Math 330 - Additional Material

Student edition with proofs

Michael Fochler
Department of Mathematics
Binghamton University

This document contains chapter 13 of the Math 330 lecture notes.

Last update: February 17, 2026

Contents

13 Metric Spaces and Topological Spaces – Part II

13.1 Continuity

13.1.1 Definition and Characterizations of Continuous Functions

We have briefly discussed in ch.**??** on p.**??**. the continuity of functions with arguments and values in $\mathbb{R}$. We now extend this definition to functions that map from metric spaces to metric spaces and, more generally, from topological spaces to topological spaces.

Definition 13.1 (Sequence continuity)**.**

Given are two metric spaces (X, d_1) and (Y, d_2). Let $A \subseteq X$, $x_0 \in A$ and let $f : A \to Y$ be a mapping from A to Y. We say that f is **sequence continuous at x_0** and we write

$$\lim_{x \to x_0} f(x) = f(x_0)\,, \tag{13.1}$$

if the following is true for any sequence (x_n) with values in A:

$$\text{if } x_n \to x_0 \text{ then } f(x_n) \to f(x_0). \tag{13.2}$$

In other words, the following must be true for any sequence (x_n) in A and $x_0 \in A$:

$$\lim_{n\to\infty} x_n = x_0 \quad \Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x_0). \tag{13.3}$$

We say that f is **sequence continuous** if f is sequence continuous at x_0 for all $x_0 \in A$. ☐

Remark 13.1.

Important points to notice:

a) It is not enough for the above to be true for some sequences that converge to x_0. Rather, this must be true for ALL such sequences!

b) We restrict our universe to the domain A of f: x_0 and the entire sequence $(x_n)_{n\in\mathbb{N}}$ must belong to A, because we need function values for all x-values. In other words, f is continuous at $x_0 \in A$ if and only if f is continuous at x_0 in the metric subspace $(A, d\big|_{A\times A})$. ☐

Definition 13.2 (ε-δ continuity)**.**

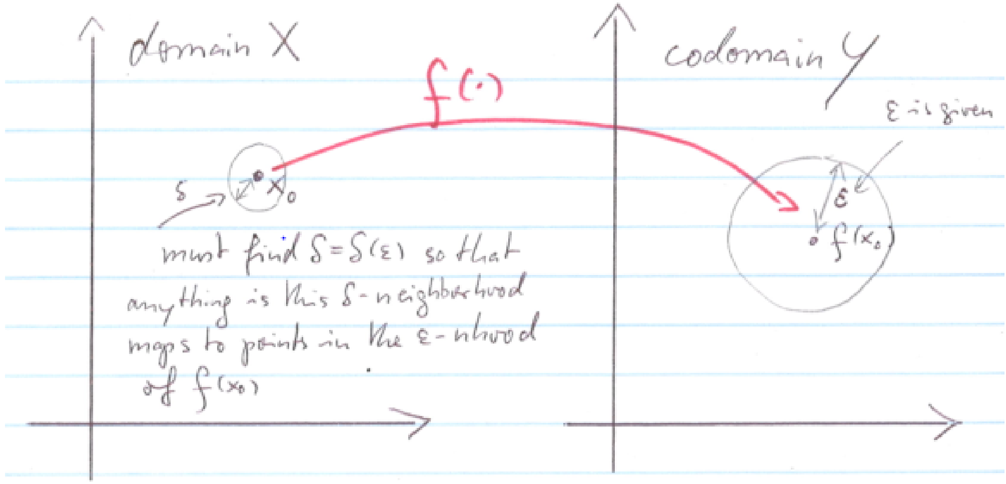
Given are two metric spaces (X, d_1) and (Y, d_2). Let $A \subseteq X$, $x_0 \in A$ and let $f(\cdot) : A \to Y$ be a mapping from A to Y. We say that $f(\cdot)$ is ε–δ **continuous at x_0** if the following is true: For any (whatever small) $\varepsilon > 0$ there exists $\delta > 0$ such that either one of the following equivalent statements is satisfied:

$$f\big(N_\delta(x_0)\cap A\big) \ \subseteq\ N_\varepsilon(f(x_0)), \tag{13.4}$$

$$d_1(x,x_0)<\delta \ \Rightarrow\ d_2(f(x),f(x_0))<\varepsilon \ \text{ for all } x\in A. \tag{13.5}$$

We say that $f(\cdot)$ is ε-δ **continuous** if $f(\cdot)$ is ε-δ continuous at a for all $a\in A$. ☐

Figure 13.1: ε-δ continuity



Remark 13.2.
We recall from thm.**??** on p.**??** that

$$N_\delta(a)\cap A \ =\ N_\delta^A(a) \ =\ \{x\in A : d\big|_{A\times A}(x,a)<\delta\}. \tag{13.6}$$

Hence (13.4) states that

f is ε-δ continuous at x_0 $\Leftrightarrow$ for all $\varepsilon>0$ there exists $\delta>0$ s.t. $f\big(N_\delta^A(x_0)\big)\ \subseteq\ N_\varepsilon(f(x_0))$. ☐

Theorem 13.1 (Continuity criterion)**.**

Let (X,d_1) and (Y,d_2) be two metric spaces. Let $A\subseteq X$, $x_0\in A$ and let $f(\cdot):A\to Y$. Then,

- *f is sequence continuous at x_0 $\quad\Leftrightarrow\quad$ f is ε-δ continuous at x_0.*
- *In particular f is sequence continuous (on A) if and only if f is ε-δ continuous.*

PROOF:

a) **⇒**: Proof that sequence continuity implies ε-δ-continuity:
We assume to the contrary that there exists some function f which is sequence continuous but not ε-δ-continuous at x_0, i.e., there exists some $\varepsilon > 0$ such that neither (13.4) nor the equivalent (**??**) is true for any $\delta > 0$.

a.1. In other words, No matter how small δ is choosen, there is at least one $x = x(\delta) \in A$ such that $d_1(x, x_0) < \delta$ but $d_2(f(x), f(x_0)) \geq \varepsilon$. In particular we obtain for $\delta := 1/m (m \in \mathbb{N})$ that

$$\text{(13.7)} \qquad \text{there exists some } x_m \in N_{1/m}(x_0) \cap A \text{ such that } d_2(f(x_m), f(x_0)) \geq \varepsilon.$$

a.2. It follows from prop.**??** on p.**??** that the sequence $(x_m)_{m \in \mathbb{N}}$ converges to x_0.

a.3. Clearly $\big(f(x_m)\big)_{m \in \mathbb{N}}$ does not converge to $f(x_0)$, as that requires $d_2(f(x_m), f(x_0)) < \varepsilon$ for all sufficiently big m, contrary to (13.7) which implies that there is not even one such m. In other words, the function f is not sequence continuous, contrary to our assumption. We have our contradiction.

b) **⇐**: Proof that ε-δ-continuity implies sequence continuity:
Let $x_n \to x_0$. Let $y_n := f(x_n)$ and $y_0 := f(x_0)$. We must prove that $y_n \to y_0$ as $n \to \infty$.

b.1. Let $\varepsilon > 0$. We can find $\delta > 0$ such that (13.4) and hence (**??**) is satisfied. Since we assumed that $x_n \to x_0$ there exists $N := N(\delta) \in \mathbb{N}$ such that $d_1(x_n, x_0) < \delta$ for all $n \geq N$.

b.2. It follows from (**??**) that $d_2(y_n, y) = d_2(f(x_n), f(x_0)) < \varepsilon$ for all $n \geq N$. In other words, $y_n \to y_0$ as $n \to \infty$ and the proof of "⇐" is finished.

