Math 330 - Additional Material

Student edition with proofs

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Contents

10 Cardinality II: Comparing Uncountable Sets

If we want to compare sets based on their sizes then we have a good idea how to go about it, at least as far as finite sets (excluding the empty set) and countable sets are concerned. We can biject them to a subset of the natural numbers, and then compare their images in $\mathbb{N}$: If X_1 and X_2 are finite sets such that we have bijective functions $X_1 \xrightarrow{f_1} [n_1]$ and $X_2 \xrightarrow{f_2} [n_2]$ such that $n_1 \leq n_2$, and if C is countably infinite with a bijection $C \xrightarrow{c} \mathbb{N}$, then we have

$$0 \;=\; |\emptyset| \;<\; |X_1| \;=\; n_1 \;\leq\; |X_2| \;=\; n_2 \;<\; |C| \;=\; \infty. \tag{10.1}$$

At times, when we compare sets, we want to know more about them than just a single number (if we think of infinity as a number). Note that we have a corresponding chain of injections

$$\emptyset \;\rightarrowtail\; X_1 \;\rightarrowtail\; X_2 \;\rightarrowtail\; C. \tag{10.2}$$

Would it be feasible to use injective functions as a means to compare sets as far as their size is concerned? Of course we lose information if we boil down the information about two sets to whether or not there exists an injection from one of them to the other, but for many purposes it turns out to be fruitful to know whether such is the case, and still not worry about any more detail. For example, all countably infinite and uncountable sets have the same size ∞, but it turns out that there are many degrees of uncountability if one considers a set X no biger than a set Y if there exists an injection $X \rightarrowtail Y$.

The above leads us to the definition of cardinality.

10.1 The Cardinality of a Set

Definition 10.1 (Cardinality Comparisons).

Given are two arbitrary sets X and Y. We say that

- **(a)** X, Y **have same cardinality**, and we write $\mathbf{card}(X) = \mathbf{card}(Y)$, if either both $X, Y \neq \emptyset$ and there is a bijection $f : X \xrightarrow{\sim} Y$, or if both X and Y are empty. Otherwise we write $\mathbf{card}(X) \neq \mathbf{card}(Y)$
- **(b)** the **cardinality of** X **is less than or equal to the cardinality of** Y, and we write
  $\mathbf{card}(X) \leq \mathbf{card}(Y)$, if there is an injective mapping $f : X \to Y$ or if X is empty.
- **(c)** the **cardinality of** X **is less than the cardinality of** Y, and we write
  $\mathbf{card}(X) < \mathbf{card}(Y)$, if both $\text{card}(X) \leq \text{card}(Y)$ and $\text{card}(Y) \neq \text{card}(X)$, i.e., if either $X = \emptyset$ and $Y \neq \emptyset$, or there is an injective mapping but not a bijection $f : X \to Y$.
- **(d)** the **cardinality of** X **is greater than or equal to the cardinality of** Y, and we write
  $\mathbf{card}(X) \geq \mathbf{card}(Y)$, if $\text{card}(Y) \leq \text{card}(X)$.
- **(e)** the **cardinality of** X **is greater than the cardinality of** Y, and we write
  $\mathbf{card}(X) > \mathbf{card}(Y)$, if $\text{card}(Y) < \text{card}(X)$. $\square$

Note the following concerning the above definition.

(a) It does not specify how $\mathbf{card}(X)$ itself is defined. This will be done in Definition 10.2 on p.455.

(b) It covers the cases $X = \emptyset$ and/or $Y = \emptyset$. We have $\mathbf{card}(\emptyset) \leq \mathbf{card}(Y)$ for any set Y, $\mathbf{card}(\emptyset) \neq \mathbf{card}(Y)$ if Y is not empty, and $\mathbf{card}(X) = \mathbf{card}(Y)$ if $X = Y = \emptyset$. □

Example 10.1. Let A, B be two sets such that $A \subseteq B$. Then $\text{card}(A) \leq \text{card}(B)$.

PROOF:

Case 1: $A = \emptyset$. It then is true by definition that $\text{card}(\emptyset) \leq \text{card}(B)$ for any set B.

Case 2: $A \neq \emptyset$. It follows from $B \supseteq A$ that $B \neq \emptyset$. Further the mapping $x \mapsto x$ is injective. This proves $\text{card}(A) \leq \text{card}(B)$. ■

Theorem 10.1 (B/G thm.13.31)**.**

Let X be a set. Then $\mathbf{card}(X) < \mathbf{card}(2^X)$.
In other words, X can be injected into 2^X, but it is not possible to find bijective $f : X \xrightarrow{\sim} 2^X$.

Proof: The function $x \mapsto \{x\}$ is an injection from X into 2^X, hence $\mathbf{card}(X) \leq \mathbf{card}(2^X)$.

It remains to show that there is no bijective function $f : X \xrightarrow{\sim} 2^X$. We will show that it is not even possible to find surjective f with domain X and codomain 2^X.

We assume to the contrary that such f exists. Let

$$\Gamma := \{x \in X : x \notin f(x)\}.$$

Obviously $\Gamma \subseteq X$, i.e., $\Gamma \in 2^X$. f is surjective, hence there exists $x_0 \in X$ such that $f(x_0) = \Gamma$.

Case 1: Assume $x_0 \in \Gamma$. Then $x_0 \notin f(x_0)$, i.e., $x_0 \notin \Gamma$. We have a contradiction.

Case 2: Assume $x_0 \notin \Gamma$. Then $x_0 \in f(x_0)$, i.e., $x_0 \in \Gamma$. Again, we have a contradiction.

We conclude that there is no surjective $f : X \to 2^X$.

Proposition 10.1.

Let X, Y be two sets such that $\text{card}(X) = \text{card}(Y)$. Then $\text{card}(2^X) = \text{card}(2^Y)$.

PROOF: The proof is left as exercise 10.2. ■

10.2 Cardinality as a Partial Ordering

We assume in this subchapter that all sets are subsets of a universal set Ω. Having such a universal set allows us to declare on its power set 2^Ω equivalence relations. If we work with specific sets, e.g. the set $\mathbb{R}$ of all real numbers, we assume implicitly that those sets are contained in Ω.

We defined in Definition 10.1 on p.453 the meaning of $\mathbf{card}(X) = \mathbf{card}(Y)$ and $\mathbf{card}(X) \leq \mathbf{card}(Y)$ for two sets X and Y but we never defined the expression $\mathbf{card}(X)$ per se. This will be done now.

Definition 10.2 (Cardinality as an Equivalence Class)**.**

$\star$ Let $X, Y \subseteq \Omega$.

We call X and Y equivalent and we write $X \sim Y$, if and only if $\mathbf{card}(X) = \mathbf{card}(Y)$, i.e., either both X and Y are empty, or both are not empty and there is a bijection $f : X \xrightarrow{\sim} Y$.

The proposition following this definition shows that "$\sim$" is indeed an equivalence relation on 2^Ω. This justifies to define for a set $X \subseteq \Omega$ its **cardinality** as follows:

$$\mathbf{card}(X) := [X] \quad \text{(the equivalence class of } X \text{ w.r.t "}\sim\text{")} . \tag{10.3}$$

In other words,

$$\mathbf{card}(\emptyset) := \{\emptyset\}, \tag{10.4}$$

$$\mathbf{card}(X) := \{Y \subseteq \Omega : \exists \text{ bijection } X \to Y\} \ \text{ if } X \neq \emptyset. \ \square \tag{10.5}$$

Proposition 10.2.

$X \sim Y$ as defined above is an equivalence relation on 2^Ω.

