod_{i\in I} X_i\,. \ \square \tag{8.9}$$

The symbol "$\prod$" is the greek "upper case" letter "Pi" (whose lower case incarnation "π" you are probably more familiar with). We have used it previously in Definition **??** (Definition of $\prod\limits_{j=k}^{n} x_j$) on p.**??**.

Remark 8.5.
Note that, because I is not empty, $\prod\limits_{i\in I} X_i = \emptyset \Leftrightarrow$ there exists some $i \in I$ such that $X_i = \emptyset$.

Further, two families are equal in the sense of the above definition if and only if they are equal in the sense of Definition **??** on p.**??**. $\square$

Remark 8.6. We note that each element $(y_x)_{x\in X}$ of the cartesian product Y^X is the function

$$y(\cdot) : X \to Y, \qquad x \mapsto y_x$$

(see Definition **??** (indexed families) and the subsequent remarks concerining the equivalence of functions and families). In other words,

$$Y^X = \{f : f \text{ is a function with domain } X \text{ and codomain } Y\}. \ \square \tag{8.10}$$

8.4 Set Operations involving Direct Images and Preimages

Let X, Y be two nonempty sets and let $f : X \to Y$ be an arbitrary function with domain X and codomain Y. Let $A \subseteq X$ and $B \subseteq Y$. We recall from Definition **??** on p.**??** that

$$\begin{aligned} f(A) &= \{f(x) : x \in A\} \text{ is the direct image of } A,\\ f^{-1}(B) &= \{x \in X : f(x) \in B\} \text{ is the indirect image or preimage of } B. \end{aligned}$$

We now will examine to which extent direct and indirect images are compatible with unions, intersections, and other basic set operations.

> Unless stated otherwise, X, Y and f are as defined above for the remainder of this chapter: $f : X \to Y$ is a function with domain X and codomain Y.

Proposition 8.4 (f^{-1} is compatible with all basic set ops)**.**

Let J be an arbitrary index set. Let $B \subseteq Y$, $B_j \subseteq Y$ for all j. Then

$$f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j) \tag{8.11}$$

$$f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j) \tag{8.12}$$

$$f^{-1}(B^{\complement}) = \left(f^{-1}(B)\right)^{\complement} \tag{8.13}$$

$$f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2) \tag{8.14}$$

$$f^{-1}(B_1 \Delta B_2) = f^{-1}(B_1) \Delta f^{-1}(B_2) \tag{8.15}$$

PROOF of (8.11): Let $x \in X$. Then

$$\begin{aligned} x \in f^{-1}(\bigcap_{j\in J} B_j) &\Leftrightarrow f(x) \in \bigcap_{j\in J} B_j \quad \text{(def preimage)} \\ &\Leftrightarrow \forall j\ f(x) \in B_j \quad \text{(def } \cap\text{)} \\ &\Leftrightarrow \forall j\ x \in f^{-1}(B_j) \quad \text{(def preimage)} \\ &\Leftrightarrow x \in \bigcap_{j\in J} f^{-1}(B_j) \quad \text{(def } \cap\text{)} \end{aligned} \tag{8.16}$$

PROOF of (8.12): Let $x \in X$. Then

$$\begin{aligned} x \in f^{-1}(\bigcup_{j\in J} B_j) &\Leftrightarrow f(x) \in \bigcup_{j\in J} B_j \quad \text{(def preimage)} \\ &\Leftrightarrow \exists j_0 :\ f(x) \in B_{j_0} \quad \text{(def } \cup\text{)} \\ &\Leftrightarrow \exists j_0 : x \in f^{-1}(B_{j_0}) \quad \text{(def preimage)} \\ &\Leftrightarrow x \in \bigcup_{j\in J} f^{-1}(B_j) \quad \text{(def } \cup\text{)} \end{aligned} \tag{8.17}$$

PROOF of (8.13): Let $x \in X$. Then

$$\begin{aligned} x \in f^{-1}(B^{\complement}) &\Leftrightarrow f(x) \in B^{\complement} \quad \text{(def preimage)} \\ &\Leftrightarrow f(x) \notin B \quad \text{(def } \complement\text{)} \\ &\Leftrightarrow x \notin f^{-1}(B) \quad \text{(def preimage)} \\ &\Leftrightarrow x \in f^{-1}(B)^{\complement} \quad \text{(def } \complement\text{)} \end{aligned} \tag{8.18}$$

PROOF of (8.14): Let $x \in X$. Then

$$
\begin{aligned}
x \in f^{-1}(B_1 \setminus B_2) &\Leftrightarrow x \in f^{-1}(B_1 \cap B_2^{\complement}) \quad (\text{def } \setminus)\\
&\Leftrightarrow x \in f^{-1}(B_1) \cap f^{-1}(B_2^{\complement}) \quad (\text{see } (8.11))\\
&\Leftrightarrow x \in f^{-1}(B_1) \cap f^{-1}(B_2)^{\complement} \quad (\text{see } (8.13))\\
&\Leftrightarrow x \in f^{-1}(B_1) \setminus f^{-1}(B_2) \quad (\text{def } \setminus)
\end{aligned} \tag{8.19}
$$