It follows from the proofs of **(a)** and **(b)** that f is sequence continuous $\Leftrightarrow$ f is ε-δ continuous. ■

Definition 13.3 (Continuity in metric spaces)**.**

From now on we can use the terms "ε-δ continuous at x_0" and "sequence continuous at x_0" interchangeably for functions between metric spaces and we will simply speak about **continuity of f at x_0**. □

Remark 13.3 (continuity for real–valued functions of real numbers)**.** Let $(X, d_1) = (Y, d_2) = \mathbb{R}$. In this case equation (**??**) on p.**??** becomes

$$|x - x_0| < \delta \ \Rightarrow \ |f(x) - f(x_0)| < \varepsilon.$$

See thm.**??** on p.**??**. □

We saw in the ε-δ continuity definition of a function with metric spaces for both domain and codomain and the subsequent remark 13.2 that continuity of $f : (A, d_1\big|_{A \times A}) \to (Y, d_2)$ in $x_0 \in A$ was equivalent to demanding that for any ε–neighborhood of $f(x_0)$ there is a δ–neighborhood of x_0 such that

$$f\big(N_\delta^A(x_0)\big) \ \subseteq \ N_\varepsilon(f(x_0)).$$

The fact that <u>any neighborhood</u> of a point z in a metric space contains a γ–neighborhood of z for suitably small γ, is at the basis of the following theorem.

Theorem 13.2 (Neighborhood characterization of continuity)**.**

> *Let (X, d_1) and (Y, d_2) be two metric spaces. Let $A \subseteq X$, $x_0 \in A$, and let $f(\cdot) : A \to Y$ be a mapping from A to Y. Then*
>
> *f is continuous at x_0 if and only if for any neighborhood $V_{f(x_0)}$ of $f(x_0)$, there exists a neighborhood U_{x_0} of x_0 in the metric space (X, d_1), such that*
>
> $$f(U_{x_0} \cap A) \subseteq V_{f(x_0)}\,. \tag{13.8}$$
>
> *Equivalently, (13.8) can be stated in terms of the subspace $(A, d_1\big|_{A\times A})$ as follows.*
>
> *for any neighborhood $V_{f(x_0)}$ of $f(x_0)$ there exists a neighborhood $U^A_{x_0}$ of x_0 in the metric space $(A, d_1\big|_{A\times A})$ such that*
>
> $$f(U^A_{x_0}) \subseteq V_{f(x_0)}. \tag{13.9}$$

PROOF:

a) **⇒**): Assume that f is continuous, i.e., ε-δ continuous at a. Let $V_{f(x_0)}$ be a neighborhood of $f(x_0)$. Then $f(x_0)$ is interior point of $V_{f(x_0)}$ and we can find suitable $\varepsilon > 0$ such that $N_\varepsilon(f(x_0)) \subseteq V_{f(x_0)}$. ε-δ continuity at a implies the existence of $\delta > 0$ such that $f\big(N_\delta(x_0) \cap A\big) \subseteq N_\varepsilon(f(x_0))$, hence $f\big(N_\delta(x_0) \cap A\big) \subseteq V_{f(x_0)}$.

This proves both (13.8) (choose $U_{x_0} := N_\delta(x_0)$) and (13.9) (choose $U^A_{x_0} := N_\delta(x_0) \cap A$).

b) **⇐**): Assume that (13.8) is satisfied for any arbitrary neighborhood $V_{f(x_0)}$ of $f(x_0)$.

Let $\varepsilon > 0$. We need to show that there exists $\delta > 0$ such that

$$f(N_\delta(x_0) \cap A) \subseteq N_\varepsilon(f(x_0)). \tag{13.10}$$

$N_\varepsilon(f(x_0))$ is a neighborhood of $f(x_0)$. It follows from (13.8) that there exists a neighborhood U_{x_0} of x_0 such that

$$f(U_{x_0} \cap A) \subseteq N_\varepsilon(f(x_0)). \tag{13.11}$$

x_0 is interior point of any of its neighborhoods. In particular, it is interior to U_{x_0}.

Accordingly, there exists $\delta > 0$ such that $N_\delta(x_0) \subseteq U_{x_0}$, hence $N_\delta(x_0) \cap A \subseteq U_{x_0} \cap A$. It follows from the monotonicity of the direct image $\Gamma \mapsto f(\Gamma)$ that

$$f\big(N_\delta(x_0) \cap A\big) \subseteq f\big(U_{x_0} \cap A\big) \subseteq N_\varepsilon(f(x_0)). \tag{13.12}$$

The second inclusion relation follows from (13.11). We have proved the existence of $\delta > 0$ such that (13.10) is satisfied. This finishes the proof of "⇐)". ■

Before we generalize continuity to topological spaces we will now generalize thm.**??** of ch.**??** which was stated for real–valued function with domain $A \subseteq \mathbb{R}$. to real–valued function with domain $A \subseteq (X, d)$ where (X, d) is a metric space. The proof of this theorem demonstrates how to work with the definitions

Theorem 13.3 (Rules of arithmetic for continuous real–valued functions)**.**

Given is a metric space (X, d). Let the functions

$$f(\cdot), g(\cdot), f_1(\cdot), f_2(\cdot), f_3(\cdot), \dots, f_n(\cdot) : A \longrightarrow \mathbb{R}$$

all be continuous at $x_0 \in A \subseteq X$. Then

- ***(a)*** *Constant functions are continuous everywhere on A.*
- ***(b)*** *The product $fg(\cdot) : x \mapsto f(x)g(x)$ is continuous at x_0. Specifically, $\alpha f(\cdot) : x \mapsto \alpha \cdot f(x)$ where $\alpha \in \mathbb{R}$ is continuous at x_0. In particular ($\alpha = -1$) the function $-f(\cdot) : x \mapsto -f(x)$ is continuous at x_0.*
- ***(c)*** *The sum $f + g(\cdot) : x \mapsto f(x) + g(x)$ is continuous at x_0.*
- ***(d)*** *If $g(x_0) \neq 0$ then the quotient $f/g(\cdot) : x \mapsto f(x)/g(x)$ is continuous at x_0.*
- ***(e)*** *Any linear combination [1] $\sum_{j=0}^{n} a_j f_j(\cdot) : x \mapsto \sum_{j=0}^{n} a_j f_j(x)$ is continuous in x_0.*

PROOF of **(a)**:

Let $f : A \to \mathbb{R}$; $x \mapsto \alpha$ for some $\alpha \in \mathbb{R}$. Let $x_n \in A$ for all $n \in \mathbb{N}$ such that $x_n \to x_0$ as $n \to \infty$. Then $f(x_n) = f(x_0) = \alpha$ for all $n \in \mathbb{N}$, i.e., the sequence $f(x_n)_n$ is constant with value $f(x_0) = \alpha$, and it thus converges to $f(x_0)$ by prop.**??** on p.**??**. This proves **(a)**.