PROOF:

Proof strategy: How about this? The equals relation is reflexive symmetric and transitive. Since $X \sim Y$, if and only if $\mathbf{card}(X) = \mathbf{card}(Y)$, $X \sim Y$ inherits those properties from the equals relation $\mathbf{card}(X) = \mathbf{card}(Y)$.

Here is the problem. When we defined $\mathbf{card}(X) = \mathbf{card}(Y)$ in Definition 10.1 on p.453 we did so without giving any meaning to the expressions card(X) and card(Y). Rather, we defined this expression, and hence $X \sim Y$, to mean the following:

$$X \sim Y \quad \Leftrightarrow \quad \text{either} \quad X = Y = \emptyset \quad \text{or} \quad \left[\, X, Y \neq \emptyset \text{ and there exists a bijection } X \xrightarrow{\sim} Y \,\right] \tag{10.6}$$

For this reason a correct proof of this proposition must refer to (10.6).

Let $X, Y, Z \subseteq \Omega$.

Case 1. $X = \emptyset$.

Reflexivity: Clearly, $X = X = \emptyset$, hence $X \sim X$.

Symmetry: If $X \sim Y$ then it follows from (10.6). that $Y = \emptyset$. Thus $X = Y = \emptyset$, thus $Y \sim X$.

Transitivity: Assume that $X \sim Y$ and $Y \sim Z$. Since $X = \emptyset$ and $X \sim Y$ it follows from (10.6) that $Y = \emptyset$. Since $Y = \emptyset$ and $Y \sim Z$ it follows from (10.6) that $Z = \emptyset$. Thus $X = Z = \emptyset$, thus $X \sim Z$.

Case 2. $X \neq \emptyset$.

Reflexivity:	$id_X : x \mapsto x$ is a bijection $X \to X$, hence $X \sim X$.
Symmetry:	If $X \sim Y$ then it follows from (10.6) and $X \neq \emptyset$ that $Y \neq \emptyset$ and that there exists a bijection $f : X \xrightarrow{\sim} Y$. The inverse $f^{-1} : Y \to X$ then is a bijection $Y \xrightarrow{\sim} X$. It follows that $Y \sim X$
Transitivity:	Assume that $X \sim Y$ and $Y \sim Z$. Since $X \neq \emptyset$ and $X \sim Y$ it follows from (10.6) that $Y \neq \emptyset$ and that there exists a bijection $f : X \xrightarrow{\sim} Y$. Since $Y \neq \emptyset$ and $Y \sim Z$ it follows from (10.6) that $Z \neq \emptyset$ and that there exists a bijection $g : Y \xrightarrow{\sim} Z$. It follows from prop.**??(c)** that the composition $g \circ f$ of the two bijective functions f and g is a bijection $X \xrightarrow{\sim} Z$. ■

Next, we collect some material to prove the Cantor–Schröder–Bernstein Theorem. That theorem allows us to prove antisymmetry of the relation

$$\mathbf{card}(X) \leq \mathbf{card}(Y), \quad \text{defined on the set} \quad \mathscr{A} := \{\mathbf{card}(X) : X \subseteq \Omega\}.$$

Proposition 10.3.

Let X', X'', Y', Y'' be nonempty sets such that $X' \cap X'' = \emptyset$ and $Y' \cap Y'' = \emptyset$. Let $f' : X' \to Y'$ and $f'' : X'' \to Y''$. Then the function

$$f : X' \biguplus X'' \longrightarrow Y' \biguplus Y''; \qquad x \ \mapsto \begin{cases} f'(x) & if x \in X', \\ f''(x) & if x \in X'', \end{cases}$$

satisfies the following:
- ***(a)*** *If f' and f'' are injective then f is injective.*
- ***(b)*** *If f' and f'' are surjective then f is surjective.*
- ***(c)*** *If f' and f'' are bijective then f is bijective.*

The proof is left as exercise 10.3 (see p.468). ■

The following theorem from Tarski and its use in a stunningly simple proof of the subsequent Cantor–Schröder–Bernstein Theorem have been found in the online article

https://chiasme.wordpress.com/2013/11/20/a-short-proof-of-cantor-bernstein-schroeder-theorem/

(A short proof of Cantor–Bernstein–Schröder Theorem). The suggestion to prove Cantor–Schröder–Bernstein with help of Tarski's Theorem was given to the author by David Biddle.

Theorem 10.2 (Tarski's Fixed Point Theorem)**.**

Let Ω be a set and let $\varphi : 2^\Omega \longrightarrow 2^\Omega$ be nondecreasing with respect to "$\subseteq$", i.e.,

$$A, B \subseteq \Omega \text{ and } A \subseteq B \quad \Rightarrow \quad \varphi(A) \subseteq \varphi(B).$$

*Then φ has a **fixed point**, i.e., there exists an argument $A_0 \in 2^\Omega$ such that $\varphi(A_0) = A_0$.*

PROOF: Let
$$\mathfrak{F} \ := \ \{A \in 2^{\Omega} : A \subseteq \varphi(A)\}\,, \qquad A_0 := \bigcup [A : A \in \mathfrak{F}]\,.$$
We will show that A_0 is a fixed point for φ. First we prove

$$(\mathbf{A}) \qquad A_0 \ \subseteq \ \varphi(A_0)\,.$$

To see this we observe that

$$A \in \mathfrak{F} \ \Rightarrow \ A \subseteq \varphi(A)\,. \quad \text{Since } \varphi \text{ is nondecreasing and } A \subseteq A_0, \quad \varphi(A) \subseteq \varphi(A_0)\,.$$
$$\text{Thus } A \subseteq \varphi(A_0) \quad \text{for all } A \in \mathfrak{F}\,, \quad \text{Thus } A_0 \ = \ \bigcup [A : A \in \mathfrak{F}] \ \subseteq \ \varphi(A_0)\,.$$

We have shown **(A)**. It remains to prove

$$(\mathbf{B}) \qquad \varphi(A_0) \ \subseteq \ A_0\,.$$

We just proved that $A_0 \subseteq \varphi(A_0)$. Since φ is nondecreasing, $\varphi(A_0) \subseteq \varphi\big(\varphi(A_0)\big)$.

$$\text{Thus } \varphi(A_0) \in \mathfrak{F}, \ \ \text{thus } \varphi(A_0) \subseteq \bigcup [A : A \in \mathfrak{F}]\,, \ \ \text{i.e., } \varphi(A_0) \subseteq A_0\,.$$

We have shown that A_0 is a fixed point for φ. ■

Example 10.2. Can one replace in Tarski's fixed point theorem the assumption φ nondecreasing with φ nonincreasing? In other words, if $\varphi : 2^{\Omega} \longrightarrow 2^{\Omega}$ satisfies

$$A, B \subseteq \Omega \text{ and } A \subseteq B \quad \Rightarrow \quad \varphi(A) \supseteq \varphi(B)\,,$$

will there be $A_0 \subseteq \Omega$ such that $\varphi(A_0) = A_0$?
Here is a counterexample. Assume that Ω contains at least two arguments. Let $a_* \in \Omega$ and define

$$\varphi(A) \ := \ \begin{cases} \emptyset & \text{if } a_* \in A\,, \\ \Omega & \text{if } a_* \notin A\,, \end{cases} \qquad (A \subseteq \Omega).$$

The table below shows both that φ is nonincreasing and that there is no fixed point.
Let $B \subseteq A \subseteq \Omega$. We look at all possible cases, depending on whether or not $a_* \in B$, $a_* \in A$. Note that $a_* \in B$ and $a_* \notin A$ cannot happen, since $B \subseteq A$.