PROOF of (8.15): This follows from $B_1 \Delta B_2 = (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$ and (8.12) and (8.14). ■

Proposition 8.5 (Properties of the direct image)**.**

Let J be an arbitrary index set. Let $A \subseteq X$, $A_j \subseteq X$ for all j. Then

$$f(\bigcap_{j \in J} A_j) \subseteq \bigcap_{j \in J} f(A_j) \tag{8.20}$$

$$f(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} f(A_j) \tag{8.21}$$

PROOF of (8.20): This follows from the monotonicity of the direct image (see **??**):

$$
\begin{aligned}
\bigcap_{j \in J} A_j \subseteq A_i \ \forall i \in J &\Rightarrow f(\bigcap_{j \in J} A_j) \subseteq f(A_i) \ \forall i \in J\\
&\Rightarrow f(\bigcap_{j \in J} A_j) \subseteq \bigcap_{i \in J} f(A_i) \quad (\text{def } \cap)
\end{aligned}
$$

First proof of (8.21)) - "Expert proof":

$$y \in f(\bigcup_{j \in J} A_j) \Leftrightarrow \exists\, x \in X: \ f(x) = y \text{ and } x \in \bigcup_{j \in J} A_j \quad (\text{def } f(A)) \tag{8.22}$$

$$\Leftrightarrow \exists\, x \in X \text{ and } j_0 \in J: \ f(x) = y \text{ and } x \in A_{j_0} \quad (\text{def } \cup) \tag{8.23}$$

$$\Leftrightarrow \exists\, x \in X \text{ and } j_0 \in J: \ f(x) = y \text{ and } f(x) \in f(A_{j_0}) \quad (\text{def } f(A)) \tag{8.24}$$

$$\Leftrightarrow \exists\, j_0 \in J: \ y \in f(A_{j_0}) \quad (\text{def } f(A)) \tag{8.25}$$

$$\Leftrightarrow y \in \bigcup_{j \in J} f(A_j) \quad (\text{def } \cup) \tag{8.26}$$

Alternate proof of (8.21) - Proving each inclusion separately. Unless you have a lot of practice reading and writing proofs whose subject is the equality of two sets you should write your proof the following way:

A. Proof of "$\subseteq$":

$$y \in f(\bigcup_{j \in J} A_j) \Rightarrow \exists\, x \in X: \ f(x) = y \text{ and } x \in \bigcup_{j \in J} A_j \quad (\text{def } f(A)) \tag{8.27}$$

$$\Rightarrow \exists\, j_0 \in J: \ f(x) = y \text{ and } x \in A_{j_0} \quad (\text{def } \cup) \tag{8.28}$$

$$\Rightarrow y = f(x) \in f(A_{j_0}) (\text{def } f(A)) \tag{8.29}$$

$$\Rightarrow y \in \bigcup_{j \in J} f(A_j) \quad (\text{def } \cup) \tag{8.30}$$

B. Proof of "⊇":

This follows from the monotonicity of $A \mapsto f(A)$ (see **??**):

$$A_i \subseteq \bigcup_{j \in J} A_j \;\forall\, i \in J \;\Rightarrow f(A_i) \subseteq f\big(\bigcup_{j \in J} A_j\big) \;\forall\, i \in J \tag{8.31}$$

$$\Rightarrow \bigcup_{i \in J} f(A_i) \subseteq f\big(\bigcup_{j \in J} A_j\big) \;\forall\, i \in J \quad (\text{def} \cup) \;\blacksquare \tag{8.32}$$

The "elementary" proof is barely longer than the first one, but it is so much easier to understand!

Remark 8.7. In general, there will not be equality in (8.20). Counterexample: $f(x) = x^2$ with domain $\mathbb{R}$: Let $A_1 :=]-\infty, 0]$ and $A_2; = [0, \infty[$. Then $A_1 \cap A_2 \;=\; \{0\}$, hence $f(A_1 \cap A_2) \;=\; f(\{0\}) \;=\; \{0\}$. On the other hand, $f(A_1) = f(A_2) = [0, \infty]$, hence $f(A_1) \cap f(A_2) \;=\; [0, \infty]$. Clearly, $\{0\} \subsetneq [0, \infty]$. □

Proposition 8.6 (Direct images and preimages of function composition)**.**

> *Let X, Y, Z be arbitrary, nonempty sets.*
> *Let $f : X \to Y$ and $g : Y \to Z$, and let $U \subseteq X$ and $W \subseteq Z$. Then*
>
> $$(g \circ f)(U) \;=\; g\big(f(U)\big) \text{ for all } U \subseteq X\,. \tag{8.33}$$
>
> $$(g \circ f)^{-1}(W) \;=\; f^{-1}\big(g^{-1}(W)\big) \text{ for all } W \subseteq Z, \text{ i.e., } \quad (g \circ f)^{-1} \;=\; f^{-1} \circ g^{-1}\,. \tag{8.34}$$

PROOF of (8.33): Left as exercise 8.10.