PROOF of **(b)**: Since it follows from the already proven part **(a)** that the constant function $x \mapsto \alpha$ and thus in particular the function $x \mapsto -1$ are continuous everywhere on A it remains to prove that fg is continuous at x_0.

Let $x_n \in A$ for all $n \in \mathbb{N}$ such that $x_n \to x_0$ as $n \to \infty$. All we need to show is convergence $f(x_n)g(x_n) \to f(x_0)g(x_0)$. This follows from prop.**??** (Rules of arithmetic for limits) on p.**??**, thus we have shown that fg is continuous at x_0. We have proven **(b)**.

PROOF of **(c)**: Let $x_n \in A$ for all $n \in \mathbb{N}$ such that $x_n \to x_0$ as $n \to \infty$. We must show convergence $f(x_n) + g(x_n) \to f(x_0) + g(x_0)$. This again follows from prop.**??** and we have proved **(c)**.

proof of **(d)** (outline): The proof is done by (strong) induction.

Base case: For $n = 2$ the proof is obvious from parts **(a)**, **(b)** and **(c)**.

Induction step: Write

$$\sum_{j=0}^{n+1} a_j f_j(x) \;=\; \Big(\sum_{j=0}^{n} a_j f_j(x) \Big) \;+\; a_{n+1} f_{n+1}(x) \;=\; U \;+\; V.$$

The left term U is continuous by the induction assumption, thus the sum $U + V$ is continuous as the sum of two continuous functions (we showed this in **(c)**). This proves **(d)**. ■

The last theorem allows us to conclude that certain sets of continuous functions are vector spaces since sums $f + g$ and scalar products αf involving continuous functions are continuous.

Example 13.1 (Vector spaces of continuous real–valued functions)**.** Let (X, d) be a metric space. Then

$$\mathscr{C}(X, \mathbb{R}) \;:=\; \{ f(\cdot) : f(\cdot) \text{ is a continuous real–valued function on } X \} \tag{13.13}$$

of all real continuous functions on X is a vector space. Note that we have seen this before in example **??** (Vector spaces of real–valued functions) on p.**??** for the special case of $X \subseteq (\mathbb{R}, d_{|\cdot|})$.

The sup–norm

$$\|f(\cdot)\|_\infty \;=\; \sup\{|f(x)| : x \in X\}$$

(see (**??**) on p.**??**) is **not a real–valued function** on all of $\mathscr{C}(X,\mathbb{R})$ because $\|f(\cdot)\|_\infty = +\infty$ for any unbounded $f(\cdot) \in \mathscr{C}(X,\mathbb{R})$. To avoid complications from dealing with infinity, we often restrict the scope to the subspace

$$\mathscr{C}_{\mathscr{B}}(X,\mathbb{R}) \;:=\; \{h : h \text{ is a bounded continuous real–valued function on } X\}$$

(see prop.**??** on p. **??**) of the normed vector space $\mathscr{B}(X,\mathbb{R})$ of all bounded real–valued functions on X. On this subspace the sup–norm truly is a real–valued function since $\|f(\cdot)\|_\infty < \infty$. ☐

Remark 13.4.

The equivalence of (13.8) and (13.9) in thm.13.2 (neighborhood characterization of continuity) has some profound consequences:

Assume that we have proven a statement about continuity at $x_0 \in X$ for all functions which have metric spaces as domain and codomain. Let's say we use the notation $f : (X,d) \to (Y,d')$. This statement then remains true for all functions $g : (A, d\big|_{A\times A}) \to (Y,d')$ as long as the proof does not make use of a property of X which its subset A does not satisfy.

A good example for this is thm.13.3 (rules of arithmetic for continuous real–valued functions). For example, if a function φ is continuous on all of X then its restriction $\varphi\big|_A$ to $A \subseteq X$ does not lose this property, and if it satisfies in addition $\varphi(x_0) \neq 0$ for some $x_0 \in A$ then it remains true that $\varphi\big|_A(x_0) \neq 0$.

Here is a somewhat contrived counterexample. If the assumptions state that X must be unbounded, i.e., $\text{diam}(X) = \infty$, then the validity of the statement does not necessarily extend to bounded subsets of X. [2] ☐

The last theorem allows us to generalize the notion of continuity to functions between abstract topological spaces.

Definition 13.4 (Continuity for topological spaces)**.**

Given are two topological spaces $(X,\mathfrak{U}_1)$ and $(Y,\mathfrak{U}_2)$. Let $A \subseteq X$, $x_0 \in A$ and let $f : A \to Y$ be a mapping from A to Y.

We say that f is **continuous at x_0** if the following is true:

For any neighborhood $V_{f(x_0)}$ of $f(x_0)$, there exists a neighborhood U_{x_0} of x_0 in the topological space $(X,\mathfrak{U}_1)$, such that

$$f(U_{x_0} \cap A) \;\subseteq\; V_{f(x_0)}\,. \tag{13.14}$$

Equivalently, continuity at x_0 can be stated in terms of the subspace $(A,\mathfrak{U}_{\mathbf{1}A})$ as follows.

For any neighborhood $V_{f(x_0)}$ of $f(x_0)$ there is a neighborhood $U^A_{x_0}$ of x_0 in $(A,\mathfrak{U}_{\mathbf{1}A})$ such that

$$f(U^A_{x_0}) \;\subseteq\; V_{f(x_0)}. \tag{13.15}$$

We say that f is **continuous on** A if f is continuous at a for all $a \in A$. ☐

---

[2] Better counterexamples involve completeness and compactness, important subjects you will learn about later. It is possible for the entire space to be complete and/or compact and for certain subsets not to have that property.

Remark 13.5.
Let (X,d) and (Y,d') be metric spaces with associated metric topologies $\mathfrak{U}_d$ and $\mathfrak{U}_{d'}$.[3] Let $A \subseteq X$ and $f : A \to Y$. Since the condition (13.8) for continuity at $x_0 \in A$ of f as a function between the metric spaces (X,d) and (Y,d') is identical to the condition (13.14) for continuity at $x_0 \in A$ of f as a function between the associated topological spaces $(X,\mathfrak{U}_d)$ and $(Y,\mathfrak{U}_{d'})$ it follows that any statement that we prove for continuity in topological spaces is automatically true for continuity in metric spaces. $\square$

[1] B/G: Art of Proof defines in appendix A, p.136, continuity of a function f as follows: "$f^{-1}(\text{open}) = \text{open}$". The following proposition proves that their definition coincides with the one given here: the validity of (13.14) for all $x_0 \in X$.