$a_* \in B$	$a_* \in A$	$\varphi(B) = \emptyset$	$\varphi(A) = \emptyset$
$a_* \notin B$	$a_* \notin A$	$\varphi(B) = \Omega$	$\varphi(A) = \Omega$
$a_* \notin B$	$a_* \in A$	$\varphi(B) = \Omega$	$\varphi(A) = \emptyset$

□

The next theorem is named after the German mathematicians Friedrich Wilhelm Karl Ernst Schröder (1841 – 1902), Georg Ferdinand Ludwig Philipp Cantor (1845 – 1918), Felix Bernstein (1878 – 1956)

Theorem 10.3 (Cantor–Schröder–Bernstein's Theorem)**.**

Let X and Y be nonempty sets. Let there be injective functions

$$f : X \rightarrowtail Y \qquad \textit{and} \qquad g : Y \rightarrowtail X\,.$$

Then there exists a bijection $X \xrightarrow{\sim} Y$.

PROOF: The proof given here is based on the Tarski Fixed Point Theorem. Let

$$\varphi : 2^X \longrightarrow 2^X ; \qquad A \mapsto g\big(Y \setminus f(X \setminus A)\big) .$$

Since $A \mapsto X \setminus A$ is nonincreasing and the direct image function $U \mapsto f(U)$ is nondecreasing, $A \mapsto f(X \setminus A)$ is nonincreasing, thus $A \mapsto Y \setminus f(X \setminus A)$ is nondecreasing.

Since the direct image function $V \mapsto g(V)$ is nondecreasing, $\varphi : A \mapsto g\big(Y \setminus f(X \setminus A)\big)$ is nondecreasing. It follows from Theorem 10.2 (Tarski's Fixed Point Theorem) on p.456 that there exists $A_0 \in X$ such that $\varphi(A_0) = A_0$.

Let $X_* \subseteq X$ and $Y_* \subseteq Y$. Since f and g are injective, it follows from Proposition **??** on p.**??** that the restrictions

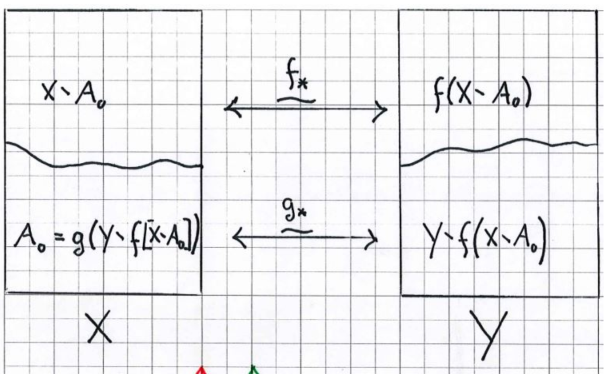
$$f_* : X_* \longrightarrow f(X_*) ; \quad x \mapsto f(x) \qquad \text{and} \qquad g_* : Y_* \longrightarrow g(Y_*) ; \quad y \mapsto g(y)$$

are bijections. We apply this to the sets $X_* := X \setminus A_0$ and $Y_* := Y \setminus f(X \setminus A_0)$:

$$g_* : Y \setminus f(X \setminus A_0) \xrightarrow{\sim} g\big(Y \setminus f(X \setminus A_0)\big) \qquad \text{and} \qquad f_* : X \setminus A_0 \xrightarrow{\sim} f(X \setminus A_0)$$

thus are bijections. Since $A_0 = \varphi(A_0) = g\big(Y \setminus f(X \setminus A_0)\big)$, we have bijections

(**A**) $$(g_*)^{-1} : A_0 \xrightarrow{\sim} Y \setminus f(X \setminus A_0) \qquad \text{and} \qquad f_* : X \setminus A_0 \xrightarrow{\sim} f(X \setminus A_0).$$



The picture to the right illustrates those bijections.

Next, we observe that any two sets U and V satisfy $U \cup V = (U \setminus V) \uplus U$. Hence,

- if $V \subseteq U$ then $U = U \cup V = (U \setminus V) \uplus U$.

It follows that we have unions of disjoint sets

- $X = (X \setminus A_0) \uplus A_0$
- $Y = \big(Y \setminus f(X \setminus A_0)\big) \uplus f(X \setminus A_0)$.

By Proposition 10.3 on p.456, the bijections $(g_*)^{-1}$ and f_* in **(A)** can be combined into a bijection

$$h : A_0 \biguplus X(\setminus A_0) \xrightarrow{\sim} (Y \setminus f(X \setminus A_0)) \biguplus f(X \setminus A_0) \qquad \text{i.e.,} \qquad X \xrightarrow{\sim} Y . \ \blacksquare$$

Corollary 10.1.

> *The relation* $\boldsymbol{card}(X) \leq \ \boldsymbol{card}(Y)$ *partially orders the set* $\mathscr{A} := \{\boldsymbol{card}(X) : X \subseteq \Omega\}$.

PROOF: We must show reflexivity, antisymmetry, and transitivity.

Case 1: None of the sets involved is empty.

Reflexivity is obvious, antisymmetry follows from Cantor-Schröder-Bernstein and transitivity follows from prop.**??(a)**: The composition of two injective functions is injective.

Case 2: At least one of the sets involved is empty:

We use the fact that if $\mathbf{card}(A) \leq \mathbf{card}(B)$ and $B = \emptyset$ then $A = \emptyset$.

As an example we prove antisymmetry: If $X = \emptyset$ and $\mathbf{card}(X) \leq \mathbf{card}(Y)$ and $\mathbf{card}(Y) \leq \mathbf{card}(X)$ then the second "$\leq$" implies $Y = \emptyset$, i.e., $X = Y$. On the other hand, If $Y = \emptyset$ and $\mathbf{card}(X) \leq \mathbf{card}(Y)$ and $\mathbf{card}(Y) \leq \mathbf{card}(X)$ then the first "$\leq$" implies $X = \emptyset$, i.e., $X = Y$. ■

Theorem 10.4.

Let $X, Y \subseteq \Omega$. Then

$$\mathbf{card}(X) \leq \mathbf{card}(Y) \qquad \text{or} \qquad \mathbf{card}(Y) \leq \mathbf{card}(X)$$

In other words, "$\leq$" is a total ordering [1] on the set of all cardinalities for subsets of Ω.

PROOF: The proof will be given in thm.**??**, p.**??**, of ch.**??** (Applications of Zorn's Lemma). ■

As an application of the Cantor–Schröder–Bernstein theorem we will prove that one can biject any two intervals of real numbers, no matter whether one or both of them are open, closed, or half–open.

Theorem 10.5.

Let $a, b \in \mathbb{R}$ such that $a < b$. Let A be one of $]a, b[,\ ]a, b\,],\ [\,a, b[,\ [\,a, b\,]$.

$$\text{Then} \quad card(A) = card(\mathbb{R})\,.$$

PROOF:

(a) $F : \mathbb{R} \to\,]-1, 1[\,;\ \ x \mapsto \frac{x}{|x|+1}$ has the function $G(y) := \begin{cases} \frac{y}{1-y} & \text{if}\, x \geq 0, \\ \frac{y}{1+y} & \text{if}\, x < 0. \end{cases}$ as an inverse. The proof of this is tedious but elementary if one observes that $y \geq 0$ if and only if $x \geq 0$. This makes it easy to solve $y = \frac{x}{|x|+1}$ for x. The details are left to the reader. It follows that $\text{card}(]-1, 1[) = \text{card}(\mathbb{R})$.

