Math 330 - Additional Material

Student edition with proofs

Michael Fochler
Department of Mathematics
Binghamton University

This document contains chapter 12 of the Math 330 lecture notes.

Last update: February 17, 2026

Contents

12 Metric Spaces and Topological Spaces – Part I

There is a branch of Mathematics, called topology, which deals with the concept of closeness. The definition of the limit of a sequence $(x_n)_n$ is based on closeness: The points of the sequence must get "arbitrarily close" to its limit as $n \to \infty$. Continuity also can be phrased in terms of closeness: Continuous functions map arbitrarily close elements of the domain to arbitrarily close elements of the codomain. In the most general setting Topology is about neighborhoods of a point without having the concept of measuring the distance of two points. We mostly won't deal with such a level of generality in this document. Instead we'll we'll focus on metric spaces (X, d): sets X that are equipped with a distance function $(x, y) \mapsto d(x, y)$. Even this limited context will significantly generalize the material of ch.**??** (Convergence and Continuity in $\mathbb{R}$) and ch.**??** (Normed Vector Spaces).

12.1 Definition and Examples of Metric Spaces

A metric is a real–valued function of two arguments which associates with any two points $x, y \in X$ their "distance" $d(x, y)$.

It is clear how you measure the distance (or closeness, depending on your point of view) of two numbers x and y: you plot them on an x–axis where the distance between two consecutive integers is exactly one inch, grab a ruler and see what you get. Alternate approach: you compute the difference. For example, the distance between $x = 12.3$ and $y = 15$ is $x - y = 12.3 - 15 = -2.7$. Actually, we have a problem: There are situations where direction matters and a negative distance is one that goes into the opposite direction of a positive distance, but we do not want that in this context and understand the distance to be always nonnegative, i.e.,

$$dist(x, y) \;=\; |y - x| \;=\; |x - y|$$

More importantly, you must forget what you learned in your in your science classes: "Never ever talk about a measure (such as distance or speed or volume) without clarifying its dimension". Is the speed measured in miles per hour our inches per second? Is the distance measured in inches or miles or micrometers? In the context of metric spaces we measure distance simply as a number, without any dimension attached to it. For the above example, you get

$$dist(12.3, 15) \;=\; |12.3 - 15| \;= 2.7.$$

In section **??** on p.**??** it is shown in great detail that the distance between two two–dimensional vectors $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$ is

$$\text{dist}(\vec{v}, \vec{w}) \;=\; \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2}$$

and the distance between two three–dimensional vectors $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ is

$$\text{dist}(\vec{v}, \vec{w}) \;=\; \sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2 + (w_3 - v_3)^2}.$$

In the next chapter we will generalize the concept of distance to more general objects.

Definition 12.1 (Metric spaces)**.**

Let X be an arbitrary, nonempty set.

A **metric** on X is a real–valued function of two arguments

$$d(\cdot,\cdot) : X \times X \to \mathbb{R}, \qquad (x,y) \mapsto d(x,y)$$

with the following three properties:

(12.1a)	$d(x,y) \geq 0 \quad \forall x,y \in X \quad \text{and} \quad d(x,y) = 0 \;\Leftrightarrow\; x = y$	**positive definiteness**
(12.1b)	$d(x,y) = d(y,x) \quad \forall x,y \in X$	**symmetry**
(12.1c)	$d(x,z) \leq d(x,y) + d(y,z) \quad \forall\, x,y,z \in X$	**triangle inequality**

Let $x, y \in X$ and $\varepsilon > 0$. We say that x and y are ε–**close** if $d(x,y) < \varepsilon$. The pair $(X, d(\cdot,\cdot))$, usually just written as (X,d), is called a **metric space**. We'll write X for short if it is clear which metric we are talking about. □

To appreciate that last sentence, you must understand that there can be more than one metric on X. See the examples below.

Remark 12.1 (Metric properties)**.** Let us examine what those properties mean.

"Positive definite":	The distance is never negative and two items x and y have distance zero if and only if they are equal.
"symmetry":	the distance from x to y is no different to that from y to x. That may come as a surprise to you if you have learned in Physics about the distance from point a to point b being the vector $\vec{v}$ that starts in a and ends in b and which is the opposite of the vector $\vec{w}$ that starts in b and ends in a, i.e., $\vec{v} = -\vec{w}$. We only care about size and not about direction.
"Triangle inequality":	If you directly drive from x to z then this will take less fuel than if you make a stopover at an intermediary y. □

Remark 12.2. Do not make the mistake and think of X as a set of numbers or vectors! For example, we might deal with

$$X \;:=\; \{\text{ all students who are currently taking this class }\}.$$

We can define the distance of any two students s_1 and s_2 as

$$d(s_1,s_2) = \begin{cases} 0 & \text{for } s_1 = s_2, \\ 1 & \text{for } s_1 \neq s_2. \end{cases}$$

We will learn later in this subchapter that the above function is called the discrete metric on X and satisfies indeed the definition of a metric. [1] □

[1] see Definition 12.3 on p.556 and prop.12.2 directly thereafter.

The triangle inequality generalizes to more than two terms.

Proposition 12.1.

Let (X, d) be a metric space. Let $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$. Then

$$d(x_1, x_n) \leq \sum_{j=1}^{n-1} d(x_j, x_{j+1}) = d(x_1, x_2) + d(x_2, x_3) + d(x_{n-1}, x_n). \tag{12.2}$$

The proof is left as exercise 12.1 on p.596. ■

Before we give some examples of metric spaces, here is a theorem that tells you that a vector space with a norm (see Definition **??** on p.**??**), becomes a metric space as follows:

Theorem 12.1 (Norms define metric spaces)**.**

Let $(V, \|\cdot\|)$ be a normed vector space. Then the function

$$d_{\|\cdot\|}(\cdot,\cdot) : V \times V \to \mathbb{R}_{\geq 0}; \qquad (x, y) \mapsto d_{\|\cdot\|}(x, y) := \|y - x\| \tag{12.3}$$

defines a metric space $(V, d_{\|\cdot\|})$.

PROOF: The proof is left as exercise 12.2 on p.596. ■

Compare the following definition to Definition **??** (Norm for an inner product) on p.**??**.

Definition 12.2 (Metric induced by a norm)**.**

We say that the metric $d_{\|\cdot\|}(\cdot,\cdot)$ defined by (12.3) is **induced by the norm** $\|\cdot\|$. and that $d_{\|\cdot\|}(\cdot,\cdot)$ is **derived from the norm** $\|\cdot\|$, or that $d_{\|\cdot\|}(\cdot,\cdot)$ is **associated with the norm** $\|\cdot\|$. □

Here are some examples of metric spaces.

Example 12.1 ($(\mathbb{R}$ with $d_{|\cdot|}(a, b) = |b-a|)$**.** According to thm.12.1 $(\mathbb{R}, d_{|\cdot|})$ is a metric space because the Euclidean norm $|\cdot|$ is a norm on $\mathbb{R} = \mathbb{R}^1$.

Here is a direct proof; It is obvious that if x, y are real numbers then the difference $x - y$, and hence its absolute value, is zero if and only if $x = y$ and that proves positive definiteness.

Symmetry follows from $d_{|\cdot|}(x, y) = |x - y| = |-(y - x)| = |y - x| = d_{|\cdot|}(y, x)$.

The triangle inequality for a metric follows from $|a + b| \leq |a| + |b|$ (see prop.**??** on p.**??**):

$$\begin{aligned} d_{|\cdot|}(x, z) &= |x - z| = |(x - y) - (z - y)| \\ &\leq |x - y| + |z - y| = d_{|\cdot|}(x, y) + d_{|\cdot|}(z, y) = d_{|\cdot|}(x, y) + d_{|\cdot|}(y, z). \end{aligned}$$ □

Example 12.2 (bounded real–valued functions with $d_{\|\cdot\|_\infty} f, g)$ = sup–norm of $g(\cdot) - f(\cdot)$).

$$d_{\|\cdot\|_\infty}(f,g) \;=\; \|g-f\|_\infty \;=\; \sup\{|g(x)-f(x)| : x \in X\} \tag{12.4}$$

is a metric on the set $\mathscr{B}(X,\mathbb{R})$ of all bounded real–valued functions on X. This follows from thm.12.1 and prop.?? on p. ??, according to which $(\mathscr{B}(X,\mathbb{R}), \|\cdot\|_\infty)$ is a normed vector space. □

Example 12.3 (continuous real–valued functions on [a,b]with $d_{\|\cdot\|_{L^2}}(f,g) = \|g-f\|_{L^2}$). We will see in ch.12.2 on p.557 that $\|g-f\|_\infty$ is a good measure for the difference of the functions f and g and that an often even better measure is that of the area difference between their graphs which is given by the netric

$$d_{\|\cdot\|_{L^2}}(f,g) \;=\; \|g-f\|_{L^2} \;=\; \sqrt{\int_a^b \big(g(x)-f(x)\big)^2 dx}. \tag{12.5}$$

(See Definition ?? on p.??). □

Example 12.4 ($\mathbb{R}^n$ with the Euclidean metric).

$$d_{\|\cdot\|_2}(\vec{x},\vec{y}) \;=\; \sqrt{(y_1-x_1)^2+(y_2-x_2)^2+\ldots+(y_n-x_n)^2} \;=\; \sqrt{\sum_{j=1}^n (y_j-x_j)^2}$$

This follows from the fact that the Euclidean norm is a norm on the vector space $\mathbb{R}^n$ (see cor.?? on p.??.) □

Just in case you think that all metrics are derived from norms, here is an extremely important counterexample.

Definition 12.3 (Discrete metric).

Let X be nonempty. Then the function

$$d(x,y) = \begin{cases} 0 & \text{for } x = y \\ 1 & \text{for } x \neq y \end{cases}$$

on $X \times X$ is called the **discrete metric** on X. □

The above definition makes sense because of the following proposition.

Proposition 12.2.

The discrete metric satisfies the properties of a metric.

PROOF: Obviously the function is nonnegative and it is zero if and only if $x = y$. Symmetry is obvious too.

The triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$ is certainly true in the special case $x = z$. (Why?) So let us assume $x \neq z$. But then $x \neq y$ or $y \neq z$ or both must be true. (Why?) That means that

$$d(x, z) = 1 \leq d(x, y) + d(y, z),$$

and this proves the triangle inequality. ■

12.2 Measuring the Distance of Real–Valued Functions

How do we compare two functions? Let us make our lives easier: How do we compare two real–valued functions $f(\cdot)$ and $g(\cdot)$? One answer is to look at a picture with the graphs of $f(\cdot)$ and $g(\cdot)$ and look at the shortest distance $|f(x) - g(x)|$ as you run through all x. That means that the distance between the functions $f(x) = x$ and $g(x) = x^2$ is zero because $f(1) = g(1) = 1$. The distance between $f(x) = x + 1$ and $g(x) = 0$ (the x–axis) is also zero because $f(-1) = g(-1) = 0$.

Do you really think this is a good way to measure closeness? You really do not want two items to have zero distance unless they coincide. It's a lot better to look for an argument x where the value $|f(x) - g(x)|$ is largest rather than smallest. [2] Now we are ready for a proper definition.

Definition 12.4 (Maximal displacement distance between real–valued functions)**.**

Let X be an arbitrary, nonempty set and let $f(\cdot), g(\cdot) : X \to \mathbb{R}$ be two real–valued functions on X. We define the **maximal displacement distance** , also called the **sup–norm distance** or $\|\cdot\|_\infty$ **distance** , between $f(\cdot)$ and $g(\cdot)$ as

$$d_\infty(f, g) := \|f(\cdot) - g(\cdot)\|_\infty = \sup\{|f(x) - g(x)| : x \in X\}, \tag{12.6}$$

i.e., as the metric induced by the sup–norm on the set $\mathscr{B}(X, \mathbb{R})$ of all bounded real–valued function on X. □

Remark 12.3.

We will see in prop.?? on p.?? of ch.?? on convergence of function sequences that the sup–norm induced metric is suitable to measure what will be called "uniform convergence" of real–valued functions. As a metric, the distance measure of two functions f, g satisfies positive definiteness, symmetry and the triangle inequality. We have seen in other contexts what those properties mean.

"Positive definite": The distance is never negative and two functions $f(\cdot)$ and $g(\cdot)$ have distance zero if and only if they are equal, i.e., if and only if $f(x) = g(x)$ for each argument $x \in X$.

[2] See example 12.2 on p.556.

"Symmetry": the distance from $f(\cdot)$ to $g(\cdot)$ is no different than that from $g(\cdot)$ to $f(\cdot)$. Symmetry implies that you do **not** obtain a negative distance if you walk in the opposite direction.

"Triangle inequality": If you directly compare the maximum deviation between two functions $f(\cdot)$ and $h(\cdot)$ then this will never be more than than using an intermediary function $g(\cdot)$ and adding the distance between $f(\cdot)$ and $g(\cdot)$ to that between $g(\cdot)$ and $h(\cdot)$. $\square$

Remark 12.4.

Figure 12.1 illustrates the last definition. Plot the graphs of f and g as usual and find the the spot x_0 on the x–axis for which the difference $|f(x_0) - g(x_0)|$ (the length of the vertical line that connects the two points with coordinates $(x_0, f(x_0))$ and $(x_0, g(x_0))$) has the largest possible value. The domain of f and g is the subset of $\mathbb{R}$ that corresponds to the thick portion of the x–axis.

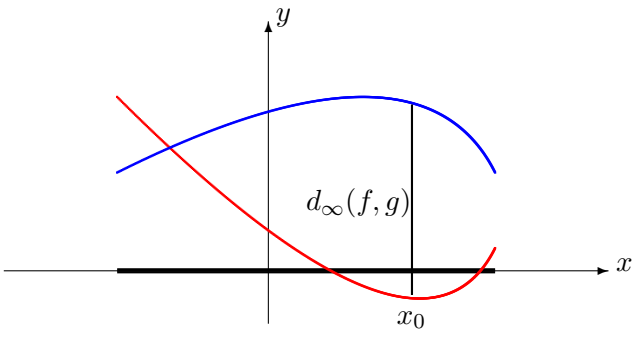


Figure 12.1: Distance of two real–valued functions.

Figure 12.2 allows you to visualize for a given $\delta > 0$ and $f : X \to \mathbb{R}$ the "δ–neighborhood" of $f(\cdot)$ defined as

$$N_\delta(f) := \{g : X \to \mathbb{R} : d(_\infty f, g) < \delta\} = \{g(\cdot) : X \to \mathbb{R} : \sup_{x \in X} |f(x) - g(x)| < \delta\}, \tag{12.7}$$

i.e., the set of all functions $g(\cdot)$ with distance less than δ from $f(\cdot)$.

You draw the graph of $f(\cdot) + \delta$ (the graph of $f(\cdot)$ shifted up by the amount of δ) and the graph of $f(\cdot) - \delta$ (the graph of $f(\cdot)$ shifted down by the amount of δ). Any function $g(\cdot)$ which stays completely inside this band, without actually touching it, belongs to the δ–neighborhood of $f(\cdot)$.

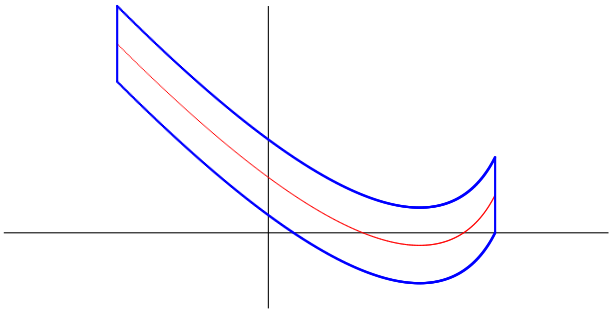


Figure 12.2: δ–neighborhood of a real–valued function.

In other words, assuming that the domain A is a single, connected chunk and not a collection of several separate intervals, the δ–neighborhood of $f(\cdot)$ is a "band" whose contours are made up on the left and right by two vertical lines and on the top and bottom by two lines that look like the graph of $f(\cdot)$ itself but have been shifted up and down by the amount of δ. □

Definition 12.5 (Mean distances between real–valued functions)**.**

Let $a, b \in \mathbb{R}$ such that $a < b$ and let $f(\cdot), g(\cdot) : X \to \mathbb{R}$ be two continuous real–valued functions on X. We define the **mean square distance** between $f(\cdot)$ and $g(\cdot)$ on $[a, b]$ as

$$d_{L^2}(f,g) := d_{\|\cdot\|_{L^2}(f,g)} = \|g-f\|_{L^2} = \left(\int_a^b (g(x)-f(x)^2 dx\right)^{1/2}, \tag{12.8}$$

i.e., as the metric induced by the L^2–norm on the set $\mathscr{C}_{\mathscr{B}}([\text{a,b}], \mathbb{R})$ of all continuous and bounded real–valued function on $[a, b]$.