Proposition 13.1 ("$f^{-1}(\text{open}) = \text{open}$" continuity)**.**

Let $(X,\mathfrak{U})$ and $(Y,\mathfrak{V})$ be two topological spaces and let $f : X \to Y$. Then

- *f is continuous (on X)* $\Leftrightarrow$ *All preimages $f^{-1}(V)$ of open $V \subseteq Y$ are open in X.*

PROOF of "$\Rightarrow$": Let V be an open set in Y. Let $U := f^{-1}(V)$, $a \in U$ and $b := f(a)$. Then $b \in V$ by the definition of inverse images. b is inner point of V because V is open. According to Definition 13.4 there exists a neighborhood U_a of a such that $f(U_a) \subseteq V$.

We conclude from the monotonicity of direct and inverse images and prop.**??** on p.**??** that

$$U_a \subseteq f^{-1}\big(f(U_a)\big) \subseteq f^{-1}(V) = U.$$

It follows that the arbitrarily chosen $a \in U$ is an interior point of U and this proves that U is open.

PROOF of "$\Leftarrow$": We now assume that all inverse images of open sets in Y are open in X.

Let $a \in X, b = f(a)$, and let V_b be a neighborhood of b. Any neighborhood of b contains an open neighborhood of b, hence we may assume that V_b is open. We are done if we can find an open neighborhood U_a of a such that

$$f(U_a) \subseteq V_b \tag{13.16}$$

Let $U_a := f^{-1}\big(V_b\big)$. Then U_a is open as the inverse image of the open set V_b It follows from the monotonicity of direct and inverse images and prop.**??** on p.**??** that

$$f(U) = f\big(f^{-1}(V_b)\big) = V_b \cap f(X) \subseteq V_b.$$

We have proved (13.16) $\blacksquare$

Note that the previous proposition only addresses "global" continuity of f for **all** $x \in X$. There is no local version which handles continuity at a specific x_0.

---

[3] See Definition **??** on p.**??**.

Note also that it is easily generalized to $f : A \to Y$ $(\emptyset \neq A \subseteq X)$ by demanding that $f^{-1}(V)$ be open in $(A, \mathfrak{U}_A)$ for all $V \in \mathfrak{V}$.

Remark 13.6. Remark 13.4 on p.608 for metric spaces can be rephrased for topological spaces as follows:

In the interest of simplicity one may assume for statements involving continuity of a function f between topological spaces$(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ that f is defined on all of X rather than assuming more generally that f is defined (only) on some arbitrary subset A of X. The general case of $f : A \to Y$ is then covered by replacing $(X, \mathfrak{U})$ with $(A, \mathfrak{U}_A)$, i.e., we deal with $f : (A, \mathfrak{U}_A) \to (Y, \mathfrak{V})$ just as long as the proof does not make use of a property of X which its subset A does not satisfy.

It is easy to see that this condition is satisfied for prop.13.2, prop.13.3, and prop.13.4 below.

The next proposition was previously stated for real–valued functions of a real variable. See prop.**??** on p.**??**.

Proposition 13.2 (The composition of continuous functions is continuous)**.**

Let $(X, \mathfrak{U})$, $(Y, \mathfrak{V})$ and $(Z, \mathfrak{W})$ be topological spaces.
Let $f : X \to Y$ be continuous at $x_0 \in X$ and $g : Y \to Z$ continuous at $f(x_0)$.

- *Then the composition $g \circ f : X \to Y$ is continuous at x_0.*

PROOF: The proof is left as exercise 13.4 (see p.636). ■

We now give some examples of continuous functions.

Proposition 13.3 (continuity of constant functions)**.**

Let $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ be topological spaces and $y_0 \in Y$.

- *Then the constant function $f : x \mapsto y_0$ is continuous.*

PROOF: It suffices to show that inverse images of open sets are open. So let $V \in \mathfrak{V}$. Then either $x_0 \in V$ in which case $f^{-1}(V) = X$, or $x_0 \notin V$ in which case $f^{-1}(V) = \emptyset$. Since both X and $\emptyset$ are open in X it follows that $f^{-1}(\text{open}) = \text{open}$, hence f is continuous. ■

Proposition 13.4 (continuity of the identity mapping)**.**

Let $(X, \mathfrak{U})$be a topological space and let

$$id_X : X \to X; \qquad x \mapsto x$$

be the identity function on X. Then id_X is continuous.

PROOF: It suffices to show that inverse images of open sets are open. So let $V \in \mathfrak{U}$. Then $id_X^{-1}(V) = V$, hence $id_X^{-1}(V)$ is open. This finishes the proof. ■

Remark 13.7.

The proof just given also applies to metric spaces, but it is instructive to give a direct proof of this proposition which works with the metric.

So let (X, d) be a metric space and let id_X be the identity function on X. Let $x \in X$ and $\varepsilon > 0$. let $\delta := \varepsilon$. If $x' \in X$ such that $d(x, x') < \delta$, then

$$d(id_X(x), id_X(x')) \ = \ d(x, x') \ < \ \delta \ = \ \varepsilon.$$

We have verified condition (**??**) of the ε–δ characterization of continuity and it follows that id_X is continuous at x. x was an arbitrary point in X, and it follows that the identity is continuous. [4] ☐

The next proposition gives a very simple example that the behavior of a function with respect to continuity strongly depends on the choice of metric on domain and/or codomain.

Proposition 13.5.

Let d be the standard Euclidean metric and let d' be the discrete metric on the set $\mathbb{R}$ of all real numbers. Let

$$f : (\mathbb{R}, d') \to (\mathbb{R}, d); \quad x \mapsto x \qquad \text{and} \qquad g : (\mathbb{R}, d) \to (\mathbb{R}, d'); \quad x \mapsto x$$

both be the identity function on $\mathbb{R}$. Then,

- *f is continuous at every point of $\mathbb{R}$*
- *g is not continuous anywhere on $\mathbb{R}$.*

The proof is left as exercise 13.6 (see p.636). ■

Because of their importance we state here once more rem.13.4, rem.13.5, and rem.13.6.

Remark 13.8.

(a) All statements about continuity proven for topological spaces are also true for the special case of metric spaces.

(b) One may assume for statements involving continuity of a function f between metric spaces (X, d) and (Y, d') or between topological spaces$(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ that f is defined on all of X rather than assuming more generally that f is defined (only) on some arbitrary subset A of X.
The general case of $f : A \to Y$ is then covered for metric spaces by replacing (X, d) with $(A, d\big|_{A \times A})$ (we deal with $f : (A, d\big|_{A \times A}) \to (Y, d')$), and it is covered for topological spaces by replacing $(X, \mathfrak{U})$ with $(A, \mathfrak{U}_A)$ (we deal with $f : (A, \mathfrak{U}_A) \to (Y, \mathfrak{V})$), just as long as the proof does not make use of a property of X which its subset A does not satisfy. ☐

---

[4] Actually, we have proved a very strong form of continuity. Generally speaking, $\delta = \delta(\varepsilon, x_0)$ is tailored not only to the given ε, but also to the particular argument x_0 at which continuity needs to be verified. We were able to find δ which does not depend on the argument x_0 but only on ε. We will learn later that this makes id_X **uniformly continuous** on its domain X. See Definition 13.5 (Uniform continuity of functions) on p.612.