(b) Let $c > 0$. The function $x \mapsto cx$ is a bijection from $]-1, 1[$ to $]-c, c[$ because it has the function $y \mapsto \frac{y}{c}$ as an inverse. It follows from part **(a)** that $\text{card}(]-c, c[) = \text{card}(\mathbb{R})$.

(c) If $\lambda \in \mathbb{R}$ then $x \mapsto x + \lambda$ bijects A to $A + \lambda$ (Inverse: $y \mapsto y - \lambda$). Let $c := \frac{b-a}{2}$ and $B := A - \frac{a+b}{2}$. Then B is the interval with endpoints $-c$ and c where either endpoint of B is included/excluded if and only if such is the case for the corresponding endpoint of A. Note that $\text{card}(B) = \text{card}(A)$ because B is the image of A under the bijection $x \mapsto x - \frac{b-a}{2}$.

(d) It follows from $]-c, c[\, \subseteq B$ and part **(b)** that $\text{card}(\mathbb{R}) = \text{card}(]-c, c[) \leq \text{card}(B)$, and it then follows from $B \subseteq \mathbb{R}$ that $\text{card}(\mathbb{R}) \leq \text{card}(B) \leq \text{card}(\mathbb{R})$.

It is a consequence of the Cantor–Schröder–Bernstein Theorem which is formulated and proved later on (ch.10.2 [2] , thm.10.3 on p.457) that there exists a bijection between B and $\mathbb{R}$. We saw in part **(c)** that $\text{card}(B) = \text{card}(A)$. This proves $\text{card}(A) = \text{card}(\mathbb{R})$. ■

[1] See Definition **??** (Linear orderings) on p.**??**.

[2] "Cardinality as a Partial Ordering". The proof of the Cantor–Schröder–Bernstein Theorem is by no means trivial.

We have previously seen that $\mathbb{R}$ is uncountable by proving that the subset of all real numbers $x = \sum_{j=1}^{\infty} d_j 10^{-j}$ such that $d_j = 3$ or $d_j = 4$ can be bijected to the uncountable set $\{3,4\}^{\mathbb{N}}$. See Theorem **??** on p.**??**. The next theorem, which is another application of the Cantor–Schröder–Bernstein Theorem, uses this fact to prove that the set of real numbers and the power set of $\mathbb{N}$ can be bijected.

Theorem 10.6.

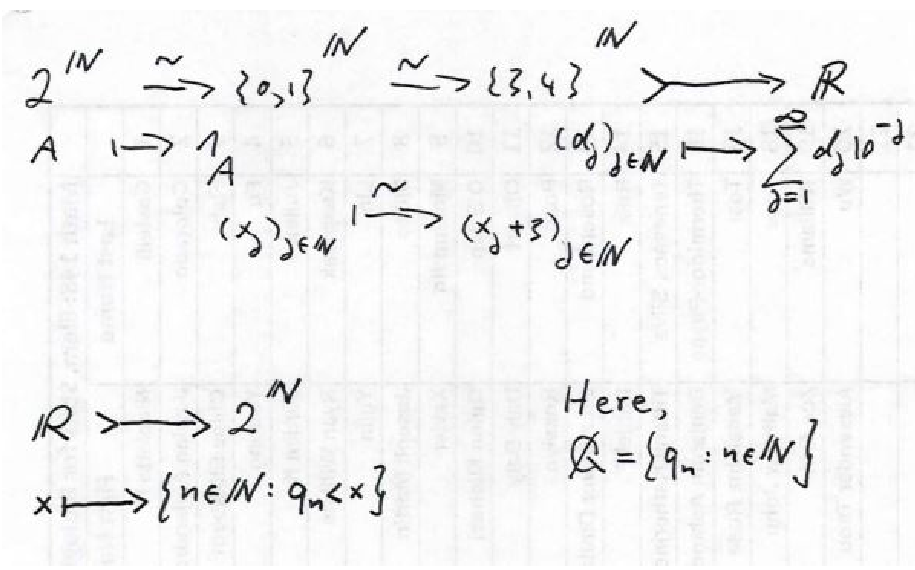
$$card(\mathbb{R}) \;=\; card(2^{\mathbb{N}})\,. \tag{10.7}$$

PROOF: First, we show that there exists an injection $f : 2^{\mathbb{N}} \to \mathbb{R}$.

- **(a)** Proposition **??** on p.**??** shows that there is a bijection $f_1 : 2^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$.
- **(b)** Clearly, the function defined by $0 \mapsto 3$ and $1 \mapsto 4$ is a bijection $f_2 : \{0,1\}^{\mathbb{N}} \to \{3,4\}^{\mathbb{N}}$.
- **(c)** Moreover, the proof of Theorem **??** on p.**??** shows that $\{3,4\}^{\mathbb{N}}$ can be bijected to the subset $\{x = 0.d_1d_2d_3\cdots : d_j = 3 \text{ or } d_j = 4\}$ of $\mathbb{R}$. Thus, we have an injection $f_3 : \{3,4\}^{\mathbb{N}} \to \mathbb{R}$.
- **(d)** It follows that $f : f_3 \circ f_2 \circ f_1$ is an injection $2^{\mathbb{N}} \to \mathbb{R}$.

Next, we show that there exists an injection $g : \mathbb{R} \to 2^{\mathbb{N}}$.

- **(e)** Since $\mathbb{Q}$ is countable, there is a sequence $q_n \in \mathbb{Q}$ such that $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$.
- **(f)** $x \mapsto \{n \in \mathbb{N} : q_n < x\}$ defines a function $g : \mathbb{R} \to 2^{\mathbb{N}}$. We claim that g is injective:
  Let $x, x' \in \mathbb{R}$ such that $x \neq x'$. We may assume that $x < x'$. According to Proposition **??** on p.**??**, there exists $n_* \in \mathbb{N}$ such that $x < q_{n_*} < x'$. Since $n_* \notin g(x_1)$ and $n_* \in g(x_2)$, it follows that $g(x_1) \neq g(x_2)$. This proves injectivity of g



It follows from **(d)** and **(g)** that there are injections $f : 2^{\mathbb{N}} \to \mathbb{R}$ and $g : \mathbb{R} \to 2^{\mathbb{N}}$. We apply Cantor–Schröder–Bernstein and conclude that $\mathbb{R}$ and $2^{\mathbb{N}}$ can be bijected. Thus, $\text{card}(\mathbb{R}) = \text{card}(2^{\mathbb{N}})$. ■

10.3 Alternate Proofs of the Cantor–Schröder–Bernstein Theorem ⋆

The author has seen different proofs of the Cantor–Schröder–Bernstein Theorem. They all are a lot more complicated than the one given in Chapter , They are given here since they are interesting exercises with respect to working with sets.

I - First alternate proof:

The following proof of the Cantor–Schröder–Bernstein Theorem and the material that precedes it closely follow Chapter 11 of [2] Chartrand, G., Polimeni, A. and Zhang, Ping: Mathematical Proofs: A Transition to Advanced Mathematics.