We further define the **mean distance** between $f(\cdot)$ and $g(\cdot)$ on $[a, b]$ as

$$d_{L^1}(f,g) := d_{\|\cdot\|_{L^1}(f,g)} = \|g-f\|_{L^1} = \int_a^b |g(x)-f(x|dx, \tag{12.9}$$

i.e., as the metric induced by the L^1–norm on the set $\mathscr{C}_{\mathscr{B}}([\text{a,b}], \mathbb{R})$. □

Remark 12.5.

We saw in Definition **??**, example **??**, and Definition **??** on pp.**??** that both

$$d_{L^1}(f,g) := d_{\|\cdot\|_{L^1}(f,g)} = \|g-f\|_{L^1} = \int_a^b |g(x)-f(x)|dx, \tag{12.10}$$

$$d_{L^2}(f,g) := d_{\|\cdot\|_{L^2}(f,g)} = \|g-f\|_{L^2} = \left(\int_a^b (g(x)-f(x)^2 dx\right)^{1/2}, \tag{12.11}$$

are often better suitable than the distance derived from the sup–norm to measure the distance of two functions. One of the drawbacks from an instructor's perspective is that there is no picture like figure 12.2 to visualize the set of all functions with an L^1–distance or L^2–distance from a given function.

12.3 Neighborhoods and Open Sets

(A) Given a point $x_0 \in \mathbb{R}$ (a real number) and $\varepsilon > 0$, we can look at

$$N_\varepsilon(x_0) = (x_0 - \varepsilon, x_0 + \varepsilon) = \{x \in \mathbb{R} : x_0 - \varepsilon < x < x_0 + \varepsilon\} = \{x \in \mathbb{R} : d(x, x_0) = |x - x_0| < \varepsilon\} \tag{12.12}$$

which is the set of all real numbers x with a distance to x_0 of strictly less than a number ε (the open interval with end points $x_0 - \varepsilon$ and $x_0 + \varepsilon$). (see example (12.1) on p.555).

(B) Given a point $\vec{x}_0 = (x_0, y_0) \in \mathbb{R}^2$ (a point in the xy–plane), we can look at

$$\begin{aligned} N_\varepsilon(\vec{x}_0) &= \{(x, y) \in \mathbb{R}^2 : (x - x_0)^2 + (y - y_0)^2 < \varepsilon^2\} \\ &= \{\vec{x} \in \mathbb{R}^2 : d_{\|\cdot\|_2}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{x}_0\|_2 < \varepsilon\} \end{aligned} \tag{12.13}$$

which is the set of all points in the plane with a distance to $\vec{x}_0$ of strictly less than a number ε (the open disc around $\vec{x}_0$ with radius ε from which the points on the boundary (those with distance equal to ε) are excluded).

(C) Given a point $\vec{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ (a point in the 3–dimensional space), we can look at

$$\begin{aligned} N_\varepsilon(\vec{x}_0) &= \{(x, y, z) \in \mathbb{R}^3 : (\vec{x} - \vec{x}_0)^2 + (\vec{y} - \vec{y}_0)^2 + (\vec{z} - \vec{z}_0)^2 < \varepsilon^2\} \\ &= \{\vec{x} \in \mathbb{R}^3 : d_{\|\cdot\|_2}(\vec{x}, \vec{y}) = \|\vec{x} - \vec{x}_0\|_2 < \varepsilon\} \end{aligned} \tag{12.14}$$

which is the set of all points in space with a distance to $\vec{x}_0$ of strictly less than a number ε (the open ball around $\vec{x}_0$ with radius ε from which the points on the boundary (those with distance equal to ε) are excluded).

(D) Given a normed vector space $(V, \|\cdot\|)$ and a vector $x_0 \in V$, we can look at

$$N_\varepsilon(x_0) = \{x \in V : \|x - x_0\| < \varepsilon\} \tag{12.15}$$

which is the set of all vectors in V with a distance to x_0 of strictly less than a number ε (the open set around x_0 with "radius" ε from which the points on the boundary (those with distance equal to ε) are excluded).

(E) Given a bounded real–valued function $f \in \mathscr{B}(X, \mathbb{R})$, we can look at the sets $N_\varepsilon(f)$ ($\varepsilon > 0$) defined in (12.7) on p.558, i.e., the set of all functions $g(\cdot)$ with distance less than ε from $f(\cdot)$.

(F) Given is a closed interval $[a, b]$ ($a, b \in \mathbb{R}$). For a continuous (hence bounded) real–valued function $f \in \mathscr{B}([a, b], \mathbb{R})$, we can look at the sets

$$N_\varepsilon(f) = \{g \in \mathscr{B}([a, b], \mathbb{R}) : \|g - f\|_{L^2} < \varepsilon\}, \tag{12.16}$$

i.e., the set of all functions $g(\cdot)$ such that $\sqrt{\int_a^b \big(g(x) - f(x)\big)^2 dx} < \varepsilon$ (see Definition **??** on p.**??**)

There is one more item more general than neighborhoods of elements belonging to normed vector spaces, and that would be neighborhoods in metric spaces. We have arrived at the final definition:

Definition 12.6 (ε-Neighborhood)**.**

Given a metric space (X, d), $x_0 \in X$ and $\varepsilon > 0$, let

$$N_\varepsilon(x_0) \;=\; \{x \in X : d(x, x_0) < \varepsilon\} \tag{12.17}$$

be the set of all elements of X with a distance to x_0 of strictly less than the number ε (the open set around x_0 with "radius" ε from which the points on the boundary (those with distance equal to ε) are excluded). We call $N_\varepsilon(x_0)$ the ε–**neighborhood** of x_0. □

The following should be intuitively clear: Look at any point $a \in N_\varepsilon(x_0)$. You can find $\delta > 0$ such that the entire δ–neighborhood $N_\delta(a)$ of a is contained inside $N_\varepsilon(x_0)$. Just in case you do not trust your intuition, this is shown in prop. 12.4 just a little bit further down.

It then follows that any $a \in N_\varepsilon(x_0)$ is an interior point of $N_\varepsilon(x_0)$ in the following sense:

Definition 12.7 (Interior points in metric spaces)**.**

Given is a metric space (X, d).
An element $a \in A \subseteq X$ is called an **inner point** or **interior point** of A if we can find some $\varepsilon > 0$ (no mattter how small), so that $N_\varepsilon(a) \subseteq A$. □

Definition 12.8 (Open sets in metric spaces)**.**

Given is a metric space (X, d).
A set all of whose members are interior points is called an **open set**. □

Proposition 12.3.

Let (X, d) be a metric space. Let $x, y \in X$ and $\varepsilon > 0$ such that $y \in N_\varepsilon(x)$.

$$\text{If } \delta > 0 \text{ Then } N_\delta(y) \subseteq N_{\delta+\varepsilon}(x)\,.$$

PROOF: Let $z \in N_\delta(y)$. Then

$$d(z, x) \;\leq\; d(z, y) + d(y, x) \;< \delta + \varepsilon.$$

In other words, each element z of $N_\delta(y)$ is $\delta + \varepsilon$–close to x. Hence $N_\delta(y) \subseteq N_{\delta+\varepsilon}(x)$. ■

Proposition 12.4.

$N_\varepsilon(x_0)$ is an open set

PROOF: It is worthwhile to examine this proof [3] closely because you can see how the triangle inequality is put to work.

$a \in N_\varepsilon(x_0)$ means that $\varepsilon - d(a, x_0) > 0$, say,

$$\varepsilon - d(a, x_0) \ = 2\delta \tag{12.18}$$

where $\delta > 0$. Let $b \in N_\delta(a)$. The claim is that any such b is an element of $N_\varepsilon(x_0)$. How so?

$$d(b, x_0) \leq d(b, a) + d(a, x_0) < \delta + (\varepsilon - 2\delta) = \varepsilon - \delta < \varepsilon$$

In the above chain, the first inequality is a consequence of the triangle inequality. The second one reflects the fact that $b \in N_\delta(a)$ and uses (12.18).

We have proved that for any $b \in N_\delta(a)$ it is true that $b \in N_\varepsilon(x_0)$ hence $N_\delta(a) \subseteq N_\varepsilon(x_0)$.

This proves that a is an interior point of $N_\varepsilon(x_0)$. But a is an arbitrary point in $N_\varepsilon(x_0)$. It follows that $N_\varepsilon(x_0)$ is open. ■

Proposition 12.5 (Open intervals are open in $(\mathbb{R}, d_{|\cdot|})$)**.**

Let $a, b \in \mathbb{R}$ such that $a < b$. Then the open interval $]a, b[$ is an open set in $(\mathbb{R}, d_{|\cdot|})$.

PROOF: The proof is left as exercise 12.3 on p.597. ■

Definition 12.9 (Neighborhoods in Metric Spaces)**.**

Let (X, d) be a metric space, $x_0 \in X$. Any open set that contains x_0 is called an **open neighborhood** of x_0. Any superset of an open neighborhood of x_0 is called a **neighborhood** of x_0. □

Remark 12.6.

(a) You will see very often that **the important neighborhoods are the small ones**, not the big ones. The definition above says that, for any neighborhood A_x of a point $x \in X$, one can find an open neighborhood U_x of x such that $U_x \subseteq A_x$. Thus, very often **the open neighborhoods are the important ones**. Accordingly, there are many theorems where it is assumed that some given neighborhood is open.

(b) The empty set is not a neighborhood of any $x \in X$, since the condition $x \in \emptyset$ is never satisfied. □

Proposition 12.6 (Metric Spaces are Hausdorff Spaces)**.**

[3] A shorter proof can be given if the previous proposition is used.

Let (X, d) be a metric space and let x, y be two different elements of X. Then there exist neighborhoods N_x of x and N_y of y such that $N_x \cap N_y = \emptyset$.

PROOF:
Let $\varepsilon := \frac{1}{2}d(x, y)$ and let $x' \in N_\varepsilon(x)$. We must show that $x' \notin N_\varepsilon(y)$, i.e., $d(x', y) \geq \varepsilon$. Assume to the contrary that $d(x', y) < \varepsilon$. It follows from $x' \in N_\varepsilon(x)$ that $d(x, x') < \varepsilon$. Thus

$$d(x, y) \leq d(x, x') + d(x', y) < \varepsilon + \varepsilon = d(x, y).$$

We have reached a contradiction. ■

Remark 12.7. ⋆ We will encounter in Chapter.12.5 objects more general than metric spaces, the so called topological spaces, which allow to define neighborhoods of a point. There are such spaces for which this proposition is not true. See Remark 12.13 on p.570 about the "indiscrete topology".
A topological space with the property that any two of its elements can be "separated" by disjoint neighborhoods is called a **Hausdorff space** [4] or also a **T2 space**. □

Theorem 12.2 (Metric spaces are topological spaces)**.**

The following is true about open sets of a metric space (X, d):

(12.19a) *An arbitrary union $\bigcup_{i \in I} U_i$ of open sets U_i is open.*

(12.19b) *A finite intersection $U_1 \cap U_2 \cap \ldots \cap U_n$ $(n \in \mathbb{N})$ of open sets is open.*

(12.19c) *The entire set X is open and the empty set $\emptyset$ is open.*

PROOF of a: Let $U := \bigcup_{i \in I} U_i$ and assume $x \in U$. We must show that x is an interior point of U. An element belongs to a union if and only if it belongs to at least one of the participating sets of the union. So there exists an index $i_0 \in I$ such that $x \in U_{i_0}$.
Because U_{i_0} is open, x is an interior point and we can find a suitable $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq U_{i_0}$. But $U_{i_0} \subseteq U$, hence $N_\varepsilon(x) \subseteq U$. It follows that x is interior point of U. But x was an arbitrary point of $U = \bigcup_{i \in I} U_i$ which therefore is shown to be an open set.
PROOF of b: Let $x \in U := U_1 \cap U_2 \cap \ldots \cap U_n$. Then $x \in U_j$ for all $1 \leq j \leq n$ according to the definition of an intersection and it is inner point of all of them because they all are open sets. Hence, for each j there is a suitable $\varepsilon_j > 0$ such that $N_{\varepsilon_j}(x) \subseteq U_j$ Now define

$$\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n\}$$

[4] so named after the German mathematician Felix Hausdorff

Then $\varepsilon > 0$ and [5]

$$N_\varepsilon(x) \subseteq N_{\varepsilon_j}(x) \subseteq U_j \ (1 \le j \le n), \quad \text{hence} \quad N_\varepsilon(x) \subseteq \bigcap_{j=1}^{n} U_j.$$

We have shown that an arbitrary $x \in U$ is interior point of U and this proves part b.

PROOF of c: First we deal with the set X. Choose any $x \in X$. No matter how small or big an $\varepsilon > 0$ you choose, $N_\varepsilon(x)$ is a subset of X. But then x is an inner point of X, so all members of x are inner points and this proves that X is open.

Now to the empty set $\emptyset$. You may have a hard time to accept the logic of this statement: All elements of $\emptyset$ are interior points. But remember, the premise "let $x \in X$" is always false and you may conclude from it whatever you please (see ch.**??** (Logic). ■

This last theorem provides the underpinnings for the definition of abstract topological spaces, a subject which will be touched upon in ch.12.5 on p.567.

12.4 Convergence

You have already encountered the precise definition of the convergence of sequences of real numbers in ch.**??**. It is only a small step to generalize this concept to all metric spaces and therefore also to all normed vector spaces.

Definition 12.10 (Convergence of Sequences in Metric Spaces)**.**

Given is a metric space (X, d). We say that a sequence (x_n) of elements of X **converges** to $a \in X$ for $n \to \infty$ if the x_n will eventually come arbitrarily close to a in the following sense:

Let δ be a (arbitrarily small) positive real number. Then there is a (possibly extremely large) integer n_0 such that all x_j belong to $N_\delta(a)$ just as long as $j \ge n_0$.

This can also be expressed as follows:

(12.20) For all $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(a, x_j) < \delta$ for all $j \ge n_0$.

Here is an yet another way of expressing convergence of $(x_n)_n$ to a:

- No matter how small a neighborhood of a is given, all members x_n will eventually be inside that neighborhood.

We write either of

(12.21) $$a = \lim_{n\to\infty} x_n \qquad \text{or} \qquad x_n \to a$$

and we call a the **limit** of the sequence (x_n) □

Theorem 12.3 (Limits in metric spaces are uniquely determined)**.**

[5] This is the exact spot where the proof breaks down if you deal with an infinite intersection of open sets: the minimum would have to be replaced by an infimum and there is no guarantee that it would be strictly larger than zero.

Let (X,d) be a metric space and let $(x_n)_n$ be a convergent sequence in X. Then its limit is uniquely determined.

PROOF: Otherwise there would be two different points $L_1, L_2 \in X$ such that both $\lim_{n\to\infty} x_n = L_1$ and $\lim_{n\to\infty} x_n = L_2$ Let $\varepsilon := d(L_1,L_2)/2$. There will be $N_1, N_2 \in \mathbb{N}$ such that

$$d(x_n, L_1) < \varepsilon \ \forall n \geq N_1 \ \text{ and } d(x_n, L_2) < \varepsilon \ \forall n \geq N_2.$$

It follows that, for $n \geq \max(N_1, N_2)$, [6]

$$d(L_1,L_2) \ \leq \ d(L_1,x_n) + d(x_n,L_2) \ < \ 2\varepsilon \ = \ d(L_1,L_2)$$

and we have reached a contradiction. ■

Proposition 12.7.

Let (X,d) be a metric space and $L, x_n \in X$ $(n \in \mathbb{N})$. Let $\delta_n \in \mathbb{R}_{>0}$ such that $\delta_n \to 0$ as $n \to \infty$. Assume further that $x_n \in N_{\delta_n}(L)$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = L$.