PROOF of (8.34):

a. "⊆": Let $W \subseteq Z$ and $x \in (g \circ f)^{-1}(W)$. Then $g\big(f(x)\big) = (g \circ f)(x) \in W$, hence $f(x) \in g^{-1}(W)$. But then $x \in f^{-1}\big(g^{-1}(W)\big)$. This proves "⊆".

b. "⊇": Let $W \subseteq Z$, $h := g \circ f$, and $x \in f^{-1}\big(g^{-1}(W)\big)$. Then $f(x) \in g^{-1}(W)$, hence $g(f(x)) \in W$, i.e., $h(x) \in W$, hence $x \in h^{-1}(W) = (g \circ f)^{-1}(W)$. This proves "⊇". ■

Proposition 8.7 (Indirect image and fibers of f)**.**

> *Let X, Y be nonempty sets and let $f : X \to Y$ be a function. We define on the domain X a relation "$\sim$" as follows:*
>
> $$x_1 \;\sim\; x_2 \quad\Leftrightarrow\quad f(x_1) \;=\; f(x_2). \tag{8.35}$$
>
> ***(a)*** *"$\sim$" is an equivalence relation. Its equivalence classes, which we denote by $[x]_f$,* [2] *are*
>
> $$[x]_f \;= \{a \in X : f(a) = f(x)\} \;=\; f^{-1}\{f(x)\}. \quad (x \in X) \tag{8.36}$$
>
> ***(b)*** *If $A \subseteq X$ then*
>
> $$f^{-1}(f(A)) \;=\; \bigcup_{a \in A} [a]_f. \tag{8.37}$$

The proof that "∼" is an equivalence relation is left as exercise 8.11.

PROOF of (8.36): The equation on the left is nothing but the definition of the equivalence classes generated by an equivalence relation. The equation on the right follows from $f(a) = f(x) \Leftrightarrow a \in f^{-1}\{f(x)\}$, which is true according to the definition of preimages.

PROOF of (8.37):

Since $f(A) = f(\bigcup_{a\in A}\{a\}) = \bigcup_{a\in A} f\{a\} = \bigcup_{a\in A}\{f(a)\}$ (see 8.21), it follows that

$$f^{-1}(f(A)) = f^{-1}(\bigcup_{a\in A}\{f(a)\}) \tag{8.38}$$

$$= \bigcup_{a\in A} f^{-1}\{f(a)\} \quad \text{(see 8.12)} \tag{8.39}$$

$$= \bigcup_{a\in A} [a]_f \quad \text{(see 8.36)} \ \blacksquare \tag{8.40}$$

Corollary 8.1.

$$\textit{If } A \subseteq X \quad \textit{then} \quad f^{-1}(f(A)) \supseteq A. \tag{8.41}$$

The proof is left as exercise 8.12 (see p.323). ■

The next example shows how to work with fibers to prove that certain relations are equivalence relations.

Example 8.5.

The following are equivalence relations on the set X.

(a) $X = \mathbb{R}$ and $x \sim y \Leftrightarrow |x| = |y|$.

(b) $X = \mathbb{R}_{\neq 0} = \{x \in \mathbb{R} : x \neq 0\}$ and $x \sim y \Leftrightarrow |xy| > 0$.

(c) $X = \mathbb{R}^3$ and $(x,y,z) \sim (u,v,w) \Leftrightarrow z\sin(xy) = w\sin(uv)$.

You can verify this by brute force, but here is an elegant way. Rewrite the equivalence relations as $\alpha \sim \beta \Leftrightarrow F(\alpha) = F(\beta)$ for a suitable function $F(\cdot)$, then apply prop.8.7 (Indirect image and fibers of f).

For the above examples you do this as follows:

(a) $F : X \to \mathbb{R}, \quad x \mapsto |x|$.

(b) $G : X \to \{-1, 1\}, \quad x \mapsto \frac{x}{|x|}$.

(b) $H : X \to \mathbb{R}, \quad (x,y,z) \mapsto z\sin(xy)$. □

Proposition 8.8.

$$\textit{If } B \subseteq Y \quad \textit{then} \quad f(f^{-1}(B)) = B \cap f(X). \tag{8.42}$$

PROOF of "$\subseteq$":

Let $y \in f(f^{-1}(B))$. There exists $x_0 \in f^{-1}(B)$ such that $f(x_0) = y$ (def direct image). We have

(a) $x_0 \in f^{-1}(B) \Rightarrow f(x_0) \in B$ (def. of preimage)

(b) Of course $x_0 \in X$. Hence $f(x_0) \in f(X)$.

(a) and **(b)** together imply that $y = f(x_0) \in B \cap f(X)$.

PROOF of "$\supseteq$":

This part of the proof is left as exercise 8.13 (see p.323). ■

Remark 8.8. Be sure to understand how the assumption $y \in f(X)$ was used. □

Corollary 8.2.

$$\text{If } B \subseteq Y \text{ then } f(f^{-1}(B)) \subseteq B. \tag{8.43}$$

Trivial as $f(f^{-1}(B)) = B \cap f(X) \subseteq B$. ■

Proposition 8.9.