13.1.2 Uniform Continuity

It will be proved in theorem **??** (Uniform continuity on sequence compact spaces) on p.**??** [5] that continuous real–valued functions on the compact set $[0,1]$ are uniformly continuous in the sense of the following definition. [6]

Definition 13.5 (Uniform continuity of functions)**.**

Let (X,d_1), (Y,d_2) be metric spaces and let A be a subset of X. A function

$$f(\cdot) : A \to Y \qquad \text{is called } \textbf{uniformly continuous}$$

if, for any $\varepsilon > 0$, there exists a (possibly very small) $\delta > 0$ such that

$$d_2(f(x) - f(y)) \ < \varepsilon \quad \text{for any } x, y \in A \text{ such that } d_1(x,y) \ < \ \delta. \ \square \tag{13.17}$$

Remark 13.9. $f : (X,d_1) \to (Y,d_2)$ is uniformly continuous on $A \subseteq X$ if and only if the following is true: For all $\varepsilon > 0$ there is $\delta > 0$ such that

$$\big[\ B \subseteq A \text{ such that } \operatorname{diam}(B) \leq \delta \ \Rightarrow \ \operatorname{diam}\big(f(B)\big) \leq \varepsilon\ \big]\,. \ \square$$

Remark 13.10 (Uniform continuity vs. continuity)**.** Note the following:

a. Condition (13.17) for uniform continuity looks very close to the ε–δ characterization of ordinary continuity (**??**) on p.**??**. Can you spot the difference?

Uniform continuity is more demanding than plain continuity because, when dealing with the latter, you can ask for specific values of both ε and x_0 according to which you must find a suitable δ. In other words, for plain continuity

$$\delta \ = \ \delta(\varepsilon, x_0).$$

In the case of uniform continuity all you get is ε. You must come up with a suitable δ regardless of what arguments are thrown at you. To write that one in functional notation,

$$\delta \ = \ \delta(\varepsilon).$$

b. It follows that uniform continuity implies continuity but the opposite need not be true.

c. Many concepts that are defined in metric spaces can be generalized to topological spaces Examples were neigborhoods, interior points and contact points, subspaces and continuity. Uniform continuity is not a concept that can be defined without a metric. [7] $\square$

---

[5] see chapter **??** (Continuous Functions and Compact Spaces) on p.**??**

[6] For the special case of $(X,d) = (\mathbb{R}, d_{|\cdot|})$ where $d_{|\cdot|}(x,y) = |y-x|$, see [1] Beck/Geoghegan, Appendix A.3, "Uniform continuity".

[7] That is not entirely accurate: There is a notion of "uniform spaces" which generalize the concept of a metric but are less general than topological spaces and there is a notion of uniform continuity for those sets.

Example 13.2 (Uniform continuity of the identity mapping)**.** Let us have another look at rem.13.7 where we proved the continuity of the identity mapping on a metric space. We chose $\delta = \varepsilon$ no matter what value of x we were dealing with and it follows that the identity mapping is always uniformly continuous. ☐

Example 13.3. Let $f(x) := \frac{1}{x}$ on $]0,1]$ with the Euclidean metric. Then f is NOT uniformly continuous on $]0,1]$. See exercise 13.2 ☐

Remark 13.11. Now that you have learned the definitions for both continuity and uniform continuity, have a closer look at example **??**, p.**??** in ch.**??** (Quantifiers for Statement Functions of more than Two Variables) where it was explained how you could obtain one definition from the other just by switching around a ∀ quantifier and a ∃ quantifier. ☐

13.1.3 Continuity of Linear Functions

Lemma 13.1.

Let $f : (V, \|\cdot\|) \to (W, \lvert\cdot\rvert)$ be a linear function between two normed vector spaces. Let

$$\begin{aligned} a &:= \sup\{ \lvert f(x) \rvert : x \in V, \|x\| = 1\}, \\ b &:= \sup\{ \lvert f(x) \rvert : x \in V, \|x\| \leq 1\}, \\ c &:= \sup\{ \frac{\lvert f(x) \rvert}{\|x\|} : x \in V, x \neq 0\}. \end{aligned}$$

Then, $a = b = c$.

PROOF: We introduce the following three sets for this proof:

$$\begin{aligned} A &:=\{ \lvert f(x) \rvert : x \in V, \|x\| = 1\}, \\ B &:=\{ \lvert f(x) \rvert : x \in V, \|x\| \leq 1\}, \\ C &:=\left\{ \frac{\lvert f(x) \rvert}{\|x\|} : x \in V, x \neq 0\right\}. \end{aligned}$$

Proof that $a = b$:
It follows from $A \subseteq B$ that $a \leq b$. On the other hand let $x \in B$ such that $x \neq 0$ (if $x = 0$ then $f(x) = 0$ certainly could not exceed a). Let $y := \|x\|^{-1}x$. Then $y \in A$ and $\|x\|^{-1} \geq 1$, hence

$$\lvert f(y) \rvert = \lvert f\big(x/\|x\|\big) \rvert = (1/\|x\|) \lvert f(x) \rvert \geq \lvert f(x) \rvert .$$

We conclude that the sup over the bigger set B does not exceed the sup over A, hence $a = b$.

Proof that $a = c$:
Let $x \in C$ and $y := \|x\|^{-1}x$. Then $y \in A$ and

$$\lvert f(x) \rvert / \|x\| = \lvert f(x)/\|x\| \rvert = \lvert f\big(x/\|x\|\big) \rvert = \lvert f(y) \rvert .$$

It follows that the sup over the bigger set C does not exceed the sup over A, hence $c = b$. ■

Definition 13.6 (norm of linear functions)**.**

★

Let $f : (V, \| \cdot \|) \to (W, \mathbf{|} \cdot \mathbf{|})$ be a linear function between two normed vector spaces. We denote the quantity $a = b = c$ from lemma 13.1 by $\|f\|$, i.e.,

$$\begin{aligned} \|f\| &= \sup\{ \mathbf{|}\, f(x) \,\mathbf{|} : x \in V, \|x\| = 1\} \\ &= \sup\{ \mathbf{|}\, f(x) \,\mathbf{|} : x \in V, \|x\| \leq 1\} \\ &= \sup\{ \frac{\mathbf{|}\, f(x) \,\mathbf{|}}{\|x\|} : x \in V, x \neq 0\}. \end{aligned} \tag{13.18}$$

$\|f\|$ is called the **norm of the linear function** f.

$\square$

We note that $\|f\|$ need not be finite.

The justification for calling $f \mapsto \|f\|$ a norm [8] will be given in thm.13.5 on p.615.

Theorem 13.4 (Continuity criterion for linear functions)**.**

Let $f : (V, \| \cdot \|) \to (W, \mathbf{|} \cdot \mathbf{|})$ be a linear function between two normed vector spaces. Then the following are equivalent.

(A) *f is continuous at $x = 0$,*
(B) *f is continuous in all points of V,*
(C) *f is uniformly continuous on V,*
(D) *$\|f\| < \infty$.*

Moreover, such a continuous linear function satisfies the inequality

$$\mathbf{|}\, f(x) \,\mathbf{|} \leq \|f\| \cdot \|x\|, \quad \text{for all } x \in V. \tag{13.19}$$

PROOF: Clearly we have **C** $\Rightarrow$ **B** $\Rightarrow$ **A**. We now show **A** $\Rightarrow$ **D**.