PROOF:
Let $\varepsilon > 0$. It follows from $\lim_{k\to\infty} \delta_k = 0$ that there exists $n_0 \in \mathbb{N}$ such that $\delta_k < \varepsilon$ for all $k \geq n_0$. Because $x_k \in N_{\delta_k}(L)$ implies $d(x_k, L) < \delta_k$, we conclude that $d(x_k, L) < \varepsilon$ for all $k \geq n_0$, and hence that $\lim_{k\to\infty} x_n = L$. ■

For the special case $\delta_n = \frac{1}{n}$ we obtain

Corollary 12.1.

Let (X,d) be a metric space and $L, x_n \in X$ $(n \in \mathbb{N})$ such that $d(x_n, L) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = L$.

PROOF: Obvious from prop.12.7. ■

We proved for constant sequences of real numbers that they are convergent. This is also true for sequences with values in metric spaces.

Proposition 12.8.

[6] You could have used $N_1 + N_2$ instead. Do you see why?

Let (X,d) be a metric space, $L \in X$ and $x_n = L$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} x_n = L$.

PROOF:
This follows from cor.12.1 above since $x_n = L \Rightarrow d(x_n, L) = 0 \Rightarrow d(x_n, L) \leq \frac{1}{n}$ for all $n \in \mathbb{N}$ but we should be able to prove this directly from the definition of convergence. This is how:
Let $\delta > 0$ and $n_0 = 1$. Then $d(x_n, L) = 0 < \delta$ for all $n \geq 1$, i.e., (12.10) on p.564 is satisfied. ■

The next proposition shows that the limit behavior of a sequence is a property of its tail, i.e., it does not depend on the first finitely many indices. See also cor.?? on p.??.

Proposition 12.9.

Let x_n, y_n be two sequences in a metric space (X,d). Assume there is $K \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq K$. Let $L \in X$ Then

$$\lim_{n\to\infty} x_n = L \Leftrightarrow \lim_{n\to\infty} y_n = L.$$

PROOF: The proof is left as exercise 12.9 on p.598. ■

Compare the next proposition to Proposition ?? on p.??.

Proposition 12.10 (Subsequences of sequences with limits)**.**

Let $(x_n)_n$ be a sequence in a metric space (X,d) with limit $L := \lim_{n\to\infty} x_n$. Then it is true for any subsequence $(x_{n_j})_j$, that $\lim_{j\to\infty} x_{n_j} = L$.

PROOF: Let $\varepsilon > 0$. Because the sequence converges, there exists $N \in \mathbb{N}$ such that $d(x_j, L) < \varepsilon$ for all $j \geq N$. As $n_j \geq j$ for all j, we conclude that $n_j \geq N$ whenever $j \geq N$, hence $|x_{n_j} - L| < \varepsilon$ for all $j \geq N$. It follows that (x_{n_j}) has limit L. ■

The following is a simple corollary. See also prop.?? on p.??.

Proposition 12.11.

Let x_n be a convergent sequence in a metric space (X,d) with limit $L \in X$. Let $K \in \mathbb{N}$. For $n \in \mathbb{N}$ let $y_n := x_{n+K}$. Then $\lim_{n\to\infty} (y_n)_n = L$.

PROOF: The proof is left as exercise 12.10 on p.598. ■

Remark 12.8.

The following allows us to prove convergence of x_n to $L \in (X,d)$ by utilizing what we know about convergence in $(\mathbb{R}, d_{|\cdot|})$.

$$\lim_{n\to\infty} x_n \;=\; L \;\Leftrightarrow\; \lim_{n\to\infty} d(x_n, L) \;=\; 0\,. \;\square$$

Remark 12.9 (Opposite of convergence)**.**
Given a metric space (X,d), what is the opposite of $\lim_{k\to\infty} x_k = L$? Beware! It is NOT the statement that $\lim_{k\to\infty} x_k \neq L$, because such a statement would mislead you to believe that such a limit exists, it just happens not to coincide with L. Rather,

$\big[\ \lim_{k\to\infty} x_k = L$ is **NOT true** $\big]\ \Leftrightarrow$
$\big[$ there exists some $\varepsilon > 0$ such that for **all** $N \in \mathbb{N}$ there exists some natural number $j = j(N)$ such that $j \geq N$ and $d(x_j, L) \geq \varepsilon\ \big]$. $\square$

It is easy to prove from the Remark 12.9 (Opposite of convergence) the following:

Proposition 12.12 (Opposite of convergence)**.**

$\big[$ *A sequence $(x_k)_k$ with values in (X,d) does not have $L \in X$ as its limit* $\big]\ \Leftrightarrow$
$\big[$ *there exists some $\varepsilon > 0$ and $n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$ such that $d(x_{n_j}, L) \geq \varepsilon$ for **all** j.* $\big]$
In other words, there is a subsequence $(x_{n_j})_j$ which completely stays out of some ε–neighborhood of L.

PROOF: The proof is left as exercise **??**. ■

12.5 Abstract Topological spaces

Theorem 12.2 on p.563 gives us a way of defining neighborhoods for sets which do not have a metric.

Definition 12.11 (Abstract topological spaces)**.**

Let X be an arbitrary nonempty set and let $\mathfrak{U}$ be a set of subsets of X whose members satisfy the properties a, b and c of (12.19) on p.563: [7]

(12.22a) An arbitrary union $\bigcup_{i\in I} U_i$ of sets $U_i \in \mathfrak{U}$ belongs to $\mathfrak{U}$,

(12.22b) $U_1, U_2, \ldots, U_n \in \mathfrak{U}\ (n \in \mathbb{N}) \quad\Rightarrow\quad U_1 \cap U_2 \cap \ldots \cap U_n \in \mathfrak{U}$,

(12.22c) $X \in \mathfrak{U}$ and $\emptyset \in \mathfrak{U}$.

Then $(X, \mathfrak{U})$ is called a **topological space** The members of $\mathfrak{U}$ are called **open sets** of $(X, \mathfrak{U})$. The collection $\mathfrak{U}$ of open sets is called the **topology** of X. $\square$

Remark 12.10.
Let (X, d) be a metric space and let

$$\mathfrak{U}_d := \{U \subseteq X : U \text{ is an open subsets of } (X, d)\}, \tag{12.23}$$

i.e., $U \in \mathfrak{U}_d \Leftrightarrow U$ consist of interior points only: for each $x \in U$ there exist $\varepsilon > 0$ such that

$$N_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\} \subseteq U$$

(see (12.7) on p.561). Then thm.12.2 on p.563 asserts the following.

Every metric space (X, d) is a topological space in the following sense: If $\mathfrak{U}_d$ denotes the open sets of (X, d) then $(X, \mathfrak{U}_d)$ is a topological space.

Remark 12.11.
Let V be a vector space with a norm $\|\cdot\|$. We recall that this norm defines a metric $d_{\|\cdot\|}(\cdot, \cdot)$ via $d_{\|\cdot\|}(x, y) = \|x - y\|$ (see thm.12.1 on p.555). According to part **A** the norm $\|\cdot\|$ defines open sets

$$\mathfrak{U}_{\|\cdot\|} := \mathfrak{U}_{d_{\|\cdot\|}} \tag{12.24}$$

in the metric space $(V, d_{\|\cdot\|})$.

Every normed vector space $(V, \|\cdot\|)$ is a topological space in the sense that If $\mathfrak{U}_d$ denotes the open subsets of a metric space (X, d) then $(V, \mathfrak{U}_d)$ is a topological space. □

We now discuss the terminology for topologies that are the open sets of metric spaces and, in particular, normed vector spaces.

Definition 12.12 (Metric Topology and Norm Topology)**.**

⋆

(a) Let (X, d) be a metric space and let $\mathfrak{U}_d$ be as defined in (12.23). We say that $\mathfrak{U}_d$ is **induced by the metric** $d(\cdot, \cdot)$ or that it is **generated by the metric** $d(\cdot, \cdot)$. or that it is the **metric topology** of X. If it is clear which metric d on X we mean then we also simply refer to "the" metric topology.

(b) Let $(V, \|\cdot\|)$ be a normed vector space, and let $\mathfrak{U}_{\|\cdot\|}$ be as defined in (12.24), i.e., $\mathfrak{U}_{\|\cdot\|}$ is the topology defined by the metric $d_{\|\cdot\|}$. We say that this topology is **induced by the norm** $\|\cdot\|$ or that it is **generated by the norm** $\|\cdot\|$. If it is clear which norm on V we are studying then we call the topology associated with this norm the **norm topology** of V. □

[7] Note that we encountered subsets of 2^X with special properties previously when looking at rings of sets in Definition ?? (Rings, algebras, and σ–algebras of Sets) on p.??.

We defined the discrete metric earlier in this chapter. (See Definition 12.3 on p.556.)

Definition 12.13 (Discrete topology)**.**

★ Let X be a nonempty set with the discrete metric

$$d(x,y) \;=\; \begin{cases} 0 & \text{for } x = y, \\ 1 & \text{for } x \neq y. \end{cases}$$

We call the topology associated with the discrete metric the **discrete topology** of X. □

Proposition 12.13.

Let (X,d) be a metric space with the discrete metric.
Then its associated topology is

$$\mathfrak{U}_d \;=\; 2^X \;=\; \{A : A \subseteq X\}.$$

PROOF: The proof is left as exercise 12.12. ■

Remark 12.12. It follows from prop.12.13 that the discrete metric defines the biggest possible topology on X, i.e., the biggest possible collection of subsets of X whose members satisfy properties a, b, c of definition 12.11 on p.567. □

We now discuss the example of a topology which is not generated by a metric.

Proposition 12.14.

Let X be an arbitrary nonempty set and let $\mathfrak{U} := \{\emptyset, X\}$.
Then $(X, \mathfrak{U})$ is a topological space.

PROOF:
This is trivial because any intersection of members of $\mathfrak{U}$ is either $\emptyset$ (if at least one member is $\emptyset$) or X (if all members are X). Moreover, any union of members of $\mathfrak{U}$ is either $\emptyset$ (if all members are $\emptyset$) or X (if at least one member is X). ■

Definition 12.14 (Indiscrete topology)**.**

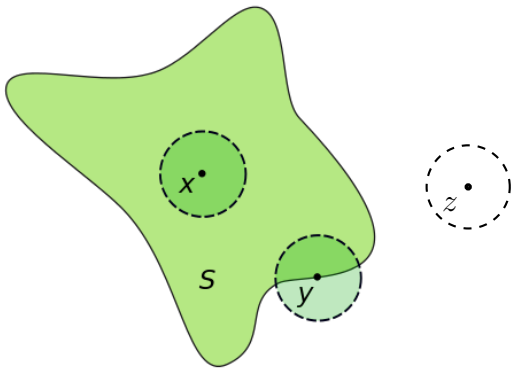
Let X be a nonempty set.
The topology $\{\emptyset, X\}$ is called the **indiscrete topology** of X. □

Remark 12.13.

(a) It follows from prop.12.14 that the indiscrete topology is the smallest possible topology on X. i.e., the smallest possible collection of subsets of X whose members satisfy properties a, b, c of definition 12.11 on p.567.

(b) Prop.12.6 (Metric Spaces are Hausdorff Spaces) on p.562 guarantees that any two different points x and y in a metric space can be separated by appropriately chosen disjoint neighborhoods. This is not true for the indiscrete topology since the only superset of a nonempty open set is X, so the only neighborhood for x is X, and the same is true for y. □

Remark 12.14.



The picture to the right [8] demonstrates that there are exactly three mutually exclusive choices how a point in $(X, \mathfrak{U})$ is related to a subset S of X:

(a) either like the point x: There exists an open set U such that $x \in U \subseteq S$,

(b) or like the point z: There exists an open set U such that $z \in U \subseteq S^{\complement}$,

(c) or like the point y: There is no open set U such that $y \in U \subseteq S$ or $y \in U \subseteq S^{\complement}$, i.e., every open set that contains y intersects both S and $S^{\complement}$.

We can classify any element $x \in X$ accordingly: x satisfies either **(a)** or **(b)** or **(c)**.

The above classification leads to the definitions of interior points, exterior points, and boundary point of $S \subseteq X$.

Definition 12.15 (Neighborhoods and interior points in topological spaces)**.**

Let $(X, \mathfrak{U})$ be a topological space, $x \in X$ and $S \subseteq X$. It is not assumed that S be open.

(a) S is called a **neighborhood** of x and x is called an **inner point** or **interior point** of S if there exists an open set U such that

$$x \in U \subseteq S.$$

We call the set $S^o := \{$ all interior points of S$\}$ the **interior** of S. An alternate but less commonly used notation for S^o is $\text{int}(S)$.

(b) x is called an **exterior point** of S if x is an inner point of $S^{\complement}$, i.e., there exists an open set U' such that

$$x \in U' \subseteq S^{\complement},$$

[8] Source: Wikipedia, https://en.wikipedia.org/wiki/Interior_(topology). The author does not like to use the letter S for subsets of topological spaces, but it came with the picture.

We call the set $\text{ext}(S) := \{$ all exterior points of S$\}$ the **open exterior** [9] of S.

(c) x is called a **boundary point** of S if any neighborhood of x intersects both S and $S^{\complement}$. We call this set the **boundary** of S and denote it ∂S. □

Remark 12.15.

Be sure you understand from the definitions of interior points and neighborhood above that the following is true:

If S is an arbitrary subset of X, U is an open subset of X, and $x \in X$, then

(a) x is an interior point of $S \Leftrightarrow S$ is a neighborhood of x.
(b) x is an interior point of $U \Leftrightarrow x \in U$.
(c) If $U \subseteq S$ then all elements of U are interior points of S, i.e., $U \subseteq S^o$.

To see that **(c)** is true observe that any $u \in U$ satisfies

$$u \in \text{ open set } U \subseteq S\,.$$

In other words, u, U, and S satisfy the relationship (**??**) of the definition of interior points. □

Remark 12.16. For metric spaces (X, d) we first defined interior points, and afterward we defined an open subset as one which consists entirely of interior points. [10] Since openness is the defining property of topological spaces and thus at the very beginning it should not come as a surprise that for such spaces we had to proceed in reverse and define interior points and neighborhoods in terms of open sets. □

Proposition 12.15.

Let $(X, \mathfrak{U})$ be a topological space and let $A \subseteq X$. Then

$$A^o = \bigcup \Big[U \in \mathfrak{U} : U \subseteq A\Big]\,. \tag{12.25}$$

In other words, the interior of A is the union of all open subsets of A.

PROOF: For convenience we abbreviate $B := \bigcup \Big[U \in \mathfrak{U} : U \subseteq A\Big]$.

We first prove that A^o is an open set, i.e., for each $x \in A^o$ there is an open set U_x such that

(a) $$x \in U_x \subseteq A^o\,.$$

[9] The expression "'open exterior" has been adopted from Wikipedia.
Source: https://en.wikipedia.org/wiki/Interior_(topology)
[10] See Definition 12.7 on p.561.

By definition of A^o, x is interior to A, thus there exists $U \in \mathfrak{U}$ such that $x \in U \subseteq A$. It follows from Remark 12.15**(c)** that $U \subseteq A_o$, thus $U_x := U$ satisfies **(a)**.

Next we show that $A^o \subseteq B$. Since A^o is open, $A^o \in \{U \in \mathfrak{U} : U \subseteq A\}$. But then,

$$A^o \subseteq \bigcup \Big[U \in \mathfrak{U} : U \subseteq A \Big], \quad \text{i.e.,} \quad A^o \subseteq B\,.$$

We finally prove that $B \subseteq A^o$. So let $x \in B$. We must show that $x \in A^o$.

By definition of B there exists $U \in \mathfrak{U}$ such that $U \subseteq A$ and $x \in U$. Again we conclude from Remark 12.15**(c)** that $U \subseteq A^o$, hence $x \in A^o$. ■

That last proposition shows that A^o is an open set which is, as a union of subsets of A, also a subset of A. Because A^o is the union of all such sets, we conclude that

> The interior A^o of A is the largest of all open subsets of A.

Proposition 12.16.

> *Let $(X, \mathfrak{U})$ be a topological space.*
>
> $$\textit{If } A \subseteq B \subseteq X \textit{ then } A^o \subseteq B^o. \ \square$$

PROOF: The proof is left as exercise 12.19. ■

The following proposition is worthwhile remembering: If we fix a subset A of a topological space X then each point $x \in X$ belongs either to the interior or the open exterior or the boundary of A.

Proposition 12.17.