(a) *Let $A \subseteq X$. If $f : X \to Y$ is injective then $f^{-1}(f(A)) = A$.*
(b) *Let $B \subseteq Y$. If $f : X \to Y$ is surjective then $f\big(f^{-1}(B)\big) = B$.*
(c) *Let $A \subseteq X$ and $B \subseteq Y$. If $f : X \to Y$ is injective and if $B = f(A)$ then $f^{-1}(B) = A$.*
(d) *Let $A \subseteq X$ and $B \subseteq Y$. If $f : X \to Y$ is surjective and if $f^{-1}(B) = A$ then $B = f(A)$.*
(e) *Let $A \subseteq X$ and $B \subseteq Y$. If $f : X \to Y$ is bijective then $B = f(A) \Leftrightarrow f^{-1}(B) = A$.*

PROOF: Left as exercise 8.14 on p.323. ■

Remark 8.9.

It follows from prop.8.9 parts **(a)** and **(b)** together with thm.**??** (Characterization of inverse functions) on p.**??** that if $f : X \to Y$ is a bijection between two nonempty sets X and Y then the direct image function $f : 2^X \to 2^Y$; $A \mapsto \{f(a) : a \in A\}$ is a bijection between the two power sets of X and Y, and its inverse is the preimage function $f^{-1} : 2^Y \to 2^X$; $B \mapsto \{x \in X : f(x) \in B\}$.

Proposition 8.10.

Let J be an arbitrary nonempty index set and let $A \subseteq X$, $A_j \subseteq X$ for all j.

Let $f : X \to Y$ be bijective. Then the following all are true:

$$f\Big(\bigcap_{j\in J} A_j\Big) = \bigcap_{j\in J} f(A_j) \tag{8.44}$$

$$f\Big(\bigcup_{j\in J} A_j\Big) = \bigcup_{j\in J} f(A_j) \tag{8.45}$$

$$f(A^{\complement}) = f(A)^{\complement} \tag{8.46}$$

$$f(A_1 \setminus A_2) = f(A_1) \setminus f(A_2) \tag{8.47}$$

$$f(A_1 \Delta A_2) = f(A_1) \Delta f(A_2) \tag{8.48}$$

PROOF: Left as exercise 8.16 on p.324. ■

Note that the remaining content of this chapter has been marked as " ⋆ " (optional)!

Proposition 8.11.

⋆ *Let $f : X \to Y$ be bijective.*
Let J be an arbitrary nonempty index set and let $(A_j)_{j\in J}$ be a partition of X, i.e., if $i \neq j$ then $A_i \cap A_j = \emptyset$ and $X = \biguplus_j A_j$. Assume further that none of the A_j are enpty. For $j \in J$ let $B_j := f(A_j)$. Then

(a) *$(B_j)_{j\in J}$ is a partition of Y.*

(b) *For $j \in J$ we look at the restriction $f\big|_{A_j} : A_j \to Y$ to A_j. Then $f\big|_{A_j}(A_j) = B_j$ and the function*

$$f_j : A_j \to B_j, \qquad x \mapsto f_j(x) := f\big|_{A_j}(x) = f(x)$$

is a bijection.

PROOF: Left as exercise 8.17 on p.324. ■

Corollary 8.3.

⋆ *Let $f : X \to Y$ be bijective. Let $A \subset X, A \neq \emptyset$ (strict inclusion, so $A^{\complement} \neq \emptyset$). Then both*

$$f_A : A \to f(A), \quad x \to f(x) \qquad \text{and} \quad f_{A^{\complement}} : A^{\complement} \to f(A^{\complement}), \quad x \to f(x)$$

are bijections.

PROOF: This follows from prop.8.11, applied to $J = \{1, 2\}, A_1 = A, A_2 = A^{\complement}$.
■

The next corollary is B/G [1] prop.13.2.

Corollary 8.4.

★ *Let $f : X \to Y$ be bijective. Let $a \subset X$ and assume that $X \neq \{a\}$. Then*

$$\tilde{f} : X \setminus \{a\} \to Y \setminus \{f(a)\}, \quad x \to f(x)$$

also is bijective.

PROOF: This follows from 8.4 applied to $A = \{a\}$ and the fact that $f(\{a\}) = \{f(a)\}$. ■

The following two propositions allow you to replace bijective and surjective functions with more suitable ones that inherit bijectivity or surjectivity. This will come in handy when we prove propositions concerning cardinality.

The first proposition shows how to preserve bijectivity if two function values need to be switched around.

Proposition 8.12.

★ *Let $X, Y \neq \emptyset$, let $f : X \to Y$ be bijective and let $x_1, x_2 \in X$. Let*

$$g(x) := \begin{cases} f(x_2) & \text{if} x = x_1, \\ f(x_1) & \text{if} x = x_2, \\ f(x) & \text{if} x \neq x_1, x_2. \end{cases} \tag{8.49}$$

(In other words, we swap two function arguments). Then $g : X \to Y$ also is bijective.