It follows from the continuity of f at 0 that there exists $\delta > 0$ such that

$$\text{if } z \in V \text{ and } \|z\| < \delta \text{ then } \mathbf{|}\, f(z) \,\mathbf{|} = \mathbf{|}\, f(z) - f(0) \,\mathbf{|} < 1. \tag{13.20}$$

Let $x \in V$ such that $\|x\| \leq 1$. Then $\|\delta/2 \cdot x\| \leq \delta/2 < \delta$, hence, according to (13.20),

$$\delta/2 \cdot \mathbf{|}\, f(x) \,\mathbf{|} = \mathbf{|}\, f(\delta/2 \cdot x) \,\mathbf{|} < 1, \quad \text{hence} \quad \mathbf{|}\, f(x) \,\mathbf{|} < 2/\delta.$$

Because this last inequality is true for all $x \in V$ with norm bounded by 1, it follows that

$$\|f\| = \sup\{ \mathbf{|}\, f(x) \,\mathbf{|} : x \in V, \|x\| \leq 1\} < 2/\delta < \infty.$$

---

[8] Note that we use the same notation $\| \cdot \|$ for both the norm on V and the norm of the linear function f. **Do not confuse the two!**

We have proved that **A** $\Rightarrow$ **D**.

We finally show **D** $\Rightarrow$ **C** and we do this in two steps.

First we show **D** $\Rightarrow$ (13.19). The inequality trivially holds for $x = 0$ because linearity of f implies $f(0) = 0$. If $x \neq 0$ then $\|x\| > 0$ (norms are positive definite) and the inequality follows from the last characterization of $\|f\|$ in (13.18).

Second step: Let $\varepsilon > 0$ and $\delta := \varepsilon/\|f\|$. Let $x, y \in V$ such that $\|x - y\| < \delta$. If we can prove that this implies $\big| f(x) - f(y) \big| < \varepsilon$, then f is indeed uniformly continuous and the proof is done. We show this as follows.

$$\big| f(x) - f(y) \big| \;=\; \big| f(x-y) \big| \;\overset{(13.19)}{\leq}\; \|f\| \cdot \|x - y\| \;<\; \|f\| \cdot \delta \;=\; \|f\| \cdot \varepsilon/\|f\| \;=\; \varepsilon. \quad \blacksquare$$

Theorem 13.5 ($\|f\|$ is a norm)**.**

★ *Let*

(13.21)
$$\mathscr{C}_{\mathit{lin}}(V,W) \;:=\; \mathscr{C}_{\mathit{lin}}\big((V, \|\cdot\|), (W, \big|\cdot\big|)\big) \;:=\; \{f : V \to W \;\; : f \textit{ is linear and continuous }\}.$$

Then, $\mathscr{C}_{\mathit{lin}}(V,W)$ *is a vector space and*

$$f \mapsto \|f\| \;=\; \sup\{\big| f(x) \big| \;:\, \|x\| = 1\} \tag{13.22}$$

defines a norm on $\mathscr{C}_{\mathit{lin}}(V,W)$*.*

PROOF:

In all of this proof let $A := x \in V : \|x\| = 1$.

(A) Proof that $\mathscr{C}_{\mathit{lin}}(V,W)$ is a vector space.

Let $f, g, \in \mathscr{C}_{\mathit{lin}}(V,W)$. We need to show that $f + g \in \mathscr{C}_{\mathit{lin}}(V,W)$, i.e., $f + g$ is both linear and continuous. Linearity is immediate. We now show continuity.

Let $x \in A$. Then

$$\big| f(x) + g(x) \big| \;\leq\; \big| f(x) \big| \;+\; \big| f(x) \big| \;\leq\; \|f\| + \|g\| \;<\; \infty. \tag{13.23}$$

The first inequality holds because the norm $\big| f(x) \big|$ satisfies the triangle inequality for norms. The second follows from (13.18) on p.614, and the finiteness of $\|f\| + \|g\|$ is, according to the continuity criterion for linear functions (thm.13.4 on p.614), equivalent to the continuity of both f and g.

We still must show that if $f \in \mathscr{C}_{\mathit{lin}}(V,W)$ and $\lambda \in \mathbb{R}$ then $\lambda f : x \mapsto \lambda f(x) \in \mathscr{C}_{\mathit{lin}}(V,W)$, i.e., we must show that this function is linear and continuous. Again, linearity is immediate. To show continuity we proceed as follows.

Let $x \in A$. $\big|\cdot\big|$ is absolutely homogeneous. Hence

$$\big| \lambda f(x) \big| \;=\; \big| |\lambda| f(x) \big| \;=\; |\lambda| \cdot \big| f(x) \big| \,. \tag{13.24}$$

It follows from prop.**??** (positive homogeneity of inf and sup) on p.**??** that

$$\|\lambda f\| \;=\; \sup\{\big| \lambda f(x) \big| \;:\, \|x\| = 1\} \;=\; \sup\{\big| |\lambda| f(x) \big| \;:\, \|x\| = 1\} \tag{13.25}$$

$$=\; |\lambda| \cdot \sup\{\big| f(x) \big| \;:\, \|x\| = 1\} \;=\; |\lambda| \cdot \|f\| \;<\; \infty. \tag{13.26}$$

This proves that λf is continuous.

(B) Proof that $\|f\|$ is a norm on $\mathscr{C}_{fin}(V,W)$.
Because (13.23) is valid for all $x \in A$, we obtain

$$\|f+g\| \;=\; \sup\{\, \big|\, f(x)+g(x) \,\big| \;: x \in A\} \;\leq\; \|f\| + \|g\|. \tag{13.27}$$

This proves the triangle inequality.

Likewise, we obtain from the validity of (13.25) for all $x \in A$,

$$\|\lambda f\| \;=\; \sup\{\, \big|\, |\lambda| f(x) \,\big| \;: x \in A\} \;=\; |\lambda| \sup\{\, \big|\, f(x) \,\big| \;: x \in A\} \;=\; |\lambda| \|f\|. \tag{13.28}$$

This proves absolute homogeneity.

Finally we show positive definiteness. Clearly $\|f\|$ is nonnegative as the sup of nonnegative numbers $\big|\, f(x) \,\big|$. Assume that $\|f\| > 0$. Then $\delta := \frac{1}{2}\|f\| > 0$ and there exists $x_0 \in A$ such that

$$\sup\{\, \big|\, f(x) \,\big| \;: x \in A\} - \big|\, f(x_0) \,\big| < \delta, \;\text{ i.e., }\; \|f\| - \big|\, f(x_0) \,\big| < \delta, \;\text{ hence }\; \big|\, f(x_0) \,\big| > \delta. \tag{13.29}$$

Positive definiteness of $\big|\cdot\big|$ implies that $f(x_0) \neq 0$ and hence $f \neq 0$. We have proved positive definiteness of $\|\cdot\|$. ■

13.2 Function Sequences and Infinite Series

13.2.1 Convergence of Function Sequences

Notation 13.1 (Functions with argument "$\cdot$").