> *Let $(X, \mathfrak{U})$ be a topological space and let $A \subseteq X$. Then,*
>
> (12.26) $$X \;=\; A^o \biguplus \mathit{ext}(A) \biguplus \partial(A)\,.$$
>
> *Thus, X is partitioned into the interior, open exterior and boundary of anyone of its subsets.*

PROOF: Obvious from the fact that any $x \in X$ falls into exactly one of the following categories:

- **(a)** either there exists an open set U such that $x \in U \subseteq S$, i.e., $x \in A^o$,
- **(b)** or there exists an open set U such that $z \in U \subseteq S^{\complement}$, i.e., $x \in \ \mathrm{ext}(A)$,
- **(c)** or there is no open set U such that $x \in U \subseteq S$ or $x \in U \subseteq S^{\complement}$, i.e., every open set that contains x intersects both S and $S^{\complement}$, i.e., i.e., $x \in \partial(A)$.

See rem.12.14 on p.570. ■

We conclude this chapter with a summary of what we have learned about the classification of sets with a concept of closeness of points.

Theorem 12.4 (Hierarchy of topological spaces)**.**

We have seen the following:

- **(a)** $\mathbb{R}^n$*, in particular* $\mathbb{R} = \mathbb{R}^1$*, is an inner product space (see prop.?? on p.??).*
- **(b)** *All inner product spaces are normed spaces (see thm.?? on p.??).*
- **(c)** *All normed spaces are metric spaces (see thm.12.1 on p.555).*
- **(d)** *All metric spaces are topological spaces (see Definition 12.11 on p.567, Definition 12.12 on p.568).*

12.6 Bases and Neighborhood Bases ⋆

This chapter has been marked as optional but we suggest that you skim its contents since some of the concepts taught here will be referred to in subsequent chapters.

Definition 12.16 (Base of the topology)**.**

Let $(X, \mathfrak{U})$ be a topological space. A subset $\mathfrak{B}$ of $\mathfrak{U}$ of open sets is called a **base of the topology** if any nonempty open set U can be written as a union of elements of $\mathfrak{B}$:

$$U = \bigcup_{i \in I} B_i \quad (B_i \in \mathfrak{B} \text{ for all } i \in I) \tag{12.27}$$

where I is a suitable index set, which of course will in general depend on U. □

We note that, because X itself is open, (12.27) implies that $X = \bigcup \big[B : B \in \mathfrak{B}\big]$.

A base of the topology is a subset of that topology, i.e., a collection of open sets, which contains enough small open sets. We can localize that definition to a point x of X by looking at collections of open neighborhoods of x which contain enough small open neighborhoods of x and we arrive at the definition of neighborhood bases of x.

Definition 12.17 (Neighborhood base of a point)**.**

Let $(X, \mathfrak{U})$ be a topological space.

- **(a)** The following set of subsets of X,

  $$\mathfrak{N}(x) := \{A \subseteq X : A \text{ is a neighborhood of } x\}, \tag{12.28}$$

  is called the **neighborhood system of** x

- **(b)** Given a point $x \in X$, any subset $\mathfrak{B} := \mathfrak{B}(x) \subseteq \mathfrak{N}(x)$ of the neighborhood system of x is called a **neighborhood base of** x if it satisfies the following condition:
  - For any $A \in \mathfrak{N}(x)$, there exists a set $B \in \mathfrak{B}(x)$ such that $B \subseteq A$. □

In many propositions where proving closeness to some element is the issue, It often suffices to show that something is true for all sets that belong to a neighborhood base of x rather than having to show it for all neighborhoods of x. The reason is that often only the small neighborhoods matter and a neighborhood base has "enough" of those.

Definition 12.18 (First axiom of countability)**.**

Let $(X, \mathfrak{U})$ be a topological space.
We say that X satisfies the **first axiom of countability** or X is **first countable** if we can find for each $x \in X$ a countable neighborhood base. □

Here are some propositions about bases, neighborhood bases, and first countability for metric spaces.

Proposition 12.18 (ε-neighborhoods are a base of the topology)**.**

Let (X, d) be a metric space. Then both

$$\mathscr{B}_1 := \{N_\varepsilon(x) : x \in X, \varepsilon > 0\} \quad \text{and} \quad \mathscr{B}_2 := \{N_{1/n}(x) : x \in X, n \in \mathbb{N}\}$$

are bases for the topology of (X, d) (see 12.16 on p.573)

PROOF: To show that $\mathscr{B}_1$ (resp., $\mathscr{B}_2$) is a base we must prove that any open subset of X can be written as a union of (open) sets all of which belong to $\mathscr{B}_1$ (resp., $\mathscr{B}_2$). We prove this for $\mathscr{B}_2$.

Let $U \subseteq X$ be open. As any $x \in U$ is an interior point of U we can find some $\varepsilon = \varepsilon(x) > 0$ such that $N_{\varepsilon(x)}(x) \subseteq U$. We note that for any such $\varepsilon(x)$ there is $n(x) \in \mathbb{N}$ such that $1/n(x) \leq \varepsilon(x)$.

We observe that $U \subseteq \bigcup \left[N_{1/n(x)}(x) : x \in U\right] \subseteq U$.

The first inclusion follows from $\{x\} \subseteq N_{1/n(x)}(x)$ for all $x \in U$ and the second inclusion follows from $N_{1/n(x)}(x) \subseteq N_{\varepsilon(x)}(x) \subseteq U$ and the inclusion lemma (lemma **??** on p.**??**).

We obtain $U = \bigcup \left[N_{1/n(x)}(x) : x \in U\right]$ and we have managed to represent our open U as a union of elements of $\mathscr{B}_2$. This proves that $\mathscr{B}_2$ is a base for the topology of (X, d).

As $\mathscr{B}_2 \subseteq \mathscr{B}_1$ it follows that $\mathscr{B}_1$ also is such a base. ■

Theorem 12.5 (Metric spaces are first countable)**.**

Let (X, d) be a metric space. Then X is first countable.

Proof (outline): For any $x \in X$ let

$$\mathfrak{B}(x) := \{ N_{1/n}(x) : n \in \mathbb{N} \}. \tag{12.29}$$

Then $\mathfrak{B}(x)$ is a neighborhood base of x because, by Definition 12.7 and Definition 12.9 (Interior points and neighborhoods in metric spaces), any neighborhood of x will contain one of the form $N_\varepsilon(x)$ and for any such $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$. ■

Proposition 12.19.

Let (X, d) be a metric space and let $\mathfrak{B} := \{N_{1/k}(x) : x \in X, k \in \mathbb{N}\}$. Then $\mathfrak{B}$ is a base of the topology for the associated topological space $(X, \mathfrak{U}_d)$.

PROOF: The proof is left as exercise 12.15 on p.599. ■

Definition 12.19 (Second axiom of countability)**.**

Let $(X, \mathfrak{U})$ be a topological space.
We say that X satisfies the **second axiom of countability** or X is **second countable** if we can find a countable base for $\mathfrak{U}$. □

The next theorem is related to the material in chapter **??** (Sequences that Enumerate Parts of $\mathbb{Q}$).

Theorem 12.6 (Euclidean space $\mathbb{R}^n$ is second countable)**.**

Let $\mathfrak{B}$ be the following collection of open subsets of $\mathbb{R}^n$:

$$\mathfrak{B} := \{ N_{1/j}(\vec{q}) : \vec{q} \in \mathbb{Q}^n,\ j \in \mathbb{N} \}. \tag{12.30}$$

Here,

$$\mathbb{Q}^n = \{\vec{q} = (q_1, \ldots, q_n) : q_j \in \mathbb{Q},\ 1 \le j \le n\}$$

is the set of all points in $\mathbb{R}^n$ with rational coordinates. Then $\mathfrak{B}$ is a countable base of $\mathbb{R}^n$.

PROOF (outline): Let $U \in \mathfrak{U}$ be an arbitrary open set in X. Any vector $\vec{x} \in U$ is interior point of U, hence we can find some $n_{\vec{x}} \in \mathbb{N}$ such that the entire $\frac{2}{n_{\vec{x}}}$–neighborhood $N_{2/(n_{\vec{x}})}(\vec{x})$ is contained within U.

As any vector can be approximated by vectors with rational coordinates, there exists $\vec{q} = \vec{q}_{\vec{x}} \in \mathbb{Q}^n$ such that $d(\vec{x}, \vec{q}_{\vec{x}}) < \frac{1}{n_{\vec{x}}}$, hence $\vec{x} \in N_{1/n_{\vec{x}}}(\vec{q}_{\vec{x}})$.

It follows from $N_{2/(n_{\vec{x}})}(\vec{x}) \subseteq U$ and prop.12.3 on p.561, applied to $\delta = \varepsilon = \frac{1}{n_{\vec{x}}}$, that

$$N_{1/n_{\vec{x}}}(\vec{q}_{\vec{x}}) \subseteq N_{2/n_{\vec{x}}}(\vec{x}) \subseteq U \text{ for all } \vec{x} \in U.$$
$$\text{Hence } U = \bigcup_{\vec{x} \in U} \{\vec{x}\} \subseteq \bigcup \left[N_{1/n_{\vec{x}}}(\vec{q}_{\vec{x}}) : \vec{x} \in U \right] \subseteq U.$$

We have managed to write the arbitrarily chosen open set U as a union of the sets $N_{1/n_{\vec{x}}}(\vec{q}_{\vec{x}})$ which belong to $\mathfrak{B}$. This proves that $\mathfrak{B}$ is a basis of the topology.

We recall from cor.?? on p. ?? that $\mathbb{Q}^n$is countable. For $j \in \mathbb{N}$ let $\mathfrak{B}_j := \{ N_{1/j}(\vec{q}) : \vec{q} \in \mathbb{Q}^n\}$. Then each $\mathfrak{B}_j$ is countable because $\vec{q} \mapsto N_{1/j}(\vec{q})$ is a surjection from the countable set $\mathbb{Q}^n$ onto $\mathfrak{B}_j$. It follows that the base of the topology $\mathfrak{B} = \bigcup_j \mathfrak{B}_{j \in \mathbb{N}}$ is countable as the countable union of countable sets. ■

12.7 Metric and Topological Subspaces

It is often advantageous to focus our attention on a subset A of a metric space (X, d) or a topological space $(X, \mathfrak{U})$. It would be nice if one could find a way to define a metric d' (a topology $\mathfrak{U}'$) on A which coexists harmoniously with the metric d (the topology $\mathfrak{U}$) defined on X.

For example let X be the real numbers with the standard metric $d(x, y) = |b-a|$ and $A = [0, 1]$. This allows us, e.g., to talk of the assignment $x \mapsto \sqrt{x}$ which cannot be extended beyond A as a function $f : (A, d') \to (\mathbb{R}, d)$ for which both domain and codomain are metric spaces.

The solution to this problem is different for metric spaces and topological spaces, but both amount to the following:

A set U will be open in A if and only if $U = V \cap A$ for some suitable set V which is open in X.

Definition 12.20 (Metric subspaces)**.**

Given is a metric space (X, d) and a nonempty $A \subseteq (X, d)$. Let

$$d\big|_{A\times A} : A \times A \to \mathbb{R}_{\geq 0} \quad \text{be the restriction} \quad d\big|_{A\times A}(x, y) \;:=\; d(x, y)(x, y \in A)$$

of the metric d to $A \times A$ (see Definition **??** on p.**??**).

It is trivial to verify that $(A, d\big|_{A\times A})$ is a metric space in the sense of Definition 12.1 on p.553. We call $(A, d\big|_{A\times A})$ a **metric subspace** of (X, d) and we call $d\big|_{A\times A}$ the **metric induced by** d or the **metric inherited from** (X, d). □

Remark 12.17.

Metric subspaces come with their own collections of open and closed sets, neighborhoods, ε-neighborhoods, convergent sequences, ...
Watch out when looking at statements and their proofs whether those concepts refer to the entire space (X, d) or to the subspace $(A, d\big|_{AxA})$. □

Notation 12.1.

a) Because the only difference between d and $d_{A\times A}$ is the domain, it is customary to write d instead of $d\big|_{A\times A}$ to make formulas look simpler, if doing so does not give rise to confusion.

b) We often shorten "open in $(A, d\big|_{A\times A})$" to "open in A", "closed in $(A, d\big|_{A\times A})$" to "closed in A", "convergent in $(A, d\big|_{A\times A})$" to "convergent in A", ... □

Example 12.5. Consider $A :=]0, 1] \cup]2, 3[\cup \{4\} \cup \{5\} \cup \{6\}$ as a metric subspace of $(\mathbb{R}, d_{|.|})$, i.e., the real numbers with the Euclidean metric. Then $\{4, 5\}$ is OPEN in A, and 1 is an interior point of $]\frac{1}{2}, 1]$, when we consider this interval as a subset of $(A, d_{|.|})$. Those assertions are true because

- $N^A_{1/2}(4) = \{4\}$ and $N^A_{1/2}(5) = \{5\}$, are δ–nhoods in $(A, d_{|.|})$ and thus open in $(A, d_{|.|})$
- $1 \in N^A_{1/2}(1) \subseteq]\frac{1}{2}, 1]$, and $N^A_{1/2}(1)$ is a δ–nhood in $(A, d_{|.|})$. Thus, 1 is interior to $]\frac{1}{2}, 1]$. □

Definition 12.21 (Traces of sets in a metric subspace)**.**

★

Let (X, d) be a metric space and $A \subseteq X$ a nonempty subset of X, viewed as a metric subspace $(A, d\big|_{A\times A})$ of (X, d). Let $Q \subseteq X$.

We call $Q \cap A$ the **trace** of Q in A.

For $\varepsilon > 0$ and $a \in A$ let $N_\varepsilon(a)$ be the ε-neighborhood of a (in (X, d)). We define

$$N^A_\varepsilon(a) \;=\; N_\varepsilon(a) \cap A. \tag{12.31}$$

i.e., $N^A_\varepsilon(a)$ is defined as the trace of $N_\varepsilon(a)$ in A. □

Proposition 12.20 (Open sets in metric subspaces are traces of open sets in X)**.**

Let (X, d) be a metric space and $A \subseteq X$ a nonempty subset of X.

(a) *Let $\varepsilon > 0$ and $a \in A$. Then*

$$N^A_\varepsilon(a) \;=\; N_\varepsilon(a) \cap A \;=\; \{x \in A : d\big|_{A\times A}(x, a) < \varepsilon\}, \tag{12.32}$$

i.e., $N^A_\varepsilon(a)$ is the "ordinary" ε–Neighborhood of a in the metric space $(A, d\big|_{A\times A})$ (as it was originally defined in Definition 12.6 on p.560). It thus follows from (12.31) that each ε-neighborhood in the subspace A is the trace of an ε-neighborhood in X.

(b) *Generalization: $U \subseteq A$ is open in $(A, d\big|_{A\times A})$ $\Leftrightarrow$ there is an open $V \subseteq (X, d)$ such that*

$$U \;=\; V \cap A. \tag{12.33}$$

In other words, U is the trace of a set V which is open in X.

PROOF of **(a)**: First we prove (12.32). As $d\big|_{A\times A}$ is the restriction of d to $A \times A$ it follows that

$$\begin{aligned} N^A_\varepsilon(a) \;&= N_\varepsilon(a) \cap A \;=\; \{x \in X : d(x, a) < \varepsilon\} \cap A \\ &= \{x \in A : d(x, a) < \varepsilon\} \;=\; \{x \in A : d\big|_{A\times A}(x, a) < \varepsilon\}. \end{aligned}$$

This finishes the proof of **(a)**

PROOF of **(b)**: First we show that if V is open in X then $U := V \cap A$ is open in the subspace A. Let $a \in U$. We must prove that a is an interior point of U with respect to $(A, d\big|_{A\times A})$. Because $a \in V$ and V is open in X, there is $\varepsilon > 0$ such that $N_\varepsilon(a) \subseteq V$. It follows that $N^A_\varepsilon(a) = N_\varepsilon(a) \cap A \subseteq V \cap A = U$. As $N^A_\varepsilon(a)$ is open in A, a is an interior point of U with respect to the subspace $(A, d\big|_{A\times A})$.