PROOF: Left as exercise 8.18 on p.324. ■

A more general version of the above shows how to preserve surjectivity if two function values need to be switched around.

Proposition 8.13.

★ *Let $X, Y \neq \emptyset$ and assume that Y contains at least two elements y_1 and y_2. Let $f : X \to Y$ be surjective.*

Let $A_1 := f^{-1}\{y_1\}$, $A_2 := f^{-1}\{y_2\}$, *and* $B := X \setminus (A_1 \cup A_2)$. *Let*

$$g(x) := \begin{cases} y_2 & if x \in A_1, \\ y_1 & if x \in A_2, \\ f(x) & if x \in B. \end{cases} \tag{8.50}$$

In other words, everything that f *maps to* y_1 *is now mapped to* y_2 *and everything that* f *maps to* y_2 *is now mapped to* y_1. *Then* $g : X \to Y$ *also is surjective.*

PROOF: Left as exercise 8.19 on p.324. ■

Proposition 8.14.

★ *Let* X, Y *be two nonempty sets and let* $f : X \to Y$ *be surjective. Let* $\emptyset \neq B \subsetneq Y$ *so that* $Y = B \uplus B^{\complement}$ *is a partitioning of* Y *into two nonempty subsets* B *and* $B^{\complement}$. *Let* $A := \{f \in B\}$. *Then the restrictions* $f_1 := f\big|_A : A \to B$ *and* $f_2 := f\big|_{A^{\complement}} : A^{\complement} \to B^{\complement}$ *of* f *to* A *and* $A^{\complement}$ *are surjections.*

PROOF: Left as exercise 8.20 on p.324. ■

8.5 Indicator Functions ★

Sometimes it is advantageous to think of the subsets of a universal set Ω as "binary" functions $\Omega \to \{0, 1\}$.

Definition 8.7 (indicator function for a set)**.**

Let Ω be "the" universal set, i.e., we restrict our scope of interest to subsets of Ω. Let $A \subseteq \Omega$. Let $\mathbf{1}_A : \Omega \to \{0, 1\}$ be the function defined as

$$\mathbf{1}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A, \\ 0 & \text{if } \omega \notin A. \end{cases} \tag{8.51}$$

$\mathbf{1}_A$ is called the **indicator function** of the set A. □

Some authors call $\mathbf{1}_A$ the **characteristic function** of A and/or write χ_A or $\mathbb{1}_A$ instead of $\mathbf{1}_A$.

Recall the following about functions and families: If X and Y are two nonempty sets then Y^X, the "X–fold cartesian product of Y", is the set of all Y–valued families $(y_x)_{x \in X}$ which are indexed by X. Equivalently Y^X is the set of all functions $f : X \to Y$ with domain X and codomain Y. See Proposition **??** (Functions are families and families are functions) on p.**??**. This will be used in the next proposition which shows that the association of a subset A of Ω with its indicator function $\mathbf{1}_A$ is a bijection.

Proposition 8.15.

Let $\mathscr{F}(\Omega, \{0,1\}) := \{0,1\}^\Omega$ denote the set of all functions $f : \Omega \to \{0,1\}$, i.e., all functions f with domain Ω for which the only possible function values $f(\omega)$ are zero or one. [3]

(a) *The mapping*

$$F : 2^\Omega \to \mathscr{F}(\Omega, \{0,1\}), \quad \text{defined as } F(A) := \mathbf{1}_A \tag{8.52}$$

which assigns to each subset of Ω its indicator function is injective.

(b) *Let $f \in \mathscr{F}(\Omega, \{0,1\})$. Further, let $A := \{f = 1\} = f^{-1}(\{1\}) = \{a \in A : f(a) = 1\}$. Then $f = \mathbf{1}_A$.*

(c) *The function F above is bijective.*

Its inverse function is

$$G : \mathscr{F}(\Omega, \{0,1\}) \to 2^\Omega, \quad \text{defined as } G(f) := \{f = 1\}. \tag{8.53}$$

PROOF of **(a)**: This follows from **(c)** which will be proved below.

PROOF of **(b)**: We have

$$\begin{aligned} f(\omega) = 1 &\Leftrightarrow \omega \in \{f = 1\} \quad \text{(def. of inverse image)} \\ &\Leftrightarrow \omega \in A \quad (\text{because } A = \{f = 1\}) \\ &\Leftrightarrow \mathbf{1}_A(\omega) = 1 \quad \text{(def. of indicator function).} \end{aligned}$$

It follows that $f(\omega) = 1$ if and only if $\mathbf{1}_A(\omega) = 1$. Since the only other possible function value is 0 we conclude that $f(\omega) = 0$ if and only if $\mathbf{1}_A(\omega) = 0$. It follows that $f(\omega) = \mathbf{1}_A(\omega)$ for all $\omega \in \Omega$, i.e., $f = \mathbf{1}_A$. This proves **(b)**.