Definition 10.3. Let $\emptyset \neq Y \subseteq X$ and $f : X \to Y$. Then we define for each $n \in [0, \infty[_{\mathbb{Z}}$,

$$f^n : X \longrightarrow Y\,; \qquad f^0(x) := x\,, \qquad f^n(x) := f\big(f^{n-1}(x)\big) \text{ if } n \in \mathbb{N}\,.$$

We call f^n the n**–th iterate of** f. ☐

Remark 10.1. It follows from $Y \subseteq X$ and the monotonicity of the direct image function that

$$X \supseteq Y \supseteq f(X)\,, \text{ thus } f(X) \supseteq f(f(X)) = f^2(X)\,, \text{ thus } f^2(X) \supseteq f(f^2(X)) = f^3(X)\,, \dots$$

It follows that Y is big enough to function as codomain for each iterate $f^1, f^2, \dots$ since

$$Y \supseteq f(X) \supseteq f^m(X) \supseteq f^n(X)\,, \qquad \text{for all integers } 1 \leq m < n\,. \tag{10.8}$$ ☐

Example 10.3.
Let $f : [0, 1/2] \to [0, 1/3]; x \mapsto x^2$.
Note that $[0, 1/3] \subseteq [0, 1/2]$ Then $f^0(x) = f(x) = x^2$ and $f^n(x) = x^{2n}$.
Also note that $0 \leq x \leq 1/2$ implies $0 \leq x^{2n} \leq 1/(2^{2n}) < 1/3$ and thus $[0, 1/3]$ is sufficiently large as codomain for each f^n (except $n = 0$). ☐

Lemma 10.1. *Let $\emptyset \neq B \subseteq A$ such that there exists an injection $f : A \rightarrowtail B$. Then there exists a bijection between A and B.*

PROOF: If $B = A$ then the identity id_A bijects A to B and the proof is finished. We thus assume $B \subsetneq A$, hence $A \setminus B \neq \emptyset$. Since nothing needs to be shown if f is surjective we also assume that

$$f(A) \neq B\,, \qquad \text{i.e.,} \quad B \setminus f(A) \neq \emptyset\,. \tag{A}$$

$$\text{Let} \quad B' := \{f^n(x) : x \in A \setminus B,\, n \in \mathbb{N}\} \tag{B}$$

We obtain from **(A)** that $B \setminus f(A) \neq \emptyset$, hence $B' \neq \emptyset$.
It follows from (10.8) on p.461 that $f^n(x) \in f(A)$ for all $x \in A$ and $n \in \mathbb{N}$, hence

$$B' \subseteq f(A)\,, \text{ for all } n \in \mathbb{N}\,. \tag{C}$$

$$\textbf{(D)} \qquad \text{Let} \quad C := B' \cup (A \setminus B) \qquad \text{and} \qquad D := B \setminus B'.$$

It follows from $f^0(x) = x$, hence $f^0(A \setminus B) = A \setminus B$, that

$$\textbf{(E)} \qquad C = \{ f^n(x) : x \in A \setminus B,\, n \in [0, \infty[_{\mathbb{Z}} \}.$$

We want to define functions which have C and D as their domains and/or codomains, and this requires $C \neq \emptyset$ and $D \neq \emptyset$. Clearly, $C \neq \emptyset$ because $\emptyset \neq A \setminus B \subseteq C$.

To see that $D \neq \emptyset$, note that **(A)** and **(C)** yield $B \setminus f(A) \neq \emptyset$ and $B' \subseteq f(A)$. Thus

$$\emptyset \neq B \setminus f(A) \subseteq B \setminus B' = D.$$

Next we show that $f(C) \subseteq B'$, i.e., if $x \in C$ then $f(x) \in B'$. We separately consider $x \in A \setminus B$ and $x \in B'$.

(i) $x \in A \setminus B$: Since $f(x) = f^1(x)$, we obtain from **(B)** that $f(x) \in B'$.

(ii) $x \in B'$: It follows from **(B)** that there exists $n \in \mathbb{N}$ and $\tilde{x} \in A \setminus B$ such that $x = f^n(\tilde{x})$, thus, again by **(B)**, $f(x) = f^{n+1}(\tilde{x}) \in B'$.

Since $f(C) \subseteq B'$ we can downsize the codomain of f to B' if we restrict f to C. In other words, the following is a valid definition of a function.

$$\textbf{(F)} \qquad f_C : C \longrightarrow B' \qquad f_C(x) := f(x).$$

Next we prove that f_C is bijective, i.e., f_C is surjective and injective.

Let $y \in B'$. It follows from **(B)** that there exists $n \in \mathbb{N}$ and $x \in A \setminus B$ such that $y = f^n(x)$.

(i) $n = 1$: Then $y = f(x)$. Since $x \in A \setminus B$ and $A \setminus B \subseteq C$ and thus $f_C(x) = f(x)$, we found $x \in C$ such that $f_C(x) = y$.

(ii) $n > 1$, i.e., $n = k + 1$ for some $k \in \mathbb{N}$: Let $\tilde{x} = f^k(x)$. Then $\tilde{x} \in B'$ by **(B)**. Since $B' \subseteq C$ and $f_C(c) = f(c)$ for all $c \in C$, we found $\tilde{x} \in C$ such that

$$f_C(\tilde{x}) = f(\tilde{x}) = f\big(f^{k+1}(x)\big) = f\big(f^n(x)\big) = y.$$

(i) and **(ii)** imply that f_C is surjective. Moreover, f_C is injective as a restriction of the injection f to a smaller domain. Thus f_C is bijective. Next we show

$$\textbf{(G)} \qquad B' \cap D = \emptyset \qquad \text{and} \qquad C \cap D = \emptyset.$$

It is trivial that $B' \cap D = \emptyset$ since $D = B \setminus B'$ by **(D)** shows that D and B' have no elements in common. This also helps to see that $C \cap D = \emptyset$:

$$D \cap C = (B \setminus B') \cap \big((A \setminus B) \cup B'\big) = B \cap (A \setminus B') \cap \big((A \setminus B) \cup B'\big) \subseteq B \cap (A \setminus B').$$

Assume $x \in D$. In other words, $x \in B \setminus B'$. Then **(i)**: $x \in B$, thus $x \notin A \setminus B$. Also, **(ii)**: $x \notin B'$. Thus **(i)** and **(ii)** together imply that $x \notin (A \setminus B) \cup B'$, i.e., $x \notin C$

We have shown that $x \in D \Rightarrow x \notin C$ and thus $C \cap D = \emptyset$. We have proved **(G)**. We recall that f_C bijects C to B'. Obviously the identity function $id_D : x \mapsto x$ bijects D to itself. By Proposition 10.3 on p.456, the function

$$\textbf{(H)} \qquad h : C \biguplus D \longrightarrow B' \uplus D; \qquad x \mapsto \begin{cases} f_C(x) & \text{if } x \in C, \\ x & \text{if } x \in D, \end{cases}$$

bijects $C \uplus D$ to $B' \uplus D$

Next we show that $C \uplus D = A$. It follows from **(E)** that

$$\textbf{(I)} \qquad C \supseteq \{f^0(x) : x \in A \setminus B\}.C = \{x : x \in A \setminus B\} = A \setminus B\,.$$

Moreover, since $C \supseteq B'$,

$$\textbf{(J)} \qquad C \cup D = C \cup (B \setminus B') \supseteq C \cup (B \setminus C) \supseteq C \cup B\,.$$

Since $C \supseteq A \setminus B$ by **(I)** and $C \cup D \supseteq C \cup B$ by **(J)**,

$$C \cup D = C \cup B \supseteq (A \setminus B) \cup B \supseteq A\,.$$

But all sets occurring above are subsets of A. It follows that $C \uplus D = A$.