Finally we prove that if $U \subseteq A$ is open in A then there is $V \subseteq X$ open in X such that $U = V \cap A$: We can write $U = \bigcup \left[N^A_{\varepsilon(a)}(a) : a \in U\right]$ for suitable $\varepsilon(a) > 0$ (see the proof of prop.12.18 on p.574). Let $V := \bigcup \left[N_{\varepsilon(a)}(a) : a \in U\right]$. V is open in (X, d) as union of the open sets $N_{\varepsilon(a)}(a)$. Further,

$$\begin{aligned} V \cap A &= A \cap \bigcup \left[N_{\varepsilon(x)}(x) : x \in U\right] = \bigcup \left[N_{\varepsilon(x)}(x) \cap A : x \in U\right] \\ &= \bigcup \left[N^A_{\varepsilon(x)}(x) : x \in U\right] = U \end{aligned}$$

(the second equalitity follows from prop.?? on p.??). This finishes the proof. ■

Remark 12.18 (Convergence does not necessarily extend to metric subspaces)**.**

Let (X, d) be a metric space, $A \subseteq (X, d)$ and $a_n \in A$ for all $n \in \mathbb{N}$. Be aware that convergence of the sequence (a_n) in the space (X, d) (i.e., there exists $x \in X$ such that $x = \lim\limits_{n\to\infty} a_n$) does **NOT** imply convergence of the sequence in the subspace $(A, d\big|_{A\times A})$! Rather, we have the following dichotomy:

(a) $x \in A$: Then a_n converges to x in the subspace $(A, d\big|_{A\times A})$ (and also in (X, d)).

(b) $x \in A^{\complement}$: Then a_n converges to x in (X, d) but not in $(A, d\big|_{A\times A})$. □

Proposition 12.20 (Open sets in metric subspaces are traces of open sets in X) justifies to define subspaces of abstract topological spaces as follows.

Definition 12.22 (Topological subspaces)**.**

⋆

Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. We say that $V \subseteq A$ is **open in A** if V is the trace of an open set in X, i.e., if there is some $U \in \mathfrak{U}$ such that $V = U \cap A$. We denote the collection of all open sets in A as $\mathfrak{U}_A$. In other words,

$$\mathfrak{U}_A = \{V \cap A : V \in \mathfrak{U}\}.$$

We call $(A, \mathfrak{U}_A)$ a **topological subspace** or also just a **subspace** of $(X, \mathfrak{U})$ and we call $\mathfrak{U}_A$ the **subspace topology induced by** $(X, \mathfrak{U})$ or the **subspace topology inherited from** $(X, \mathfrak{U})$. □

Proposition 12.21 (Topological subspaces are topological spaces)**.**

Let $(X, \mathfrak{U})$ be a topological space, $A \subseteq X$, and let $\mathfrak{U}_A$ be the collection of all open sets in A. Then $(A, \mathfrak{U}_A)$ is a topological space, i.e., it satisfies Definition 12.11 on p.567 of an abstract topological space.

PROOF:

(a) Let $(U_i)_{i\in I}$ be a family of open sets in A. For each U_i there exists V_i open in X such that $U_i = V_i \cap A$. According to prop.**??** (Distributivity of unions and intersections) on p.**??** we obtain

$$A \cap \bigcup_{i\in I} V_i \;=\; \bigcup_{i\in I} (A \cap V_i) \;=\; \bigcup_{i\in I} U_i$$

and this proves that $\bigcup_{i\in I} U_i$ is the trace of the open set $\bigcup_{i\in I} V_i$ in A, hence open in A.

(b) Let $U_1, U_2, \ldots U_n$ $(n \in \mathbb{N})$ be open in A. For each U_i there exists V_i open in X such that $U_i = V_i \cap A$. Because the intersection of sets is commutative we obtain

$$U_1 \cap \ldots U_n \;=\; (V_1 \cap A) \cap \ldots (V_n \cap A) \;=\; (A \cap \cdots \cap A) \cap (V_1 \cap \cdots \cap V_n) \;=\; A \cap (V_1 \cap \cdots \cap V_n)$$

and this proves that $U_1 \cap \ldots U_n$ is the trace of the open set $V_1 \cap \cdots \cap V_n$ in A, hence open in A.

(c) It follows from $\emptyset = \emptyset \cap A$, $A = X \cap A$, and $X, \emptyset \in \mathfrak{U}$, that $\emptyset, A \in \mathfrak{U}_A$. ■

12.8 Contact Points and Closed Sets

If you look at any **closed interval** $[a, b] = \{y \in \mathbb{R} : a \le y \le b\}$ of real numbers, then all of its points are interior points, except for the end points a and b. Moreover a and b are contact points according to the following definition which makes sense for any abstract topological space.

Definition 12.23 (Contact points)**.**

Given is a topological space $(X, \mathfrak{U})$.
Let $A \subseteq X$ and $x \in X$ (x may or may not to belong to A). x is called a **contact point**, of A if

(12.34) $\qquad A \cap N \neq \emptyset$ for any neighborhood N of x. □

Note 12.1.

Note that any $a \in A$ is a contact point of A but not necessarily the other way around:

(a) Let $a \in A$. Then any neighborhood U_a of a contains a, hence $U_A \cap A$ is not empty, hence a is a contact point of A. This proves that any $a \in A$ is a contact point of A.

(b) Here is a counterexample which shows that the converse need not be true.
Let $(X, d) := \mathbb{R}$ with the standard Euclidean metric and let A be the subset $]0, 1[$. We show now that 0 is a contact point of A.
Any neighborhood A_0 of 0 contains for some small enough $\delta > 0$ the entire interval $]-\delta, \delta[$.
Let $x := \min(\delta/2, 1/2)$.
Clearly, $x \in]-\delta, \delta[\subseteq A_0$ and $x \in]0, 1[= A$.
It follows that $x \in A \cap A_0$. As A_0 was an arbitrary neighborhood of 0, we have proved that 0 is a contact point of A, even though $0 \notin A$.

(c) The above counterexample can be proven much faster if the criterion for contact points in metric spaces is employed: Let $x_n := 1/n$ $(n \ge 2)$ Then $x_n \in]0, 1[$ for all n and the sequence converges to 0. It follows that 0 is a contact point of $]0, 1[$. □

Definition 12.24 (Closed sets)**.**

Let $(X, \mathfrak{U})$ be topological space and $A \subseteq X$. Let the set $\bar{A}$ be

$$\bar{A} := \{x \in X : x \text{ is a contact point of } A\}. \tag{12.35}$$

We call $\bar{A}$ the **closure** of A. A set that contains all its contact points is called a **closed set**. □

Proposition 12.22.

If A is a subset of a topological space then

$$\bar{A} = A \cup \partial(A) = A^o \cup \partial(A). \tag{12.36}$$

PROOF of $\bar{A} = A^o \cup \partial(A)$:
We recall from prop.12.16 on p.572 that any $x \in X$ either belongs to the interior A^o or to the open exterior $\text{ext}(A)$ or to the boundary $\partial(A)$. Since it is precisely the set $\text{ext}(A)$ for whose elements one can find neighborhoods of x which do not intersect with A, we obtain

$$x \in \bar{A} \Leftrightarrow x \notin \text{ext}(A) \Leftrightarrow x \in A^o \cup \partial(A).$$

This proves the assertion.
PROOF of $\bar{A} = A \cup \partial(A)$:
It follows from the definitions of A^o and $\bar{A}$ that $A^o \subseteq A \subseteq \bar{A}$, thus

$$A^o \cup \partial(A) \subseteq A \cup \partial(A) \subseteq \bar{A} \cup \partial(A). \tag{12.37}$$

Since $\bar{A} = A^o \cup \partial(A)$ formula (12.37) yields

$$\bar{A} = A^o \cup \partial(A) \subseteq A \cup \partial(A) \subseteq \bar{A} \cup \partial(A) = \bar{A}. \tag{12.38}$$

We obtained the last equation from $\bar{A} = A^o \cup \partial(A)$ since this equation implies

$$\partial(A) \subseteq \bar{A}, \qquad \text{thus} \qquad \partial(A) \cup \bar{A} \subseteq \bar{A} \cup \bar{A} = \bar{A}.$$

Formula (12.38) shows that the set $A \cup \partial(A)$ is both subset and superset of $\bar{A}$. It follows that $A \cup \partial(A) = \bar{A}$. ■

Remark 12.19.
It follows from note 12.1**(a)** that $A \subseteq \bar{A}$. ; □

The following theorem shows that we can characterize contact points of subsets of metric spaces by means of sequences.

Theorem 12.7 (Sequence criterion for contact points in metric spaces)**.**

Given is a metric space (X,d). Let $A \subseteq X$ and $x \in X$. Then x is a contact point of A if and only if there exists a sequence $x_1, x_2, x_3, \ldots$ of members of A which converges to x.

PROOF of "⇒": Let $x \in X$ be such that $N \cap A \neq \emptyset$ for any neighborhood N of x. Let $x_n \in N_{1/n}(x) \cap A$. Such x_n exists because the neighborhood $N_{1/n}(x)$ has nonempty intersection with A.

Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be chosen such that $\frac{1}{N} < \varepsilon$. This is possible because $\mathbb{N}$ is not bounded (above) in $\mathbb{R}$.

For any $j \geq N$ we obtain $d(x_j, x) < 1/j \leq 1/N < \varepsilon$. This proves convergence $x_n \to x$.

PROOF of "⇐" Let $x \in X$ and assume there is $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in A$ for all n and $x_n \to x$.

We must show that if U_x is a (open) neighborhood of x then $U_x \cap A \neq \emptyset$. Let $\varepsilon > 0$ such that $N_\varepsilon(x) \subseteq U_x$.

It follows from $x_n \to x$ that there is $N = N(\varepsilon)$ such that $x_n \in N_\varepsilon(x)$ for all $n \geq N$, especially, $x_N \in N_\varepsilon(x)$. By assumption, $x_N \in A$, hence $x_N \in N_\varepsilon(x) \cap A \subseteq U_x \cap A$, hence $U_x \cap A \neq \emptyset$. ■

Remark 12.20.

Many books define closed sets as the complements of open sets and only afterwards define contact points as we did. No surprise then that our definition of closed sets becomes their theorem: It is then proven from those definitions that closed sets are exactly those that contain all their contact points. □

Theorem 12.8 (Open iff complement is closed and vice versa)**.**

Let (X,d) be a metric space and $A \subseteq X$. Then,

(a) *A is open if and only if $A^{\complement}$ is closed.*

(b) *A is closed if and only if $A^{\complement}$ is open.*

PROOF: For the following, let $A, B, F, U \subseteq X$. Note that, since $\overline{A} = A^o \uplus \partial A$ and $X = A^o \uplus \big(A^{\complement}\big)^o \uplus \partial A$,

(⋆) $X = \overline{A} \uplus \big(A^{\complement}\big)^o$, for all $A \subseteq X$

(A) We prove that, if F is closed, then $F^{\complement}$ is open.

It follows from $F = \overline{F}$ and **(⋆)** that $X = F \uplus \big(F^{\complement}\big)^o$ Since also $X = F \uplus F^{\complement}$, we obtain that $F^{\complement} = \big(F^{\complement}\big)^o$. Since A^o is open for all $A \subseteq X$, $F^{\complement}$ is open.

(B) We prove that, if U is open, then $U^{\complement}$ is closed.

Let $F := U^{\complement}$. This is equivalent to $U = F^{\complement}$. Also, $U^o = U$, because U is open. Hence,

(⋆⋆) $X \overset{(\star)}{=} \overline{F} \uplus \big(F^{\complement}\big)^o = \overline{F} \uplus U^o = \overline{F} \uplus U = \overline{F} \uplus F^{\complement}$

It follows from **(⋆⋆)** that $X = \overline{F} \uplus F^{\complement}$. On the other hand, $X = F \uplus F^{\complement}$.

A comparison of those two equations shows that $F = \overline{F}$. this proves that F is closed, i.e., $U^{\complement}$ is closed.

(C) We prove that, if $U^{\complement}$ is closed, then U is open.

By **(A)**, $U^{\complement}$ being closed implies that $\left(U^{\complement}\right)^{\complement}$ is open, i.e., U is open.

(D) We prove that, if $F^{\complement}$ is open, then F is closed.

By **(B)**, $F^{\complement}$ being open implies that $\left(F^{\complement}\right)^{\complement}$ is closed, i.e., F is closed. ■

Here is an easy consequence of the fact that open sets are the complements of closed sets and vice versa.

Proposition 12.23.

Let $(X, \mathfrak{U})$ be a topological space. The closed sets of X satisfy the following:

(12.39)
- ***(a)*** *An arbitrary intersection of closed sets is closed.*
- ***(b)*** *A finite union of closed sets is closed.*
- ***(c)*** *The entire set X is closed and $\emptyset$ is closed.*

The proofs of **(a)** and **(b)** follow easily from De Morgan's law (the duality principle for sets: see (**??**) on p.**??**). Observe that X plays the role of a universal set because all members U of $\mathfrak{U}$ and their complements $U^{\complement}$ are subsets of X.

PROOF of **(a)**: Let (C_α) be an arbitrary familiy of closed sets. Then $U_\alpha := C_\alpha^{\complement}$ is an open set for each α. Observe that $C_\alpha^{\complement} = U_\alpha$ because the complement of the complement of any set gives you back that set. Let $C := \bigcap_\alpha C_\alpha$. Then

$$C^{\complement} \;=\; \left(\bigcap_\alpha C_\alpha\right)^{\complement} = \bigcup_\alpha C_\alpha^{\complement} = \bigcup_\alpha U_\alpha.$$

In other words $C^{\complement}$ is an arbitrary union of open sets which is open by the very definition of open sets of a topological space. We have proved **(a)**.

PROOF of **(b)**: Let $C_1, C_2, \ldots C_n$ be closed sets. Then $U_j := C_j^{\complement}$ is an open set for each j. Let $C := \bigcup_{1\le j\le n} C_j$. Then

$$C^{\complement} \;=\; \left(\bigcup_j C_j\right)^{\complement} = \bigcap_j C_j^{\complement}. = \bigcap_j U_j$$

Hence, $C^{\complement}$ is the intersection of finitely many open sets. This shows that $C^{\complement}$ is open, i.e., C is closed. We have proved **(b)**.

PROOF of **(c)**: Trivial because

$$X^{\complement} = \emptyset \qquad \text{and} \qquad \emptyset^{\complement} = X. \;\blacksquare$$

Next, derive some immediate properties of closures.

Proposition 12.24.

Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq B \subseteq X$. Then $\bar{A} \subseteq \bar{B}$.

PROOF: The proof is left as exercise 12.18 on p.599. ■

Proposition 12.25.

Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. Then,

$$\partial A \;=\; \bar{A} \cap \overline{A^{\complement}}. \tag{12.40}$$

In other words, $x \in X$ is a boundary point of A if and only if x is a contact point of both A and $A^{\complement}$.

PROOF: Left as exercise 12.17 on p.599. ■

Proposition 12.26 (Minimality of the closure of a set)**.**

Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. Then

$$\bar{A} \;=\; \bigcap \Big[C \supseteq A : C \text{ is closed} \Big]. \tag{12.41}$$

In other words, the closure $\bar{A}$ of A is the smallest of all closed supersets of A.

PROOF: Let $\mathfrak{C} := \{C \supset A : C \text{ is closed }\}$ and let $F := \bigcap \mathfrak{C}$. We need to show that $\bar{A} = F$.
It follows from prop.12.23**(a)** that F is closed, hence $F = \bar{F}$. It follows from $C \supseteq A$ for all $C \in \mathfrak{C}$ that $F \supseteq A$, hence $F = \bar{F} \supseteq \bar{A}$.
It remains to be shown that $F \subseteq \bar{A}$. It is true that $\bar{A} \in \mathfrak{C}$ because $\bar{A}$ is a closed set which contains A, hence $\bar{A} \supseteq \bigcap \mathfrak{C} = F$. (See prop.12.24 on p.583). ■

Proposition 12.27 (Closure of a set as a hull operator)**.**

Let $(X, \mathfrak{U})$ be a topological space. Consider the closure of sets as a function

$$^{-} : 2^X \longrightarrow 2^X; \qquad A \mapsto \bar{A}.$$

Then this function has the following properties for all $A, B \subseteq X$:

(a) $\bar{\emptyset} = \emptyset$, ***(b)*** $A \subseteq \bar{A}$, ***(c)*** $\bar{\bar{A}} = \bar{A}$, ***(d)*** $\overline{A \cup B} = \bar{A} \cup \bar{B}$.

PROOF: **(a)** follows from (12.39)**(c)** and **(b)** follows from remark 12.19.
The proof of **(c)** and **(d)** is left as exercise 12.20 on p.599.