PROOF of **(c)**: According to theorem **??** on p.**??** about the characterization of inverse functions **(c)** is proved if we can demonstrate that F and G are inverse to each other. To prove this it suffices to show that

$$G \circ F = id_{2^\Omega} \quad \text{and} \quad F \circ G = id_{\mathscr{F}(\Omega, \{0,1\})}. \tag{8.54}$$

Let $A \in 2^\Omega$, i.e., $A \subseteq \Omega$. Then

$$G \circ F(A) = G(\mathbf{1}_A) = \{\mathbf{1}_A = 1\} = \{\omega \in \Omega : \mathbf{1}_A(\omega) = 1\} = \{\omega \in \Omega : \omega \in A\} = A.$$

This proves $G \circ F = id_{2^\Omega}$. Now let $f \in \mathscr{F}(\Omega, \{0,1\}$ and $\omega \in \Omega$. Then

$$\begin{aligned} \big(F \circ G(f)\big)(\omega) &= F(\{f = 1\})(\omega) = \mathbf{1}_{\{f=1\}}(\omega) \\ &= \begin{cases} 1 & \text{iff } \omega \in \{f = 1\}, \\ 0 & \text{iff } \omega \notin \{f = 1\} \end{cases} = \begin{cases} 1 & \text{iff } f(\omega) = 1, \\ 0 & \text{iff } f(\omega) \neq 1 \end{cases} = \begin{cases} 1 & \text{iff } f(\omega) = 1, \\ 0 & \text{iff } f(\omega) = 0 \end{cases} = f(\omega). \end{aligned}$$

[3] See remark 8.6 on p.311, ch.8.3 (Cartesian Products of More Than Two Sets).

The equation next to the last results from the fact that the only possible function values for f are 0 and 1; the equation before that follows from (**??**) (definition of the preimage). It follows from the above chain of equations that $F \circ G(f) = f = id_{\mathscr{F}(\Omega,\{0,1\})}(f)$ for all $f \in \mathscr{F}(\Omega,\{0,1\})$, hence $F \circ G = id_{\mathscr{F}(\Omega,\{0,1\})}$. We have proved (8.54) and hence **(c)**. ■

Let $m, n \in \mathbb{Z}$. We recall from Definition **??** (Equivalence Modulo n) (p.**??** of ch. **??**) that $m+n$ mod 2 (the sum mod 2 of m and n) is given by

$$m + n \mod 2 = \begin{cases} 0 & \Leftrightarrow \quad (m+n)/2 \text{ has remainder } 0, \text{ i.e., } m+n \text{ is even,} \\ 1 & \Leftrightarrow \quad (m+n)/2 \text{ has remainder } 1, \text{ i.e., } m+n \text{ is odd.} \end{cases} \tag{8.55}$$

Proposition 8.16. *Let $m, n, p \in \mathbb{Z}$. Then addition mod 2 is associative, i.e.,*

$$\big(m + n \mod 2\big) + p \mod 2 = m + \big(n + p \mod 2\big) \mod 2. \tag{8.56}$$

PROOF: This follows from Theorem **??** on p.**??** ($\mathbb{Z}_n$ is a commutative ring with unit). [4] ■

Proposition 8.17.

Let A, B, C be subsets of Ω. Then

$$\mathbb{1}_{A\cup B} = \max(\mathbb{1}_A, \mathbb{1}_B), \tag{8.57}$$
$$\mathbb{1}_{A\cap B} = \min(\mathbb{1}_A, \mathbb{1}_B), \tag{8.58}$$
$$\mathbb{1}_{A^\complement} = 1 - \mathbb{1}_A, \tag{8.59}$$
$$\mathbb{1}_{A\triangle B} = \mathbb{1}_A + \mathbb{1}_B \mod 2. \tag{8.60}$$

PROOF: The proof of the first three equations is left as an exercise.
PROOF of (8.60): This follows easily from the the fact that

$$(A\triangle B)^\complement = \{\omega \in \Omega : [\text{either } \omega \in A \cap B] \text{ or } [\text{neither } \omega \in A \text{ nor } \omega \in B]\}$$ ■

Prop.8.16 above helps us to prove associativity of symmetric set differences.

Proposition 8.18 (Symmetric set differences $A\triangle B$ are associative)**.**

Let $A, B, C \subseteq \Omega$. Then

$$(A\triangle B)\triangle C = A\triangle(B\triangle C). \tag{8.61}$$

[4] There also are elementary proofs for this proposition. See exercise 8.23 on p.325.

PROOF: This follows easily from (8.60) and and the associativity of $a \oplus b := a + b \mod 2$ as follows. Let $\omega \in \Omega$. Then

$$\begin{aligned} \omega \in (A \triangle B) \triangle C \ &\Leftrightarrow \mathbb{1}_{(A \triangle B) \triangle C}(\omega) = 1 \\ &\Leftrightarrow \big(\mathbb{1}_A(\omega) \oplus \mathbb{1}_B(\omega)\big) \oplus \mathbb{1}_C(\omega) = 1 \\ &\Leftrightarrow \mathbb{1}_A(\omega) \oplus \big(\mathbb{1}_B(\omega) \oplus \mathbb{1}_C(\omega)\big) = 1 \\ &\Leftrightarrow \mathbb{1}_{A \triangle (B \triangle C)}(\omega) = 1 \ \Leftrightarrow \ \omega \in A \triangle (B \triangle C). \end{aligned}$$

We obtained the equivalence in the middle from prop.8.16. ■

8.6 Exercises for Ch.8

Exercise 8.1.