Next we show that $B' \uplus D = B$. It follows from **(C)** that $B' \subseteq f(A)$. Since B is the codomain of f we further have $f(A) \subseteq B$. Thus $B' \subseteq B$, thus $B' \cup (B \setminus B') = B$, thus

$$B' \cup D = B' \cup (B \setminus B') = B\,.$$

It follows that the function h defined in **(H)** is a bijection

$$h : A \xrightarrow{\sim} B\,.$$

We have proved the lemma. ■

With the help of Lemma 10.1 the proof of the Cantor–Schröder–Bernstein Theorem is a simple affair.

ALTERNATE PROOF I of Theorem 10.3:

Since g is injective, the function

$$g_* : Y \longrightarrow g(Y)\,; \qquad y \mapsto g(y)$$

is bijective.

Since the function $\varphi := g_* \circ f : X \longrightarrow g(Y)$ is injective as the composition of two injective functions

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g_* \circ f"'] & Y \arrow[d, "g_*"] \\ & g(Y) \end{tikzcd}$$

and since $g(Y) \subseteq X$ we can apply Lemma 10.1 with X instead of A and $g(Y)$ instead of B. It follows that there exists a bijection

$$h : X \xrightarrow{\sim} g(Y)\,.$$

The function g_* is bijective, hence it has an inverse $g_*^{-1} : g(Y) \xrightarrow{\sim} Y$. Thus $g_*^{-1} \circ h : X \xrightarrow{\sim} Y$ is bijective as the composition of two bijections. ■

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "g_*^{-1} \circ h"'] & g(Y) \arrow[d, "g_*^{-1}"] \\ & Y \end{tikzcd}$$

II - Second alternate proof:

The next proof of the Cantor–Schröder–Bernstein Theorem and the material that precedes it closely follow the presentation in [3] Haaser/Sullivan: Real Analysis.

Lemma 10.2. *Let A_1, A_2, A_3 be nonempty sets such that $A_1 \supseteq A_2 \supseteq A_3$. and such that there exists a bijection $f : A_1 \xrightarrow{\sim} A_3$. Then there exists a bijection $g : A_1 \xrightarrow{\sim} A_2$.*

PROOF: We define a sequence of sets A_n for $n \geq 4$ as $A_n := f(A_{n-2})$. Note that if follows from the surjectivity of f that $f(A_1) = A_3$ and thus we obtain

$$A_n \ = \ f(A_{n-2}) \quad \text{for} \quad n \geq 3.$$

Next we define sets B_k as follows:

$$B_0 \ := \ \bigcap_{n=1}^{\infty} A_n; \qquad B_k := A_k \setminus A_{k+1} \quad \text{for} \quad n \in \mathbb{N}.$$

We will prove the following:

(a) The sequence $(A_n)_{n=1}^{\infty}$ is nonincreasing, i.e., $k < n \ \Rightarrow \ A_k \supseteq A_n$.

(b) The sets B_k are mutually disjoint: $i, j \in [0, \infty[\ \Rightarrow \ A_i \cap A_j = \emptyset$.

(c) **c1.** $A_1 \ = \ \bigcup\limits_{k=0}^{\infty} B_k$; **c2.** $A_2 \ = \ \bigcup \big[B_k : k \geq 0, k \neq 1\big]$.

(d) $k \in \mathbb{N} \ \Rightarrow \ f(B_k) = B_{k+2}$.

(e) The function g defined as $g(x) \ := \ \begin{cases} f(x) & \text{if } x \in \biguplus\big[B_{2k-1} : k \geq 1\big], \\ x & \text{if } x \in \biguplus\big[B_{2k} : k \geq 0\big] \end{cases}$ is a bijection $A_1 \xrightarrow{\sim} A_2$.

The function g defined in **(e)** above is the bijection which this lemma claims to exists.

PROOF of **(a)**:

We reformulate **(a)** as follows: Let $n \in \mathbb{N}$. For all $i < j \leq n$ it is true that $A_i \supseteq A_j$. The proof is done by strong induction on n.

Base cases: $n \leq 3$ The above is true for $n = 1, 2, 3$ since we assumed $A_1 \supseteq A_2 \supseteq A_3$.

Induction assumption: Since the base cases cover $n + 1 \leq 3$ we may assume that $n + 1 \geq 4$, i.e., $n \geq 3$:

We have some $n \geq 3$ such that if $1 \leq i < j \leq n$ then $A_i \supseteq A_j$. (IA)

We must prove that if $1 \leq i, j \leq n + 1$ then $A_i \supseteq A_j$. $(\star)$.

There is nothing to prove if $i = j$. We may also assume that $j = n + 1$ since otherwise $1 \leq i < j \leq n$ and the induction assumption allows us to conclude that $A_i \supseteq A_j$. There is no need for the variable j since it has been fixed to $n + 1$.

We moreover may assume that $i \geq 3$: If we can prove $(\star)$ for $3 \leq i < n + 1$ then it is true that $A_3 \supseteq A_{n+1}$. Since $A_1 \supseteq A_2 \supseteq A_3$. $A_i \supseteq A_{n+1}$ will also be true for $i = 1$ and $i = 2$.

Thus it suffices to prove that if $3 \leq i < n + 1$ then $A_i \supseteq A_{n+1}$. $(\star)$.

That is trivial: Since $1 \leq i - 2 < n - 1$ the induction assumption yields $A_{i-2} \supseteq A_{n-1}$. Since the direct image function is monotone (see **??** on p.**??**) it follows that $f(A_{i-2}) \supseteq f(A_{n-1})$, i.e., $A_i \supseteq A_{n+1}$.

PROOF of **(b)**:

Let $1 \leq i < j$. Then

$$B_i \cap B_j \ = \ (A_i \setminus A_{i+1}) \cap (A_j \setminus A_{j+1}) \ \subseteq \ (A_i \setminus A_{i+1}) \cap A_j \ \subseteq \ (A_i \setminus A_{i+1}) \cap A_{i+1} \ = \ \emptyset.$$

Here the last "$\subseteq$" follows from the fact that A_n is nonincreasing and $j \geq i + 1$, and the last equation follows from the definition of "$\setminus$".

PROOF of **c1**: For the proof of "$\supseteq$" we note that $A_1 \supseteq A_k \supseteq A_k \setminus A_{k+1} = B_k$ for all $k \in \mathbb{N}$, and that trivially $A_1 \supseteq B_0 = \bigcap\limits_{k=1}^{\infty} A_k$.

For the reverse inclusion let $x \in A_1$ and $J := \{j \in \mathbb{N} : x \in A_j\}$. There are two cases: **Case 1**: J is unbounded. Then $J = \mathbb{N}$ since the sets A_j are nonincreasing. Thus $x \in A_j$ for all $j \geq 0$, thus $x \in \bigcap\limits_{n=1}^{\infty} A_n$, i.e., $x \in B_0$, thus $x \in \bigcup\limits_{k=0}^{\infty} B_k$. **Case 2**: J is bounded. Note that $J \neq \emptyset$ since $1 \in J$, thus $j^* := \max(J)$ exists according to the extended well-ordering principle. It follows from $x \in A_{j^*}$ and $x \notin A_{j^*+1}$ that $x \in B_{j^*} = A_{j^*} \setminus A_{j^*+1}$, thus $x \in \bigcup\limits_{k=0}^{\infty} B_k$.

PROOF of **c2**: For the proof of "$\supseteq$" we note that $A_2 \supseteq A_k \supseteq A_k \setminus A_{k+1} = B_k$ for all $k \geq 2$, and that trivially $A_2 \supseteq B_0 = \bigcap\limits_{k=1}^{\infty} A_k$.