The last proposition states that the closure is a so–called **closure operator** which is defined to be a function

$$\mathcal{cl} : 2^X \to 2^X; \qquad A \mapsto \quad \mathcal{cl}(A) := \bar{A}$$

on some abstract, nonempty set X (which need not be a topological space) such that the following are satisfied:

(a) $\mathcal{cl}(\emptyset) = \emptyset$, **(b)** $A \subseteq \quad \mathcal{cl}(A)$, **(c)** $\mathcal{cl}(\mathcal{cl}(A)) = \mathcal{cl}(A)$, **(d)** $\mathcal{cl}(A \cup B) = \mathcal{cl}(A) \cup \quad \mathcal{cl}(B)$.

It can be shown that if we define

$$\mathfrak{U} \;:=\; \{A^{\complement} : \mathcal{cl}(A) = A\}$$

then $(X, \mathfrak{U})$ satisfies the properties of a topological space.

Besides contact points there also is the concept of a limit point. We will not work with limit points in this document and only give its definition to make the reader aware that those two concepts are different and s/he must be mindful of this fact because many other writers work exclusively with limit points and often do not define contact points.
Here is the definition (see [1] Munkres, a standard book on topology):

Definition 12.25 (Contact points vs Limit points)**.**

★

Given is a topological space $(X, \mathfrak{U})$. Let $A \subseteq X$ and $x_0 \in X$. x_0 is called a **limit point** or **cluster point** or **point of accumulation** of A if every neighborhood U of x_0 intersects A in at least one point other than x_0, i.e.,

$$U \cap (A \setminus x_0) \neq \emptyset. \; \square$$

Remark 12.21. Not every element of a set $A \subseteq (X, \mathfrak{U})$ is necessarily a limit point of A. An example for this are the so called isolated points. [11] $\square$

We close this section with alternate proofs of Theorem 12.9 (Open iff complement is closed) on p.585. They are not as elegant and brief, but they are worthwhile studying to build more geometric intuition about the interplay of neighborhoods and contact points.

Proposition 12.28. *The complement of an open set is closed.*

PROOF of 12.28: Let A be an open set in a topological space $(X, \mathfrak{U})$. and assume $x \in X$ is a contact point of $A^{\complement}$. We want to prove that $A^{\complement}$ is a closed set, so we must show that $x \in A^{\complement}$.
We assume to the contrary that x is a contact point of $A^{\complement}$ such that $x \notin A^{\complement}$. Then $x \in A$.

[11] $a \in A$ is called an **isolated point** of A if there is a neighborhood U of a such that $U \cap A = \{a\}$, i.e., the "**punctured neighborhood**" $U \setminus \{a\}$ of a has empty intersection with A. Here is an example: Let $X := \mathbb{R}$ with the topology induced by the Euclidean metric $d(x, x') := |x - x'|$. Then the subset $A := [0, 1] \cup \{3\} \cup \{4\}$ possesses 3 and 4 as isolated points. You should convince yourself that those two elements of A are NOT limit points of A.

A is open, so x is an interior point of A. Hence there is a neighborhood N_x that contains x and is entirely contained in A, hence $N_x \cap A^{\complement} = \emptyset$.

We also assumed that x is a contact point of $A^{\complement}$. This implies that $N_x \cap A^{\complement} \neq \emptyset$. We have reached a contradiction. ■

Proposition 12.29. *The complement of a closed set is open.*

PROOF: We will give two proofs of the above.

(a) First proof of prop.12.29, valid for all topological spaces:

Let A be closed set and $b \in A^{\complement}$.

The closed set A contains all its contact points, so $b \notin A$ implies that b is not a contact point of A. According to Definition 12.23 there exists some neighborhood V of b such that $V \cap A = \emptyset$, i.e., $V \subseteq A^{\complement}$.

We have shown that an arbitrary $b \in A^{\complement}$ is an interior point of $A^{\complement}$, i.e., the complement of the closed set A is open. This proves the proposition.

(b) Alternate proof of prop.12.29, valid for metric spaces only, since it works with sequences. We give it to illustrate the use of Theorem 12.7, the sequence criterion for contact points.

Let $A \subset (X, d)$ be closed. If $A^{\complement}$ is not open then there must some be $b \in A^{\complement}$ which is not an interior point of $A^{\complement}$.

We show that this assumption leads to a contradiction. Because b is not an interior point of $A^{\complement}$, there is no δ-neighborhood, for whatever small δ, that entirely belongs to $A^{\complement}$. So, for each $j \in \mathbb{N}$, there is an $x_j \in N_{1/j}(b)$ which does not belong to $A^{\complement}$, i.e., $x_j \in A$.

We have constructed a sequence x_j which is entirely contained in A and which converges to b. The latter is true because, for any j, all but finitely many members are contained in $N_{1/j}(b)$.

The closed set A contains all its contact points and it follows from the criterion for contact points that $b \in A$.

But we had assumed at the outset that $b \in A^{\complement}$ and we have a contradiction. ■

Theorem 12.9 (Open iff complement is closed)**.**

Let (X, d) be a metric space and $A \subseteq X$. Then A is open if and only if $A^{\complement}$ is closed.

PROOF: Immediate from prop.12.28 and prop.12.29 ■

12.9 Bounded Sets and Bounded Functions in Metric Spaces

Definition 12.26 (bounded sets)**.**

Given is a subset A of a metric space (X, d).

The **diameter** of A is defined as

(12.42) $\qquad \operatorname{diam}(\emptyset) := 0, \qquad \operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$ if $A \neq \emptyset$.

We call A a **bounded set** if $\operatorname{diam}(A) < \infty$. □

Remark 12.22.

(a) Note that we needed a metric $d(x, y)$ to define the boundedness of a set. We cannot generalize this concept to topological spaces.

(b) A set can be bounded in one metric and unbounded in another. For example, let d be the Euclidean metric on $\mathbb{R}$ and let d' be the discrete metric on $\mathbb{R}$. Then each of the sets $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ is bounded in $(\mathbb{R}, d')$ (by the number 1), but it is unbounded in $(\mathbb{R}, d)$. □

Proposition 12.30.

Given is a metric space (X, d) and a nonempty subset A. The following are equivalent:

(12.43)

(a) *$\mathrm{diam}(A) < \infty$, i.e., A is bounded.*

(b) *There exists $\gamma > 0$ and $x_0 \in X$ such that $A \subseteq N_\gamma(x_0)$.*

(c) *For all $x \in X$ there exists $\gamma > 0$ such that $A \subseteq N_\gamma(x)$.*

PROOF of "**(b)** ⇒ **(a)**": For any $x, y \in A$ we have

$$d(x, y) \;\le\; d(x, x_0) + d(x_0, y) \;\le\; 2\gamma$$

and it follows that $\mathrm{diam}(A) \le 2\gamma$.

PROOF of "**(a)** ⇒ **(b)**": Pick an arbitrary $x_0 \in A$ and let $\gamma := \mathrm{diam}(A)$. Then for all $a \in A$

$$d(x_0, a) \;\le\; \sup_{x \in A} d(x, a) \;\le\; \sup_{x, z \in A} d(x, z) \;=\; \mathrm{diam}(A) \;=\; \gamma.$$

It follows that $A \subseteq N_\gamma(x_0)$.

PROOF of "**(c)** ⇒ **(a)**": We pick an arbitrary $x_0 \in A$ which is possible as A is not empty. Then there is $\gamma = \gamma(x_0)$ such that $A \subseteq N_\gamma(x_0)$. For any $y, z \in A$ we then have

$$d(y, z) \;\le\; d(y, x_0) + d(x_0, z) \;\le\; 2\gamma$$

and it follows that $\mathrm{diam}(A) \le 2\gamma < \infty$.

PROOF of "**(a)** ⇒ **(c)**": Given $x \in X$, pick an arbitrary $x_0 \in A$ and let $\gamma := d(x, x_0) + \mathrm{diam}(A) + 1$. Then

$$\begin{aligned} y \in A \quad \Rightarrow \quad d(x, y) \;&\le\; d(x, x_0) + d(x_0, y) \;\le\; d(x, x_0) + \sup_{u \in A} d(u, y) \\ &\le d(x, x_0) + \sup_{u, z \in A} d(u, z) \;=\; d(x, x_0) + \mathrm{diam}(A) \;=\; \gamma. \end{aligned}$$

It follows that $A \subseteq N_\gamma(x)$. ■

Proposition 12.31.

Let (X,d) be a metric space. For $n\in\mathbb{N}$ let $A_n\subseteq X$ such that $\delta_n := \operatorname{diam}(A_n)\to 0$ as $n\to\infty$. Let $A:=\bigcap_n A_n$. Then,

$$\text{either}\quad A=\emptyset,\qquad \text{or}\quad \text{there is some } a\in X \text{ such that } A=\{a\}.$$

PROOF: Let $a,a'\in A$ and let $\delta := d(a,a')$. It follows from $A\subseteq A_n$ that

$$d(a,a') \le \sup\{d(x,x') : x,x'\in A_n\} = \operatorname{diam}(A_n), \text{ i.e., } d(a,a')\le\delta_n$$

for all $n\in\mathbb{N}$. It follows from $\delta_n\to 0$ that $d(a,a')=0$, i.e., $a=a'$. We have shown that A contains at most one element. ■

In metric spaces the points in the closure of a set A can be approaced by sequences that live in A. It should not come as a surprise that $\operatorname{diam}(A)$ does not increase when one replaces A by its closure.

Proposition 12.32.

Let (X,d) be a metric space and $A\subseteq X$. Then,

$$\operatorname{diam}(A) = \operatorname{diam}(\bar{A}).$$

PROOF:

It follows from $A\subseteq\bar{A}$ that $\operatorname{diam}(A)\le\operatorname{diam}(\bar{A})$. It remains to prove that $\operatorname{diam}(\bar{A})\le\operatorname{diam}(A)$.

Nothing needs to be shown if A is unbounded, i.e., $\operatorname{diam}(A)=\infty$, because $x\le\infty$ is true for any $x\in\mathbb{R}\biguplus\{\pm\infty\}$. We hence may assume that A is bounded.

Let $\varepsilon>0$ and $x,y\in\bar{A}$. It follows from Thm.12.7 (Sequence criterion for contact points in metric spaces) on p.580 that there are sequences $(x_n)_n$ and $(y_n)_n$ in A such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Thus there exist $N_x,N_y\in\mathbb{N}$ such that $d(x_j,x)<\frac{\varepsilon}{2}$ for all $j\ge N_x$ and $d(y_j,y)<\frac{\varepsilon}{2}$ for all $j\ge N_y$. Let $n:=\max(N_x,N_y)$. Then

$$d(x,y) \le d(x,x_n)+d(x_n,y_n)+d(y_n,y) < d(x_n,y_n)+\varepsilon \le \operatorname{diam}(A)+\varepsilon.$$

The inequality $d(x,y)\le\operatorname{diam}(A)+\varepsilon$ is true for arbitrary $x,y\in\bar{A}$, hence $\operatorname{diam}(A)+\varepsilon$ is an upper bound for the set $\{d(x,y):x,y\in\bar{A}\}$. We conclude that

$$\operatorname{diam}(\bar{A}) = \sup\big(\{d(x,y):x,y\in\bar{A}\}\big) \le \operatorname{diam}(A)+\varepsilon$$

. The above holds for arbitrary $\varepsilon>0$, and we conclude that $\operatorname{diam}(\bar{A})\le\operatorname{diam}(A)$. ■

Proposition 12.33.

Let (X,d) be a metric space. Let $A_1\supseteq A_2\supseteq\ldots$ be subsets of X such that $\operatorname{diam}(A_n)\to 0$ as $n\to\infty$ and let $A:=\bigcap_j \bar{A}_j$. Let $x_n\in A_n$ for all n. Then

- *$(x_n)_n$ converges if and only if A is not empty.*
- *If $A\neq\emptyset$, then A is the singleton set $A=\left\{\lim_{n\to\infty} x_n\right\}$.*

PROOF:

The proof is done in two stages.

(a) We first prove that if x_n has a limit $x \in X$ then $A = \{x\}$.

Let $n, k \in \mathbb{N}$ such that $k \geq n$. It follows from $x_k \in A_k \subseteq A_n$ that $x_k \in A_n$. Thm.12.7 (Sequence criterion for contact points in metric spaces) on p.580 yields $x = \lim_{n\to\infty} x_n \in \bar{A}_n$. As n was arbitrary, we obtain $x \in \bigcap_j \bar{A}_j$, i.e., $x \in A$.

It follows from Proposition 12.32 and $\operatorname{diam}(A_n) \to 0$, that $\operatorname{diam}(\bar{A}_n) \to 0$. It now follows from Proposition 12.31 that the intersection A has size zero or one. Since $x \in A, A = \{x\}$. This concludes the proof of **b1**.

(b) It remains to prove that if $A \neq \emptyset$ then $(x_n)_n$ converges.

Let $x \in A$. Then $x \in \bar{A}_n$ for all n. We use again that $\operatorname{diam}(A_n) = \operatorname{diam}(\bar{A}_n)$ and obtain

$$d(x_n, x) \;\leq\; \operatorname{diam}(\bar{A}_n) \;=\; \operatorname{diam}(A_n) \;=\; \delta_n \;\to\; 0 \;\text{ as } n \to \infty. \tag{12.44}$$

Let $\varepsilon > 0$. It follows from $\lim_{k\to\infty} \delta_k = 0$ that there exists $n_0 \in \mathbb{N}$ such that $\delta_k < \varepsilon$ for all $k \geq n_0$. We conclude from (12.44) that $d(x_k, x) \;< \varepsilon$ for all $k \geq n_0$, and hence that $\lim_{k\to\infty} x_n = x$. ■

Remark 12.23. You may wonder whether it can ever happen that a sequence of nonempty sets $A_1 \supseteq A_2 \supseteq \cdots$ and $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$ can ever have a non–empty intersection A.

Yes, it can: Let (X, d) be the rational numbers with the standard metric $d(q, q') = |q - q'|$. Let

$$A_n := \{\, q \in]0, \infty[_{\mathbb{Q}} \colon 2 - \frac{1}{n} \;<\; q^2 \;< 2 + \frac{1}{n} \,\}$$

In other words, A_n is the open neighborhood of $\sqrt{2}$ with radius $1/n$. Clearly, $\bigcap_n A_n = \{\sqrt{2}\}$. Well, since $\sqrt{2} \in \mathbb{R}$, this would be correct in the metric space $(\mathbb{R}, d)$. But, since $\sqrt{2} \notin \mathbb{Q}$, $\bigcap_n A_n = \emptyset$! □

12.10 Completeness in Metric Spaces

Often you are faced with a situation where you need to find a contact point a and all you have is a sequence which behaves like one converging to a contact point in the sense of inequality 12.20 (page 564)

Definition 12.27 (Cauchy sequences [12])**.**

Given is a metric space (X, d). A sequence (x_n) in X is called a **Cauchy sequence** or, in short, it is Cauchy if for any $\varepsilon > 0$ (no matter how small), there exists some index $n_0 \in \mathbb{N}$ such that

$$d(x_i, x_j) \;<\; \varepsilon \qquad \text{for all } i, j \geq n_0 \tag{12.45}$$

This is called the **Cauchy criterion for convergence** of a sequence. □

[12] Cauchy sequence are named after the great french mathematician Augustin–Louis Cauchy (1789–1857) who contributed massively to the most fundamental ideas of Calculus.

Example 12.6 (Cauchy criterion for real numbers)**.** In $\mathbb{R}$ we have $d(x,y) = |x-y|$ and the Cauchy criterion requires for any given $\varepsilon > 0$ the existence of $n_0 \in \mathbb{N}$ such that

$$|x_i - x_j| < \varepsilon \qquad \text{for all } i,j \geq n_0. \quad \square \tag{12.46}$$

Proposition 12.34.

Let (X,d) be a metric space and $x_n \in X$ $(n \in \mathbb{N})$. Then the following are equivalent:

- ***(a)*** *$(x_n)_n$ is Cauchy.*
- ***(b)*** *The diameters of the tail sets $T_n = \{x_j : j \geq n\}$ converge to zero.*
- ***(c)*** *There exists a nonincreasing sequence $A_1 \supseteq A_2 \supseteq \ldots$ of subsets of X such that $x_n \in A_n$ and $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$.*

PROOF:

PROOF of **(a)** $\Rightarrow$ **(b)**:

Let $\varepsilon > 0$. It follows from the definition of Cauchy sequences that there exists $n_0 \in \mathbb{N}$ such that $d(x_i, x_j) < \varepsilon$ for all $i,j \geq n_0$. From this we obtain

$$\operatorname{diam}(T_{n_0}) = \sup\{d(x_i,x_j) : i,j \geq n_0\} \leq \varepsilon.$$

It follows from prop.12.7 on p.565 that $\lim_{n\to\infty} \operatorname{diam}(T_n) = 0$.