Prove **(a)** and **(b)** of prop.8.2 (Rewrite unions as disjoint unions) on p.306:

Let $(A_j)_{j \in \mathbb{N}}$ such that $A_j \subseteq \Omega$ for all $j \in \mathbb{N}$. For $n \in \mathbb{N}$ let $B_n := \bigcup_{j=1}^{n} A_j \ = \ A_1 \cup A_2 \cup \cdots \cup A_n$

Further, let $C_1 := A_1 = B_1$ and $C_{n+1} := A_{n+1} \setminus B_n$ $(n \in \mathbb{N})$. Then

(a) The sequence $(B_j)_j$ is increasing: $m < n \Rightarrow B_m \subseteq B_n$,

(b) For each $n \in \mathbb{N}$, $\bigcup_{j=1}^{n} A_j \ = \ \bigcup_{j=1}^{n} B_j$. □

Exercise 8.2.

(See example **??** on p.**??**). Let $X := [0, 2]$. For $0 \leq x \leq 2$ let $A_x := [x, 2x]$.

(a) What is $\bigcap \Big[A_x : x \in X\Big]$? **(b)** What is $\bigcup \Big[A_x : x \in X\Big]$? □

Exercise 8.3.

Prove **(b)** of thm.8.1 (De Morgan's Law):

Let $(A_\alpha)_{\alpha \in I}$ be a family of subsets of a universal set Ω. Then $\big(\bigcap_\alpha A_\alpha\big)^\complement \ = \ \bigcup_\alpha A_\alpha^\complement$. □

Exercise 8.4.

Supply the missing proofs of prop.8.3 on p.307 of this document. □

Exercise 8.5.

Prove the second formula of prop.8.1 (Distributivity of unions and intersections): Let $(A_i)_{i \in I}$ be an arbitrary family of sets and let B be a set. Then

$$\bigcap_{i \in I} (B \cup A_i) \ = B \cup \bigcap_{i \in I} A_i. \ \square$$

Exercise 8.6.

Let $(G, \diamond)$ be a group, let $(H_i)_{i \in J}$ be a family of subgroups of G, and let $H := \bigcap_{i \in J} H_i$. Then H is a subgroup of G. □

Exercise 8.7.

Let f be the function $f : [-3, 3] \to \mathbb{R}; \quad x \mapsto x^2$.

(a) Is $f \in [-3, 3]^{\mathbb{R}}$ or $f \in \mathbb{R}^{[-3,3]}$?

(b) Write f as a family. **Hint:** What is the index set? Domain or codomain? □

Exercise 8.8.

Let $f : [-2.\infty[\to \mathbb{R}; \quad x \mapsto x^2$. Compute the following.

(a) $f\big(f^{-1}([-4, 4])\big)$, **(b)** $f^{-1}\big(f([0, 3])\big)$. □

Exercise 8.9.

Prove prop.**??** on p.**??**:

(a) $f(\emptyset) = f^{-1}(\emptyset) = \emptyset$

(b) $A_1 \subseteq A_2 \subseteq X \Rightarrow f(A_1) \subseteq f(A_2)$

(c) $B_1 \subseteq B_2 \subseteq Y \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(d) $x \in X \Rightarrow f(\{x\}) = \{f(x)\}$

(e) $f(X) = Y \Leftrightarrow f$ is surjective

(f) $f^{-1}(Y) = X$ always! □

Exercise 8.10.

Prove (8.33) of prop.8.6 on p.314: Let X, Y, Z be arbitrary, nonempty sets.
Let $f : X \to Y$ and $g : Y \to Z$. Then $(g \circ f)(U) = g\big(f(U)\big)$ for all $U \subseteq X$. □

Exercise 8.11.

Let X, Y be nonempty sets and let $f : X \to Y$ be a function. We define on the domain X a relation "$\sim$" as follows:

$$x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2). \ \square$$

(See prop.8.7 (Indirect image and fibers of f) on p.314). Prove that "$\sim$" is an equivalence relation.

Exercise 8.12.

Prove cor.8.1 on p.315 of this document: If $A \subseteq X$ then $f^{-1}(f(A)) \supseteq A$. □

Exercise 8.13.

Prove prop.8.8 on p.315 of this document: If $B \subseteq Y$ then $f(f^{-1}(B)) = B \cap f(X)$. □

Exercise 8.14.

Prove prop.8.9 on p.316.
Hint: The main tools you need are prop.8.7 on p.314, prop8.8 on p.315, and their corollaries. □

Exercise 8.15. Prove the reverse directions of prop.8.9**(a)** and prop.8.9**(b)** on p.316.