This chapter makes heavy use of the notation $f(\cdot)$ instead of f for a function $X \to \mathbb{R}$ to emphasize when sequences of functions $f_n(\cdot)$ are used and when function values (real numbers) $f_n(x)$ are used. ☐

Vectors are more complicated than numbers because an n–dimensional vector $v \in \mathbb{R}^n$ represents a list of only finitely many real numbers. Any such vector $(x_1, x_2, x_3, \cdots, x_n)$ can be interpreted as a real–valued function (remember: a real–valued function is one which maps it arguments into $\mathbb{R}$)

$$f(\cdot) : \{1, 2, 3, \cdots, n\} \to \mathbb{R} \qquad j \mapsto x_j \tag{13.30}$$

(see (**??**) on p.**??**).

Next come sequences $(x_j)_{j\in\mathbb{N}}$ which can be interpreted as real–valued functions

$$g(\cdot) : \mathbb{N} \to \mathbb{R} \qquad j \mapsto x_j. \tag{13.31}$$

Finally we deal with real–valued functions

$$h(\cdot) : X \to \mathbb{R} \qquad x \mapsto h(x) \tag{13.32}$$

which are defined on an arbitrary domain X as the most general case.

Now we add more complexity by not just dealing with one or two or three real–valued functions but with an entire sequence of functions

$$f_n(\cdot) : X \to \mathbb{R} \qquad x \mapsto f_n(x) \tag{13.33}$$

For any fixed argument x_0 we have a sequence $f_1(x_0), f_2(x_0), f_3(x_0), \cdots$ of real numbers which we can examine for convergence. This sequence may converge for some or all arguments $x_0 \in X$ to some limit $L = L(x_0) \in \mathbb{R}$. [9] Examination of the limit behavior of a function sequence is not only of interest if those functions are real–valued but also if their codomain is a metric space (Y, d).

It is time now for some definitions.

Definition 13.7 (Pointwise convergence of function sequences)**.**

Let X be a nonempty set, (Y, d) a metric space and let $f_n(\cdot) : X \to Y$ and $f(\cdot) : X \to Y$ be functions on X $(n \in \mathbb{N})$. Let $A \subseteq X$ be a nonempty subset of X.

We say that $f_n(\cdot)$ **converges pointwise** or, simply, **converges** to $f(\cdot)$ on A and we write $f_n(\cdot) \to f(\cdot)$ on A as $n \to \infty$, or simply $f_n(\cdot) \to f(\cdot)$ on A, if

$$f_n(x) \to f(x) \text{ as } n \to \infty \text{ for all } x \in A. \tag{13.34}$$

We omit the phrase "on A" if it is clear how A is defined, in particular if $A = X$. ☐

Definition 13.8 (Uniform convergence of function sequences)**.**

Let X be a nonempty set, (Y, d) a metric space, let $f_n(\cdot) : X \to Y$ and $f(\cdot) : X \to Y$ be functions on X $(n \in \mathbb{N})$, and let $A \subseteq X$.

We say that $f_n(\cdot)$ **converges uniformly** to $f(\cdot)$ on A and we write

$$f_n(\cdot) \xrightarrow{uc} f(\cdot) \text{ on } A^{10} \tag{13.35}$$

if, for each $\varepsilon > 0$ (no matter how small), there exists an index n_0 which can be chosen once and for all, <u>independently of the specific argument x</u>, such that

$$d(f_n(x), f(x)) < \varepsilon \quad \text{for all } x \in A \quad \text{and } n \geq n_0. \tag{13.36}$$

We omit the phrase "on A" if it is clear how A is defined, in particular if $A = X$. ☐

Remark 13.12 (Uniform convergence implies pointwise convergence)**.** Take another look at definition Definition **??** (convergence of sequences in metric spaces) on p.**??**. Note that (13.36) implies, for any given $x \in A$, ordinary convergence $f(x) = \lim\limits_{n\to\infty} f_n(x)$. The reason is that the number $n_0 = n_0(\varepsilon)$ chosen in (13.36) will also satisfy (**??**) of that definition for $x_n = f_n(x)$ and $a = f(x)$.

---

[9] We previously examined sequences of functions in ch.**??** (Sequences of Sets and Indicator functions and their liminf and limsup) on p.**??**.

[10] Note that the notation " $f_n(\cdot) \overset{uc}{\to} f(\cdot)$ " is not very widely used.

In other words, unform convergence implies pointwise convergence. But what is the difference between pointwise and uniform convergence? The difference is that, for poinwise convergence, the number n_0 will depend on both ε and x: $n_0 = n_0(\varepsilon, x)$. In the case of uniform convergence, the number n_0 will still depend on ε but can be chosen independently of the argument $x \in A$. ☐

Example 13.4 (Constant sequence of functions)**.** Let X be a set and let $f : X \to \mathbb{R}$ be a real–valued function on X which may or may not be continuous anywhere. Define a sequence of functions

$$f_n : X \to \mathbb{R}\ (n \in \mathbb{N}) \quad \text{as} \quad f_1 \ = \ f_2 \ = \ \cdots = \ f$$

i.e.,

$$f_1(x) \ = \ f_2(x) \ = \ \cdots = f(x)\ \forall n \in \mathbb{N},\ \forall x \in X.$$

In other words, we are looking at a constant sequence of functions (not to be confused with a sequence of constant functions – seriously!).

We obtain $d(f_n(x), f(x)) = 0 < \varepsilon$ for all $x \in X$ and $\varepsilon > 0$. Thus (13.36) in the definition of uniform convergence is trivially satisfied, hence $f_n(\cdot) \overset{uc}{\to} f(\cdot)$. ☐

PROOF of the example: This is trivial. No matter how small an ε and n_0 we choose and no matter what argument $x \in X$ we are looking at, we have

$$|f_n(x) - f(x)| = 0 < \varepsilon \quad \text{for all } x \in A \text{ and } n > n_0 \ \blacksquare$$

Next comes an example of a function sequence that converges pointwise, but not uniformly. The reader is suggested to draw a picture of the functions f_n.

Example 13.5. Let $X = [0, 1]$,
i.e., X is the closed unit interval $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Let the functions f_n be defined as follows on X:

$$f_n(x) = \begin{cases} n^2 x & \text{for } 0 \leq x \leq \frac{1}{n} \\ \frac{1}{x} & \text{for } \frac{1}{n} \leq x \leq 1 \end{cases}$$

Let the function $f(\cdot) : [0, 1] \to \mathbb{R}$ be defined as follows.

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } 0 < x \leq 1 \\ 0 & \text{for } x = 0 \end{cases}$$

Then the functions $f_n(\cdot)$ converge pointwise but not uniformly to $f(\cdot)$ on the entire unit interval. ☐

PROOF of the example:

Before we start, note that both pieces of f_n fit together in the point $x = 1/n$ because the "$\frac{1}{x}$ definition" gives $f_n(a) = \frac{1}{1/n} = n$ and the "$n^2 x$ definition" gives the same value $n = n^2 \frac{1}{n}$. We encourage you to draw a picture to convince yourself that $f_n(\cdot)$ is continuous at every point of $[0, 1]$. You are asked in exercise 13.3 on p.636 to give a proof of the continuity of f_n. Finally note that the limit function f is not continuous at all points of $[0, 1]$.