PROOF of **(b)** $\Rightarrow$ **(c)**:

We choose $A_n := T_n$ as our nonincreasing sequence of sets.

PROOF of **(c)** $\Rightarrow$ **(a)**:

Let $\varepsilon > 0$. It follows from the definition of convergence $\operatorname{diam}(A_n) \to 0$ that there exists $n_0 \in \mathbb{N}$ such that $\operatorname{diam}(A_{n_0}) < \varepsilon$. Let $k \in \mathbb{N}, k \geq n_0$. Then $A_k \subseteq A_{n_0}$, hence $\operatorname{diam}(A_k) \leq \operatorname{diam}(A_{n_0}) < \varepsilon$.

Let $i,j \in \mathbb{N}$ such that $i,j \geq n_0$. By assumption, $x_i \in A_i \subseteq A_{n_0}$ and $x_j \in A_j \subseteq A_{n_0}$, hence $x_i, x_j \in A_{n_0}$, hence $d(x_i,x_j) \leq \operatorname{diam}(A_{n_0}) < \varepsilon$. This proves that $(x_n)_n$ is Cauchy. ■

Proposition 12.35.

A Cauchy sequence in a metric space is bounded.

PROOF: Let $(x_n)_n$ be a Cauchy sequence in a metric space (X,d). There is $N = N(1/2)$ such that $d(x_i,x_j) < 1/2$ for all $i,j \geq N$. In particular, $d(x_i, x_N) < 1/2$.

Let $M := \max\{d(x_j,x_N) : j < N\}$. We obtain for any two indices $i,j \in \mathbb{N}$ that

$$d(x_i,x_j) \leq d(x_i,x_N) + d(x_N,x_j).$$

$d(x_i,x_N)$ is bounded by M in case that $i < N$ and by $1/2$ if $i \geq N$; hence $d(x_i,x_N) < 1/2 + M$. We use the same reasoning to conclude that $d(x_N,x_j) < 1/2 + M$ and obtain $d(x_i,x_j) < 1 + 2M$. This proves the boundedness of $(x_n)_n$. ■

Theorem 12.10 (Convergent sequences are Cauchy)**.**

Let $(x_n)_n$ be a convergent sequence in a metric space (X, d). Then $(x_n)_n$ is Cauchy.

PROOF: Let $L \in X$ and $x_n \to L$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that

$$k \geq N \;\Rightarrow\; d(x_k, L) \;<\; \varepsilon/2. \tag{12.47}$$

It follows from (12.47) that, for any $i, j \geq N$,

$$d(x_i, x_j) \;\leq d(x_i, L) + d(L, x_j) \;<\; \varepsilon/2 + \varepsilon/2 \;=\; \varepsilon.$$

It follows that the sequence satisfies (12.45) of the definition of a Cauchy sequence (def. 12.27 on p.588). ■

Proposition 12.36.

Let $(x_n)_n$ be a Cauchy sequence in a metric space (X, d).
If some subsequence x_{n_j} converges to a limit L. Then
(a) *ANY subsequence of $(x_n)_n$ converges to L.*
(b) *$(x_n)_n$ is a convergent sequence.*
Further, any subsequence y_{n_j} of a convergent sequence $(y_n)_n$ converges to the limit of $(y_n)_n$.

PROOF of **(a)**: Let $n_1 < n_2 < n_3 \ldots$ be such that x_{n_j} converges to L. For $k \in \mathbb{N}$ let $y_k := x_{n_k}$.
Let $\varepsilon > 0$. Convergence $y_j \to L$ implies that there is $N \in \mathbb{N}$ such that

$$d(y_j, L) < \varepsilon/2 \text{ for all } j \geq N. \tag{12.48}$$

Because (x_j) is Cauchy there also exists $N' \in \mathbb{N}$ such that

$$d(x_i, x_j) < \varepsilon/2 \text{ for all } i, j \geq N'. \tag{12.49}$$

Let $K := \max(n_N, N')$ and $j \geq K$. Then

$$d(x_j, L) \;\leq\; d(x_j, y_K) + d(y_K, L)$$

It follows from $n_K \geq K$ and $j \geq K$ and (12.49) that $d(x_j, y_K) = d(x_j, x_{n_K}) < \varepsilon/2$ and it follows from (12.48) that $d(y_K, L) < \varepsilon/2$. We conclude that $d(x_j, L) < \varepsilon$ for all $j \geq K$ and this proves convergence $x_j \to L$.
PROOF of **(b)**: This is trivial: The full sequence $x_1, x_2, \ldots$ is a subsequence, and it converges by assumption to L
PROOF of the addendum: This is trivial, too: The convergent sequence $(x_n)_n$ is Cauchy, and the assumption now follows from part **(a)** ■

Here is the formal definition of a complete set in a metric space.

Definition 12.28 (Completeness in metric spaces)**.**

A subset A of a metric space (X,d) is called **complete**, if any Cauchy sequence (a_n) with elements in A converges to some $a \in A$. □

Remark 12.24.

(a) It is **NOT sufficient** that $\lim_{n\to\infty} x_n$ exists in X: It must not belong to the complement of A!
(b) In particular, X itself is complete iff any Cauchy sequence in X converges.
(c) A is complete as a subset of (X,d) iff the subspace $((A,d\big|_{A\times A})$ is complete "in itself". □

The following theorem of the completeness of the set of all real numbers [13] states that any Cauchy sequence converges to a real number. This is a big deal: To show that a sequence in $\mathbb{R}$ has a finite limit one need not provide the actual value of that limit. All one must show is that this sequence satisfies the Cauchy criterion. One can say that this preoccupation with proving existence rather than computing the actual value is one of the major points which distinguish mathematics from applied physics and the engineering disciplines.

Theorem 12.11 (Completeness of the real numbers)**.**

Let (x_n) be a Cauchy sequence in $\mathbb{R}$. then there exists a real number L such that $L = \lim_{n\to\infty} x_n$.

PROOF: It follows from prop.12.35 that x_n is bounded, hence $(x_n)_n$ possesses finite liminf and limsup. [14] We now show that $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $|x_i - x_j| \le \varepsilon$ for all $i,j \ge N$.

Let $T_n := \{x_j : j \ge n\}$ be the tail set of the sequence $(x_n)_n$. Let $\alpha_n := \inf T_n, \beta_n := \sup T_n$.

There is some $i \ge N$ such that $|x_i - \alpha_N| = x_i - \alpha_N \le \varepsilon$ and there is some $j \ge N$ such that $|\beta_N - x_j| = \beta_N - x_j \le \varepsilon$. It follows that

$$0 \le \beta_N - \alpha_N = |\beta_N - \alpha_N| \le |\beta_N - x_j| + |x_j - x_i| + |x_i - \alpha_N| \le 3\varepsilon.$$

Further, if $k \ge N$ then $T_k \subseteq T_N$, hence $\alpha_k \ge \alpha_N$ and $\beta_k \le \beta_N$. It follows that

$$0 \le \inf \beta_k - \sup \alpha_k \le \beta_N - \alpha_N \le 3\varepsilon.$$

But then

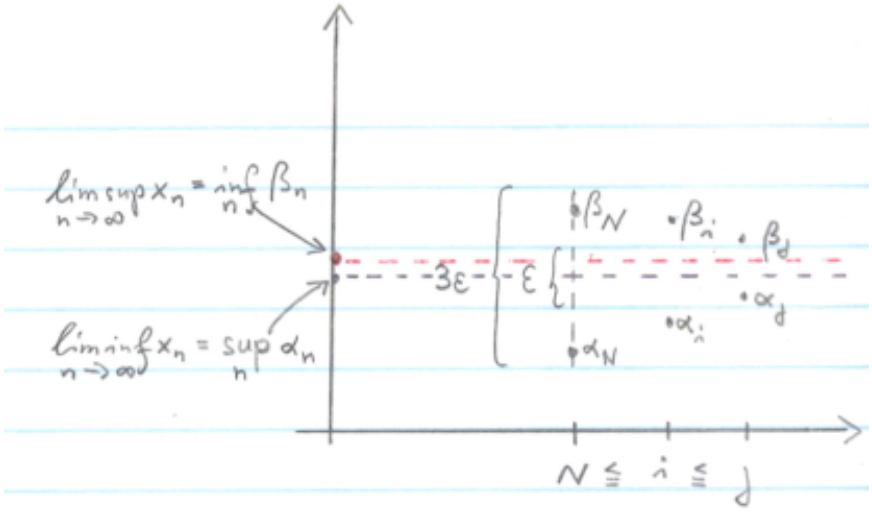
$$0 \le \limsup_{k\to\infty} x_k - \liminf_{k\to\infty} x_k = \inf_k \beta_k - \sup_k \alpha_k \le 3\varepsilon.$$

[13] Remember the completeness axiom for $\mathbb{R}$ (axiom **??(c)** on p.**??**) which states that any subset A of $\mathbb{R}$ which possesses upper bounds has a least upper bound (the supremum $\sup(A)$). This axiom was needed to establish the validity of thm.**??** (Characterization of limits via limsup and liminf) on p.**??**, a theorem which will be used in this chapter to prove the completeness of $\mathbb{R}$ as a metric space.

[14] See ch.**??** (Minima, Maxima, Infima and Suprema).

$\varepsilon > 0$ was arbitrary, hence $\limsup_{k\to\infty} x_k \;=\; \liminf_{k\to\infty} x_k$.

Figure 12.3: liminf = limsup for Cauchy sequences



Part 3: It follows from theorem **??** on p.**??** that the sequence $(x_n)_n$ converges to $L := \limsup_{k\to\infty} x_k$ and the proof is finished. ■

Now that the completeness of $\mathbb{R}$ has been established, it is not very difficult to see that n–dimensional space $\mathbb{R}^n$ also is complete.

Theorem 12.12 (Completeness of $\mathbb{R}^n$)**.**

Let $(\vec{x}_j)$ be a Cauchy sequence in $\mathbb{R}^n$. Then there exists $\vec{a} \in \mathbb{R}^n$ such that $\vec{a} = \lim_{j\to\infty} \vec{x}_j$.

PROOF (outline): Let $\vec{x}_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,n})$ be Cauchy in $\mathbb{R}^n$. For fixed k, each coordinate sequence $(x_{j,k})_j$ is Cauchy because, if $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that if $i, j \geq K$ then $\|\vec{x}_i - \vec{x}_j\|_2 < \varepsilon$. Hence

$$|x_{i,k} - x_{j,k}| \;=\; \sqrt{|x_{i,k} - x_{j,k}|^2} \;\leq\; \sqrt{\sum_{k=1}^{n} |x_{i,k} - x_{j,k}|^2} \;=\; \|\vec{x}_i - \vec{x}_j\|_2 \;<\; \varepsilon.$$

It follows from the completeness of $\mathbb{R}$ as a metric space that there exist real numbers

$$a_1, a_2, a_3, \ldots, a_n \qquad \text{such that } \; a_k \;=\; \lim_{n\to\infty} x_{n,k} \; (1 \leq k \leq n).$$

For a given $\varepsilon > 0$ we can find natural numbers $N_{0,1}, N_{0,2}, \ldots, N_{0,n}$ such that

$$|x_{n,k} - a_k| \;<\; \frac{\varepsilon}{n} \quad \text{for all } n \geq N_{0,k} \text{ and for all } 1 \leq k \leq n.$$

Let $N^\star := \max(N_{0,1}, N_{0,2}, \ldots, N_{0,n})$ and $\vec{a} := (a_1, a_2, \ldots, a_n)$. It follows that

$$d(\vec{x}_j - \vec{a}) \;=\; \sqrt{\sum_{k=1}^{n} |x_{j,k} - a_k|^2} \;\le\; \sqrt{n \cdot \left(\frac{\varepsilon}{n}\right)^2} \;=\; \frac{\varepsilon}{\sqrt{n}} \;\le\; \varepsilon \qquad \text{for all } j \ge N^\star.$$

This proves convergence of $\vec{x}_j$ to $\vec{a}$.

You have learned in multivariable calculus that the limit of a sequence of vectors can be computed as the vector of the limits, taken separately for each coordinate. The proof is very similar to that of Theorem 12.12.

Proposition 12.37.

Let $\vec{x}_j = (x_{j,1}, x_{j,2}, \ldots, x_{j,n})$ and $\vec{b} \in \mathbb{R}^n$. Then,

(12.50) $$\lim_{j\to\infty} \vec{x}_j \;=\; \vec{b} \;\Leftrightarrow\; \lim_{j\to\infty} x_{j,k} \;=\; b_k \text{ for all } 1 \le k \le n\,.$$

PROOF of "⇒": If $\vec{x}_j$ converges to $\vec{b}$ then this sequence is Cauchy. We have seen in the proof of thm.12.12 that it has as limit a vector $\vec{a} := (a_1, a_2, \ldots, a_n)$ whose k–th coordinate a_k was obtained as $a_k \;=\; \lim_{j\to\infty} x_{j,k}$. In other words, $a_k = b_k$. This proves "⇒".

PROOF of "⇐": Assume that $\lim_{j\to\infty} x_{j,k} \;=\; b_k$ for all $1 \le k \le n$. We copy word for word the second half of the proof of thm.12.12.

For a given $\varepsilon > 0$ we can find natural numbers $N_{0,1}, N_{0,2}, \ldots, N_{0,n}$ such that

$$|x_{n,k} - b_k| \;<\; \frac{\varepsilon}{n} \quad \text{for all } n \ge N_{0,j} \text{ and for all } 1 \le k \le n.$$

Let $N^\star := \max(N_{0,1}, N_{0,2}, \ldots, N_{0,n})$. It follows that

$$d(\vec{x}_j - \vec{b}) \;=\; \sqrt{\sum_{k=1}^{n} |x_{j,k} - b_k|^2} \;\le\; \sqrt{n \cdot \left(\frac{\varepsilon}{n}\right)^2} \;=\; \frac{\varepsilon}{\sqrt{n}} \;\le\; \varepsilon \qquad \text{for all } j \ge N^\star.$$

This proves convergence of $\vec{x}_j$ to $\vec{b}$ and hence "⇐". ■

Example 12.7 (Approximation of decimals)**.** The following illustrates Cauchy sequences and completeness in $\mathbb{R}$. We have seen in ch.**??** (Decimal Expansions of Real and Rational Numbers) that any real number $x \ge 0$ can be written as a decimal

$$x \;=\; m \;+\; \sum_{j=1}^{\infty} d_j \cdot 10^{-j} \quad (m \in \mathbb{Z}_{\ge 0},\; d_j \in \{0, 1, 2, \ldots, 9\}).$$

Further, any such infinite series is a real number since each partial sum $s_n = m \;+\; \sum_{j=1}^{n} d_j \cdot 10^{-j}$ is bounded (above) by $m \;+\; 9 \sum_{j=1}^{\infty} 10^{-j} \;=\; m+1$, and thus, x is a real number as the supremum of the bounded and nondecreasing sequence $(s_n)_n$.

What just has been illustrated is that there a natural way to construct for a given $x \in \mathbb{R}$ Cauchy sequences of rational numbers that converge toward x. (Each s_n is rational as the sum of the finitely many rational numbers m and $\frac{d_j}{10^j}$. The completeness of $\mathbb{R}$ states that the reverse also is true: For any Cauchy sequence $s_n \in \mathbb{Q}$ there is an element $x \in \mathbb{R}$ toward which this sequence converges. □

The existence of irrational numbers tells us that the limit of a sequence of rational partial sums need not be rational. This can be used to construct metric spaces which are not complete.

Proposition 12.38.

The metric space $(\mathbb{Q}, d_{|\cdot|})$ (Euclidean metric) is not complete.

PROOF: Let us work for the time being in the metric space $(\mathbb{R}, d_{|\cdot|})$ of all real numbers, not in the subspace $(\mathbb{Q}, d_{|\cdot|})$ which we are interested in.