(a) If $f : X \to Y$ satisfies $f^{-1}\big(f(A)\big) = A$ for all $A \subseteq X$ then f is injective.

(b) If $f : X \to Y$ satisfies $f\big(f^{-1}(B)\big) = B$ for all $B \subseteq Y$ then f is surjective.

Exercise 8.16.

Prove prop.8.10 on p.316.
Hint: Work with the inverse of f and apply prop.8.4 on p.312. $\square$

Exercise 8.17.

Prove prop.8.11 on p.317.
Hint: To prove **(a)**, use prop.8.5 on p.313. $\square$

Exercise 8.18.

Prove prop.8.12 on p.318: Let $X, Y \neq \emptyset$, let $f : X \to Y$ be bijective and let $x_1, x_2 \in X$. Let

$$g(x) := \begin{cases} f(x_2) & \text{if } x = x_1, \\ f(x_1) & \text{if } x = x_2, \\ f(x) & \text{if } x \neq x_1, x_2. \end{cases}$$

(In other words, we swap two function arguments). Then $g : X \to Y$ also is bijective. $\square$

Exercise 8.19.

Prove prop.8.13 on p.318: Let $X, Y \neq \emptyset$ and assume that Y contains at least two elements y_1 and y_2. Let $f : X \to Y$ be surjective.
Let $A_1 := f^{-1}\{y_1\}$, $A_2 := f^{-1}\{y_2\}$, and $B := X \setminus (A_1 \cup A_2)$. Let

$$g(x) := \begin{cases} y_2 & \text{if } x \in A_1, \\ y_1 & \text{if } x \in A_2, \\ f(x) & \text{if } x \in B. \end{cases}$$

In other words, everything that f maps to y_1 is now mapped to y_2 and everything that f maps to y_2 is now mapped to y_1. Then $g : X \to Y$ also is surjective. $\square$

Exercise 8.20.

Prove prop.8.14 on p.319: Let X, Y be two nonempty sets and let $f : X \to Y$ be surjective. Let $\emptyset \neq B \subsetneq Y$ so that $Y = B \uplus B^{\complement}$ is a partitioning of Y into two nonempty subsets B and $B^{\complement}$. Let $A := \{f \in B\}$. Then the restrictions $f_1 := f\big|_A : A \to B$ and $f_2 := f\big|_{A^{\complement}} : A^{\complement} \to B^{\complement}$ of f to A and to $A^{\complement}$ are surjections. $\square$

Exercise 8.21.

Prove prop.8.16 on p.321: Let $m, n, p \in \mathbb{Z}$. Then

$$\big(m + n \mod 2\big) + p \mod 2 \quad = \quad m + \big(n + p \mod 2\big) \mod 2.$$

directly, i.e., without referring to Theorem **??** on p.**??** ($\mathbb{Z}_n$ is CRU).
Hint: There are eight possible combinations of zeros and ones for the functions

$$(m, n, p) \to \big(m + n \mod 2\big) + p \mod 2 \quad \text{and} \quad (m, n, p) \to m + \big(n + p \mod 2\big) \mod 2.$$

Complete the entries in the table below and show that the entries in the two rightmost columns match. To save space, write $m \oplus n$ for $m + n \mod 2$. To get you started, the row for $m = 1, n = 0, p = 0$ has been already completed.

m	n	p	$m \oplus n$	$n \oplus p$	$(m \oplus n) \oplus p$	$m \oplus (n \oplus p$
0	0	0				
0	0	1				
0	1	0				
0	1	1				
1	0	0	1	0	1	1
1	0	1				
1	1	0				
1	1	1				

☐

Exercise 8.22.

See [1] B/G project 6.8 on p.58 for the following.

Prove that the following are equivalence relations on $\mathbb{R}^2$.

- **(a)** $(x, y) \sim (u, v) \;\Leftrightarrow\; \sqrt{x^2 + y^2} = \sqrt{u^2 + v^2}$.
- **(b)** $X = \mathbb{R}_{\neq 0} \;=\; \{x \in \mathbb{R} : x \neq 0\}$ and $x \sim y \;\Leftrightarrow\; |xy| > 0$.
- **(c)** $X = \mathbb{R}^3$ and $(x, y, z) \sim (u, v, w) \;\Leftrightarrow\; z \sin(xy) = w \sin(uv)$.

Hint: See example 8.5 on p.315. ☐

Exercise 8.23. Let $m, n, p \in \mathbb{Z}$.

Prove that addition mod 2 is associative. (see prop.8.16 on p.321) without referring to Theorem **??**. Rather inspect what happens for each of the eight possible combinations of zeros and ones for the functions $(m, n, p) \to \big(m + n \mod 2\big) + p \mod 2$ and $(m, n, p) \to m + \big(n + p \mod 2\big) \mod 2$ ☐

References

[1] Matthias Beck and Ross Geoghegan. The Art of Proof. Springer, 1st edition, 2010.

List of Symbols

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