For the reverse inclusion let $x \in A_2$ and $J := \{j \in \mathbb{Z} : j \geq 2 \text{ and } x \in A_j\}$. **Case 1**: J is unbounded. Then $J = [2, \infty[_{\mathbb{Z}}$ since the sets A_j are nonincreasing. Thus $x \in A_j$ for all $j \geq 2$, thus $x \in \bigcap\limits_{n=2}^{\infty} A_n$ which equals $\bigcap\limits_{n=1}^{\infty} A_n$ since $A_1 \supseteq \bigcap\limits_{n=2}^{\infty} A_n$, i.e., $x \in B_0$.It follows hat $x \in \bigcup \big[B_k : k \geq 0, k \neq 1\big]$. **Case 2**: J is bounded. Then $j^* := \max(J)$ exists according to the extended well-ordering principle. It follows from $x \in A_{j^*}$ and $x \notin A_{j^*+1}$, thus $x \in B_{j^*} = A_{j^*} \setminus A_{j^*+1}$, thus $x \in \bigcup \big[B_k : k \geq 0, k \neq 1\big]$.

PROOF of **(d)**: Since the bijective f is compatible with all set operation we have $f(U \setminus V) = f(U) \setminus f(V)$ for any two sets U, V (see (**??**) on p.**??**). It follows for any $k \in \mathbb{N}$ that

$$f(B_k) \;=\; f(A_k \setminus A_{k+1}) \;=\; f(A_k) \setminus f(A_{k+1}) \;=\; A_{k+2} \setminus A_{k+3} \;=\; B_{k+2}.$$

PROOF of **(e)**: Let $O := \biguplus \big[B_{2k-1} : k \geq 1\big]$ and $E := \biguplus \big[B_{2k} : k \geq 0\big]$. We note that it is appropriate to write $\uplus$ instead of $\cup$ since we proved in **(b)** that the sets B_k are mutually disjoint, and that this then implies that the assignment $x \mapsto g(x)$ is unambiguous: either $x \in O$ and $g(x) = f(x)$ or $x \in E$ and $g(x) = x$. Further we obtain from **c1** that the domain $O \uplus E = \bigcup\limits_{k=0}^{\infty} B_k$ of f equals A_1. Since f satisfies $f\Big(\bigcup\limits_i U_i\Big) = \bigcup\limits_i f(U_i)$ for any family $(U_i)_i$ such that each U_i belongs to the domain A_1 of f (see prop.**??** (Properties of the direct image) on p.**??**)) it follows that

$$g(A_1) \;=\; g(O \biguplus E) \;=\; g(O) \cup g(E) = f(O) \cup E \;=\; f\Big(\biguplus_{k\geq 1} B_{2k-1}\Big) \;\cup\; \biguplus_{k \geq 0} B_{2k} : k \geq 0.$$

We proved in **(d)** that $f(B_k) = f(B_{k+2})$ for all $k \in \mathbb{N}$. Thus

$$g(A_1) \;=\; \biguplus_{k\geq 1} f(B_{2k-1}) \;\cup\; \biguplus_{k\geq 0} B_{2k} \;=\; \biguplus_{k\geq 1} B_{2k+1} \;\uplus\; \biguplus_{k\geq 0} B_{2k} \;=\; B_0 \uplus \biguplus \big[B_j : k \geq 2\big] \;=\; A_2.$$

We have proven that the function $g : A_1 \to A_2$ is surjective. All that remains to prove the lemma, i.e., that there exists a bijection $A_1 \xrightarrow{\sim} A_2$ is to show that g is injective.

So let $x, x' \in A_1$ such that $x \neq x'$. It follows from **(b)** and **c1** that there exist unique indices $j, k \geq 0$ such that $x \in B_j$ and $x' \in B_k$. Since both mappings $x \mapsto f(x)$ and $x \mapsto x$ are injective $g(x) \neq g(x')$ in the case that $j = k$.

We may thus assume that $j \neq k$. If both j and k are odd then it follows from the injectivity of f that $g(x) = f(x) \neq f(x') = g(x')$ and we are done. If both j and k are even then $g(x) = x \neq x' = g(x')$. Again we are done.

Finally assume that j is odd and k is even. Then $j+2$ is odd and thus must be different from the even index k, thus B_{j+2} and B_k are disjoint. Since $g(x) \in g(B_j) = f(B_j) = B_{j+2}$ and $g(x') = x' \in B_k$ and those two sets have empty intersection it follows that $g(x) \neq g(x')$. We have proven injectivity and thus bijectivity of $g : A_1 \xrightarrow{\sim} A_2$. ■

ALTERNATE PROOF II of Theorem 10.3:

Let $B_1 := f'(X),\ A_2 := g'(Y),\ A_3 := g'(B_1)$.
Then $B_1 \subseteq Y$ and $A_2 \subseteq X$, thus $A_3 = g'(B_1) \subseteq g'(Y) = A_2 \subseteq X$.
Further, $g'\big(f'(X)\big) = g'(B_1) = A_3$.
Since the function $g' \circ f'$ is injective as the composition of two injective functions this proves that $g' \circ f' : X \xrightarrow{\sim} A_3$ is bijective.
It follows from lemma 10.2 above that there exists a bijection

$$f : X \xrightarrow{\sim} A_2 = g'(Y)\,.$$

Since $g' : Y \to A$ is injective we obtain from this function a bijection

$$g : Y \xrightarrow{\sim} A_2$$

by simply downsizing the codomain to $g'(Y)$ But then the function

$$g^{-1} \circ f \quad \text{is a bijection} \quad X \xrightarrow{\sim} Y$$

as the composition of two bijective functions. ■

III - Third alternate proof:

The follwing last proof of the Cantor–Schröder–Bernstein's Theorem is a more detailed version of the one found in the chapter Further Topics F: Cardinal Number and Ordinal Number of [1] B/G (Beck/Geoghegan).

ALTERNATE PROOF III of Theorem 10.3:

We have no interest in any particulars of the sets X and Y. We only are interested in establishing the existence of a bijection $h : X \to Y$. We hence may assume that X and Y are mutually disjoint, replacing X with $\{1\} \times X$ and Y with $\{2\} \times Y$ if necessary (see remark **??** on p.**??**).

Let $f : X \xrightarrow{\sim} f'(X)$ and $g : Y \xrightarrow{\sim} g'(Y)$ be the bijective functions obtained from the injections f' and g' by restricting their codomains to the images of their domains. We note that the subsets $f'(X) \subseteq Y$ and $g'(Y) \subseteq X$ also are disjoint and that $f(X) = f'(X),\ g(Y) = g'(Y)$. Let

$$\sigma : f(X) \biguplus g(Y) \ \to\ X \biguplus Y; \qquad z \ \mapsto \begin{cases} f^{-1}(z) & \text{if } z \in f(X), \\ g^{-1}(z) & \text{if } z \in g(Y), \end{cases} \tag{10.9}$$

i.e., $\sigma\big|_{f(X)} = f^{-1}$ and $\sigma\big|_{g(Y)} = g^{-1}$. Note that if $y \in f(X)$ then $\sigma(y) \in X$; if $x \in g(Y)$ then $\sigma(x) \in Y$. We can create iterates

$$\sigma^2(z) = \sigma\big(\sigma(z)\big),\ \ \sigma^3(z) = \sigma\big(\sigma^2(z)\big),\ \ \dots, \sigma^{n+1}(z) = \sigma\big(\sigma^n(z)\big),\ \ \dots,$$

just as long as $\sigma^n(z) \in f(X) \biguplus g(Y)$. We further define σ^0 for all $z \in f(X) \biguplus g(Y)$ as $\sigma^0(z) := z$. We associate with each $z \in X \biguplus Y$ a "score" $N(z) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ as follows.