Let $x \in \mathbb{R}$ be any positive irrational number, e.g., $x = \pi = 3.1415\ldots$ or $x = \sqrt{2} = 1.414\ldots$. x has a decimal representation $x = m + \sum_{j=1}^{\infty} d_j 10^{-j}$ where each $d_j \in \mathbb{Z}_{\geq 0}$ is a digit, i.e., $0 \leq d_j \leq 9$. Let $s_n = m + \sum_{j=1}^{n} d_j 10^{-j}$ Then,

$$|x - s_n| = x - s_n = \sum_{j=n+1}^{\infty} d_j 10^{-j} \leq 9 \cdot \sum_{j=n+1}^{\infty} 10^{-j} = 10^{-n}.$$

It follows, not surprisingly, that $x = \lim_{n\to\infty} x_n$.

We know from thm.12.10 (Convergent sequences are Cauchy) on p.590 that the sequence s_n is Cauchy in $(\mathbb{R}, d_{|\cdot|})$. But $s_n \in \mathbb{Q}$ for all n and the distance $d_{|\cdot|}(s_n, s_m) = |s_n - s_m|$ is the same in $(\mathbb{R}, d_{|\cdot|})$ and $(\mathbb{Q}, d_{|\cdot|})$.

It follows that s_n is Cauchy in $(\mathbb{Q}, d_{|\cdot|})$. We had constructed this sequence in such a way that it does not have a limit in $\mathbb{Q}$, and it follows that $(\mathbb{Q}, d_{|\cdot|})$ is not complete. ■

A byproduct of this next proposition is that the discrete metric is complete.

Proposition 12.39.

Let d be the discrete metric on a nonempty set X and let $(x_n)_n$ a sequence in X. Then,

$$(x_n)_n \text{ is Cauchy} \quad \Leftrightarrow \quad (x_n)_n \text{ converges} \quad \Leftrightarrow \quad (x_n)_n \text{ is constant eventually.}$$

PROOF:

We show the equivalence of **(a)** and **(c)**. It follows from the definition of Cauchy sequences that there exists $n_0 \in \mathbb{N}$ such that $d(x_i, x_j) < 1$ for all $i, j \geq n_0$ For the discrete metric $d(x_i, x_j) < 1$ means the same as $d(x_i, x_j) = 0$, thus $x_i = x_j$ for all $i, j \geq n_0$. This proves that x_n is eventually constant.

On the other hand, if x_n is eventually constant, then it follows from the definition of a property holding eventually that there exists $n_0 \in \mathbb{N}$ such that $x_i = x_j$ for all $i, j \geq n_0$ Thus $d(x_i, x_j) = 0$ for

such i and j (in any metric!). Let $\delta > 0$. It follows that $d(x_i, x_j) < \delta$ for all $i, j \geq n_0$, i.e., $(x_n)_n$ is Cauchy. Matter of fact, since $d(x_{n_0}, x_j) = 0 < \delta$ for all $j \geq n_0$, it follows that the sequence is convergent with x_{n_0} as its limit. ■

An easy corollary is

Corollary 12.2.

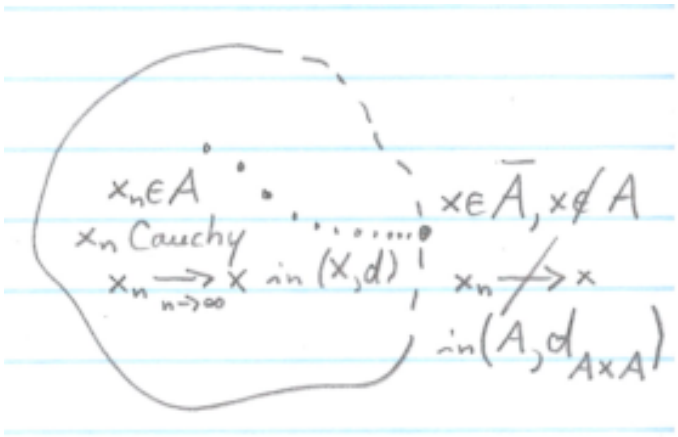
Discrete metric spaces are complete.

PROOF: We must show that all Cauchy sequences converge in discrete metric spaces. This follows from the first equivalence of prop.12.39 above. ■

Theorem 12.13.

Any complete subset of a metric space is closed.

Figure 12.4: complete ⇒ closed



PROOF: Let (X, d) be a metric space and$A \subseteq X$. Let $a \in X$ be a contact point of A. The theorem is proved if we can show that $a \in A$.

a) We employ Definition 12.23 on p.579: A point $x \in X$ is a contact point of A if and only if $A \cap V \neq \emptyset$ for any neighborhood V of x.

Let $m \in \mathbb{N}$. Then $N_{1/m}(a)$ is a neighborhood of the contact point a, hence hence $A \bigcap N_{1/m}(a) \neq \emptyset$ and we can pick a point from this intersection which we name x_m.

b) We prove next that $(x_m)_m$ is Cauchy. Let $\varepsilon > 0$ and let $N \in \mathbb{N}$ be such that $N > \frac{2}{\varepsilon}$. if $j \in \mathbb{N}$ and $k \in \mathbb{N}$ both exceed N then

$$d(x_j, x_k) \;\leq\; d(x_j, a) + d(a, x_k) \;\leq\; \frac{1}{j} + \frac{1}{k} \;\leq\; \frac{1}{N} + \frac{1}{N} < \varepsilon.$$

It follows that the sequence (x_j) is Cauchy.

c) Because A is complete, this sequence must converge to some $b \in A$. But b cannot be different from a. Otherwise we could "separate" a and b by two disjoint neighborhoods: choose the open ρ–balls $N_\rho(a)$ and $N_\rho(b)$ where ρ is one half the distance between a and b (see the proof of thm.12.3 on p.564).

Only finitely many of the x_n are allowed to be outside $N_\rho(a)$ and the same is true for $N_\rho(b)$. That is a contradiction and it follows that $b = a$, i.e., $a \in A$.

d) We summarize: if a is a contact point of A then $a \in A$. It follows that A is closed. ■

The following is the reverse of thm.12.13.

Theorem 12.14 (Closed subsets of a complete space are complete)**.**

Let (X, d) be a complete metric space and let $A \subseteq X$ be closed. Then A is complete, i.e., the metric subspace $(A, d\big|_{A\times A})$ is complete.

PROOF: Let $(x_n)_n$ be Cauchy in A. We must show that there is $a \in A$ such that $x_n \to a$. Note that (x_n) also is Cauchy in X because the Cauchy criterion is entirely specified in terms of members of the sequence (x_n).

Because X is complete there exists $x \in X$ such that $x_n \to x$. All x_n belong to A. According to thm.12.7 (Sequence criterion for contact points in metric spaces), x is a contact point of A.

As the set A is assumed to be closed, it contains all its contact points. It follows that $x \in A$, i.e., the arbitrary Cauchy sequence (x_n) in A converges to an element of A. We conclude that A is complete. ■

12.11 Exercises for Ch.12

12.11.0.1 Exercises for Ch.12.1 (Definition and Examples of Metric Spaces)

Exercise 12.1. Prove prop.12.1 on p.555: Let (X, d) be a metric space. Let $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n \in X$. Then

$$d(x_1, x_n) \leq \sum_{j=1}^{n-1} d(x_j, x_{j+1}) = d(x_1, x_2) + d(x_2, x_3) + d(x_{n-1}, x_n). \quad \square$$

Exercise 12.2.

Prove thm.12.1 (Norms define metric spaces) on p.555: Let $(V, \|\cdot\|)$ be a normed vector space. Then the function

$$d_{\|\cdot\|}(\cdot,\cdot) : V \times V \to \mathbb{R}_{\geq 0}; \qquad (x, y) \mapsto d_{\|\cdot\|}(x, y) := \|y - x\|$$

defines a metric space $(V, d_{\|\cdot\|})$.

Hint: This proof is very easy. Even the triangle inequality for the metric $d(x, y) = \|x - y\|$ follows easily from the triangle inequality for the norm. $\square$

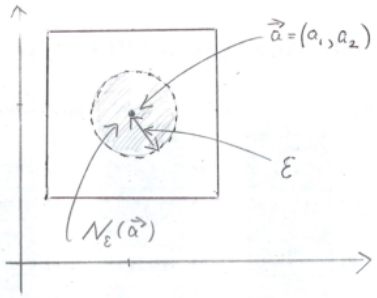
12.11.0.2 Exercises for Ch.12.2 (Measuring the Distance of Real–Valued Functions)

12.11.0.3 Exercises for Ch.12.3 (Neighborhoods and Open Sets)

Exercise 12.3. Prove prop.12.5 on p.562: Let $a, b \in \mathbb{R}$ such that $a < b$. Then the open interval $]a, b[$ is an open set in $(\mathbb{R}, d_{|\cdot|})$. □

Exercise 12.4. Let $A := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ be the first quadrant in the plane (the points on the coordinate axes are excluded). Prove that each element of A is an inner point, i.e., A is open in $\mathbb{R}^2$. See the picture for a hint.

Hint: Find for $\vec{a} = (a_1, a_2)$ small enough ε such that $N_\varepsilon(\vec{a}) \subseteq A$



□

Exercise 12.5. Let $a, b \in \mathbb{R}$ such that $a < b$.

(a) The closed interval $[\ a, b\ ]$ is not open in $(\mathbb{R}, d_{|\cdot|})$.

(b) The complement of the closed interval $[\ a, b\ ]$ is open in $(\mathbb{R}, d_{|\cdot|})$. □

Exercise 12.6.

(a) Let $m \in \mathbb{Z}$, viewed as a subset of the metric space $(\mathbb{R}, d_{|\cdot|})$. Prove that m is a boundary point of $\mathbb{Z}$.

(b) Prove that the above also holds both for the set of rational numbers: $\mathbb{Q} \subseteq \partial(\mathbb{Q})$ and for the set of all irrational numbers: $\mathbb{R} \setminus \mathbb{Q} \subseteq \partial(\mathbb{R} \setminus \mathbb{Q})$. □

12.11.0.4 Exercises for Ch.12.4 (Convergence)

Exercise 12.7. Given is a metric space (X, d).
Prove the following: A sequence (x_n) of elements of X converges to $a \in X$ as $n \to \infty$ iff for any neighborhood U of a there exists some $n_0 \in \mathbb{N}$ such that the n_0–tail set $T_{n_0} = \{x_j : j \geq n_0\}$ is contained in U (see Definition **??** (Tail sets of a sequence) on p.**??**.) □

Exercise 12.8. Prove remark 12.8 on p.566: Let (X, d) be a metric space and $x_n, L \in (X, d)$. Then

$$\lim_{n\to\infty} x_n = L \Leftrightarrow \lim_{n\to\infty} d(x_n, L) = 0\,.$$ □

Exercise 12.9.

Prove prop.12.9 on p.566:
Let x_n, y_n be two sequences in a metric space (X, d). Assume there is $K \in \mathbb{N}$ such that $x_n = y_n$ for all $n \geq K$. Let $L \in X$ Then

$$\lim_{n\to\infty} x_n = L \Leftrightarrow \lim_{n\to\infty} y_n = L. \square$$

Exercise 12.10.

Prove prop.12.11 on p.566:
Let x_n be a convergent sequence in a metric space (X, d) with limit $L \in E$. Let $K \in \mathbb{N}$. For $n \in \mathbb{N}$ let $y_n := x_{n+K}$. Then $\lim_{n\to\infty} (y_n)_n = L$. $\square$

Exercise 12.11.

Let $f_n, f \in \mathscr{B}([0,1], \mathbb{R})$ $n \in \mathbb{N}$ be continuous such that $f = \lim_{n\to\infty} f_n$ in $(\mathscr{B}([0,1], d_{\|\cdot\|_\infty})$(!) Prove $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$. You must use the ε, N definition of convergence.
Hints: **(a)** No need to mention that continuous functions are both bounded and integrable and that they attain both max and min on closed and bounded intervals. **(b)** Use the mean value theorem: For cont. $h(\cdot)$ on $[0,1]$ let $\alpha := \min_{x\in[0,1]} h(x), \beta := \max_{x\in[0,1]} h(x)$. Then $\exists\ \lambda \in [\alpha, \beta]$ such that $\lim_{n\to\infty} \int_0^1 h(x)dx = \lambda\ (= \lambda(1-0)$. **(c)** Use without proof that $\left| \int_0^1 h(x)dx \right| \leq \int_0^1 |h(x)|dx$ for any integrable $h(\cdot)$ on $[0,1]$ **(d)** Apply **(b)** and **(c)** to $h_n(x) = |f_n(x) - f(x)|$. (So you deal with $\alpha_n, \lambda_n, \beta_n$).
$\square$

12.11.0.5 Exercises for Ch.12.5 (Abstract Topological spaces)

Exercise 12.12.

It was stated in prop.12.13 on p.569 that the discrete topology which is induced by the discrete metric $d(x, y) = 1$ if $x \neq y$ and 0 if $x = y$ is the entire power set 2^X of X. Prove it. $\square$

Exercise 12.13.

Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. Prove that the open exterior of A is

$$\text{ext}(A) = \left(\overline{A^{\complement}}\right)^o. \square$$

Exercise 12.14.

Let X be a set that contains. at least two elements. [15] Prove that there is no metric d on X such that $\mathfrak{U}_d = \{\emptyset, X\}$, i.e., such that its only open sets are the empty set and X.. $\square$

[15] See exercise 12.14 on p.598

12.11.0.6 Exercises for Ch.12.6 (Bases and Neighborhood Bases)

Exercise 12.15. Let (X, d) be a metric space and let $\mathfrak{B} := \{N_{1/k}(x) : x \in X, k \in \mathbb{N}\}$. Then $\mathfrak{B}$ is a base of the topology for the associated topological space $(X, \mathfrak{U}_d)$. □

12.11.0.7 Exercises for Ch.12.7 (Metric and Topological Subspaces)

12.11.0.8 Exercises for Ch.12.9 (Bounded Sets and Bounded Functions)

12.11.0.9 Exercises for Ch.12.8 (Contact Points and Closed Sets)

Exercise 12.16.
Let $A \subseteq \mathbb{R}$ be a closed, nonempty set which is bounded above. Prove that the maximum of A exists and that $\sup(A) = \max(A)$. □

Exercise 12.17.
Prove prop.12.25 on p.583: Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq X$. Then $\partial A = \bar{A} \cap \overline{A^{\complement}}$. □

Exercise 12.18. Prove prop.12.24 on p.583: Let $(X, \mathfrak{U})$ be a topological space and $A \subseteq B \subseteq X$. Then $\bar{A} \subseteq \bar{B}$. □

Exercise 12.19. Prove prop.12.16 on p.572: Let $(X, \mathfrak{U})$ be a topological space. If $A \subseteq B \subseteq X$ then $A^o \subseteq B^o$. □

Exercise 12.20.
Prove parts **(c)** and **(d)** of prop.12.27 (Closure of a set as a hull operator) on p.583: Let A and B be subsets of a topological space $(X, \mathfrak{U})$. Then **(c)** $\bar{\bar{A}} = \bar{A}$, **(d)** $\overline{A \cup B} = \bar{A} \cup \bar{B}$. □

12.11.0.10 Exercises for Ch.12.10 (Completeness in Metric Spaces)

Exercise 12.21.
Let (X, d) be a metric space and $A \subseteq X$, $A \neq \emptyset$. Let

$$\gamma := \gamma(A) := \inf\{d(x, y) : x, y \in A \text{ and } x \neq y\}.$$

(a) Prove that if $\gamma > 0$ then A is complete.

(b) The reverse is not true. Find a counterexample. □

Exercise 12.22.
Let (X, d) be a metric space and let $A \subseteq X$ be a finite subset. Prove that A is complete. □

Exercise 12.23.
Given is $\mathbb{R}$ with the Euclidean metric $d(x, y) = |x - y|$. We look at $\mathbb{N}$ and $\mathbb{Q}$ as metric subspaces of $\mathbb{R}$. We know that $\mathbb{Q}$ is not complete.

(a) Is $\mathbb{N}$ complete as a subspace of $\mathbb{Q}$?

(b) Is $\mathbb{N}$ complete as a subspace of $\mathbb{R}$?

Prove your answer. □

Exercise 12.24.

Let X be a nonempty set with the discrete metric $d(x, y) = 1 - \mathbf{1}_{\{x\}}(y)$, i.e., $d(x, y) = 0$ if $x = y$ and 1 else. Prove that (X, d) is complete. $\square$

References

[1] James R. Munkres. Topology. Prentice-Hal, 1st edition, 2000.

List of Symbols

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