(a) If $\sigma^k(z) \in f(X) \biguplus g(Y)$ for all $k \in \mathbb{N}$ then $N(z) := \infty$.

(b) If $\sigma^k(z) \notin f(X) \biguplus g(Y)$ for some $k \in \mathbb{N}$ then $N(z) := \min\{ j \geq 0 : \sigma^j(z) \notin f(X) \biguplus g(Y) \}$.

Note that **(b)** implies the following: If $z = \sigma^0(z) \notin f(X) \biguplus g(Y)$ then $N(z) = 0$.

Depending on whether we start out with $x \in g(Y)$ or $y \in f(X)$, we obtain the following finite or infinite sequences:

$$\begin{aligned} &\text{if } x \in g(Y): \quad x \xrightarrow{\sigma} f^{-1}(x) \xrightarrow{\sigma} g^{-1}(f^{-1}(x)) \xrightarrow{\sigma} f^{-1}(g^{-1}(f^{-1}(x))) \xrightarrow{\sigma} \dots, \\ &\text{if } y \in f(X): \quad y \xrightarrow{\sigma} g^{-1}(y) \xrightarrow{\sigma} f^{-1}(g^{-1}(y)) \xrightarrow{\sigma} g^{-1}(f^{-1}(g^{-1}(y))) \xrightarrow{\sigma} \dots . \end{aligned}$$

If $N(z) < \infty$ then the sequence will terminate after $N(z)$ iterations. Let

$$\begin{aligned} X_E &:= \{x \in X : N(x) \text{ is even } \}, X_O := \{x \in X : N(x) \text{ is odd } \}, X_\infty := \{x \in X : N(x) = \infty\}, \\ Y_E &:= \{y \in Y : N(y) \text{ is even } \}, Y_O := \{y \in Y : N(y) \text{ is odd } \}, Y_\infty := \{y \in Y : N(y) = \infty\}. \end{aligned}$$

The above defines partitions $X = X_E \biguplus X_O \biguplus X_\infty$ and $Y = Y_E \biguplus Y_O \biguplus Y_\infty$ of X and Y.

Each of the functions f, g, σ changes the score of its argument from odd to even and from even to odd. Hence

$$\begin{aligned} & f(X_E) \subseteq Y_O,\ f^{-1}(Y_O) \subseteq X_E, && f(X_O) \subseteq Y_E,\ f^{-1}(Y_E) \subseteq X_O, \\ & g(Y_E) \subseteq X_O,\ g^{-1}(X_O) \subseteq Y_E, && g(Y_O) \subseteq X_E,\ g^{-1}(X_E) \subseteq Y_O, \\ & f(X_\infty) \subseteq Y_\infty,\ f^{-1}(Y_\infty) \subseteq X_\infty, && g(Y_\infty) \subseteq X_\infty,\ g^{-1}(X_\infty) \subseteq Y_\infty, \end{aligned} \tag{10.10}$$

We define a bijection $h : X \xrightarrow{\sim} Y$ as follows:

$$h : X \to Y; \qquad x \mapsto \begin{cases} \sigma(x) = g^{-1}(x) & \text{if } x \in X_O \biguplus X_\infty, \\ f(x) & \text{if } x \in X_E. \end{cases}$$

Note that $g^{-1}(x)$ is defined for all $x \in X_O \biguplus X_\infty$ because we then have $N(x) > 0$.

We show that h is injective: Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. There are four cases.

Case 1: Both $x_1, x_2 \in X_O \biguplus X_\infty$. Then $h(x_1) = g^{-1}(x_1) \neq g^{-1}(x_2) = h(x_2)$ because the bijectivity of g implies that of g^{-1}. In particular, g^{-1} is injective.

Case 2: Both $x_1, x_2 \in X_E$. Then $h(x_1) = f(x_1) \neq f(x_2) = h(x_2)$ because f is injective.

Case 3: $x_1 \in X_O \biguplus X_\infty$ and $x_2 \in X_E$. It follows from (10.10) that $h(x_1) = g^{-1}(x_1) \in Y_E \uplus Y_\infty$ and that $h(x_2) = f(x_2) \in Y_O$. Because $Y_E \uplus Y_\infty$ and Y_O have no elements in common, it follows that $h(x_1) \neq h(x_2)$. We have proved that h is injective.

Case 4: $x_2 \in X_O \biguplus X_\infty$ and $x_1 \in X_E$. Injectivity of h follows from case 3 because we can switch the roles of x_1 and x_2.

We finally show that h is surjective: Let $y \in Y$. There are two cases.

Case (i): $y \in Y_E \biguplus Y_\infty$. It follows from (10.10) that $g(y) \in X_O \biguplus X_\infty$, hence

$$h(g(y)) \;=\; \sigma(g(y)) \;=\; g^{-1}(g(y)) \;=\; y.$$

Here the second equation follows from (10.9). We have found an item in X which is mapped by h to y. and this proves that h is surjective.

Case (ii): $y \in Y_O$. It follows from (10.10) that $f^{-1}(y) \in X_E$, hence $h(f^{-1}(y)) = f(f^{-1}(y)) = y$. Again we have found an item in X which is mapped by h to y. We have proved that h is surjective also in this case. ■

10.4 Exercises for Ch.10

Exercise 10.1. Let Ω be a set and let $\varphi : 2^\Omega \to 2^\Omega$ satisfy $A, B \subseteq \Omega$ and $A \subseteq B \Rightarrow \varphi(A) \subseteq \varphi(B)$.

$$\text{Let} \quad \mathfrak{F} := \{A \in 2^\Omega : A \subseteq \varphi(A)\}, \quad A_0 := \bigcup[A : A \in \mathfrak{F}].$$

The proof of Tarski's fixed point theorem (Theorem 10.2 on p.456) shows that A_0 is a fixed point for φ, i.e., $\varphi(A_0) = A_0$. Modify this proof to show the following:

$$\text{Let} \quad \mathscr{E} := \{B \in 2^\Omega : \varphi(B) \subseteq B\}, \quad B_0 := \bigcap[B : B \in \mathscr{E}].$$

Then B_0 also is a fixed point for φ.

Exercise 10.2.

Prove prop.10.1 on p.454:

Let X, Y be two sets such that $\operatorname{card}(X) = \operatorname{card}(Y)$. Then $\operatorname{card}(2^X) = \operatorname{card}(2^Y)$. □

Exercise 10.3.

Prove prop.10.3 on p.456 of this document: Let X, X', Y, Y' be nonempty sets such that $X \cap X' = \emptyset$ and $Y \cap Y' = \emptyset$ Let $f : X \rightarrowtail Y$ and $f' : X' \rightarrowtail Y'$ injective functions. Then

$$h : X \biguplus X' \longrightarrow Y \uplus Y'; \qquad x \mapsto \begin{cases} f(x) & \text{if } x \in X, \\ f'(x) & \text{if } x \in X', \end{cases}$$

is an injection.

Note that part of the proof is showing that the relation $\{(x, y) \in (X \biguplus X') \times (Y \uplus Y') : y = h(x)\}$ indeed defines the graph of a function. □

References

[1] Matthias Beck and Ross Geoghegan. The Art of Proof. Springer, 1st edition, 2010.

[2] G. Chartrand, A. Polimeni, and Ping Zhang. Mathematical Proofs: A Transition to Advanced Mathematics. Pearson, 4th edition, 2018.

[3] Norman B. Haaser and Joseph A. Sullivan. Real Analysis. publisher = Van Nostrand, year = 1971, 1st edition.

List of Symbols

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