PROOF of pointwise convergence:
first we inspect the point $a = 0$. We have $f(0) = 0 = n^2 \cdot 0 = f^n(0)$ and the constant sequence of

zeros certainly converges to zero. Now assume $a > 0$. If $n > 1/a$ then $f_n(a) = \frac{1}{a}$ for all such n. We have a constant sequence $(\frac{1}{a})$ except for the first finitely many n and this sequence converges to $\frac{1}{a} = f(a)$. See cor.**??** on p.**??**. We have thus proved pointwise convergence.

PROOF that there is no uniform convergence:

To prove that (13.36) is not satisfied, we must find $\varepsilon > 0$ and points x_N so that for no matter how big a natural number N we choose, there will be at least one $j > N$ such that $|f_j(x_N) - f(x_N)| \geq \varepsilon$. Let $N \in \mathbb{N}$ be any natural number and let $x_N := \frac{1}{N^2}$. Then

$$\begin{aligned} f_N(x_N) &= \frac{N^2}{N^2} = 1, \\ f_{2N}(x_N) &= \frac{(2N)^2}{N^2} = 4. \end{aligned}$$

Hence

$$\left| f_{2N}(x_N) - f_N(x_N) \right| = 3.$$

To recap: We found $\varepsilon > 0$ so that for each $N \in \mathbb{N}$ there is at least one $j \geq N$ and $x_N \in [0,1]$ such that $|f_j(X_N) - f_N(x_N)| > \varepsilon$: we chose

$$\varepsilon = 2, \quad j = 2N, \quad x_N = \frac{1}{N^2}$$

We have proved that convergence is pointwise but not uniform. ■

Let X be a nonempty set and $\mathscr{B}(X, \mathbb{R})$ the set of all bounded real–valued functions on X. We recall from Theorem 13.5 ($\|f\|$ is a norm) on p.615, that $\mathscr{B}(X, \mathbb{R})$ is a vector space with the norm

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

and it is a metric space with the corresponding metric

$$d_{\|\cdot\|_\infty}(f,g) = \sup\{|g(x) - f(x)| : x \in X\}$$

(see example **??** on p.**??**).

Proposition 13.6 (Uniform convergence is $\|\cdot\|_\infty$ convergence)**.**

The following is true for any a nonempty set X and $f_n, f \in \mathscr{B}(X, \mathbb{R})$:

$$\begin{aligned} f_n(\cdot) \overset{uc}{\to} f(\cdot) \quad &\Leftrightarrow \quad f_n(\cdot) \overset{\|\cdot\|_\infty}{\to} f(\cdot), \quad \textit{i.e.,} \\ f_n(\cdot) \overset{uc}{\to} f(\cdot) \quad &\Leftrightarrow \quad f_n \textit{ converges to } f \textit{ in the metric space } \left(\mathscr{B}(X, \mathbb{R}), d_{\|\cdot\|_\infty}(\cdot,\cdot)\right). \end{aligned}$$

PROOF of "⇒": Assume that $f_n(\cdot) \overset{uc}{\to} f(\cdot)$. Let $\varepsilon > 0$. According to Definition 13.8 (Uniform convergence of function sequences) on p.617, there exists an index $n_0 = n_0(\varepsilon)$ (which does not depend on the function argument $x \in X$) such that

$$d(f_n(x), f(x)) = |f_n(x) - f(x)| < \varepsilon/2 \quad \text{for all } x \in X \quad \text{and } n \geq n_0.$$

Note that here the metric space Y in Definition 13.8 is $\mathbb{R}$, so $d(f_n(x), f(x))$ becomes $|f_n(x) - f(x)|$. We obtain

$$\|f_n - f\|_\infty \;=\; \sup\{|f_n(x) - f(x)| : x \in X\} \;\leq\; \varepsilon/2 \text{ for all } n \geq n_0,$$

i.e., $d_{\|\cdot\|_\infty}(f_n, f) < \varepsilon$ for all $n \geq n_0$. It follows that $f_n(\cdot) \overset{\|\cdot\|_\infty}{\to} f(\cdot)$.

PROOF of "$\Leftarrow$": Assume that $f_n \overset{\|\cdot\|_\infty}{\to} f$, i.e., $\lim\limits_{n\to\infty} f_n \;=\; f$ in the metric space $\big(\mathscr{B}(X,\mathbb{R}), d_{\|\cdot\|_\infty}\big)$. Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$d_{\|\cdot\|_\infty}(f_n, f) \;=\; \|f_n - f\|_\infty \;=\; \sup\{|f_n(x) - f(x)| : x \in X\} \;<\; \varepsilon \quad \text{for all } n \geq n_0$$

But then

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } x \in X \text{ and all } n \geq n_0.$$

This proves $f_n(\cdot) \overset{uc}{\to} f(\cdot)$. ■

The last proposition justifies the next definition.

Definition 13.9 (Norm and metric of uniform convergence)**.**

⋆

We also call the sup–norm on $\mathscr{B}(X,\mathbb{R})$ the **norm of uniform convergence** on X and its associated metric $d_{\|\cdot\|_\infty}(\cdot,\cdot)$ the **metric of uniform convergence** on X. ☐

Theorem 13.6 (Uniform limits of continuous functions are continuous)**.**

Let (X, d_1) and (Y, d_2) be metric spaces and let $f_n(\cdot) : X \to Y$ and $f(\cdot) : X \to Y$ be functions on X $(n \in \mathbb{N})$. Let $x_0 \in X$ and let $V \subseteq X$ be a neighborhood of x_0. Assume the following:

(a) *The functions $f_n(\cdot)$ are continuous at x_0 for all n.*

(b) *$f_n(\cdot) \overset{uc}{\to} f(\cdot)$ on V.*

Then, f is continuous at x_0

PROOF: Let $\varepsilon > 0$.

(A) Uniform convergence $f_n(\cdot) \overset{uc}{\to} f(\cdot)$ on V guarantees the existence of some $N = N(\varepsilon)$ such that

$$d_2\big(f_n(x), f(x)\big) < \frac{\varepsilon}{3} \text{ for all } x \in V \text{ and } n \geq \mathbb{N}.$$

In particular, for $n = N$,

$$d_2\big(f_N(x), f(x)\big) < \frac{\varepsilon}{3} \text{ for all } x \in V\ . \tag{13.37}$$

(B) All functions f_n and in particular f_N are continuous in V. There is $\tilde{\delta} > 0$ such that

$$d_2(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3} \text{ for all } x \in N_{\tilde{\delta}}(x_0). \tag{13.38}$$

(C) As x_0 is an interior point of V, there exists $\hat{\delta} > 0$ such that $N_{\hat{\delta}}(x_0) \subseteq V$. Let δ be the smaller of $\hat{\delta}$ and $\tilde{\delta}$.

Then (13.37) and (13.38) both hold for any $x \in N_\delta(x_0)$. Because $x_0 \in N_\delta(x_0)$ we obtain

$$d(f(x), f(x_0)) \;\leq\; d(f(x), f_N(x)) \;+\; d(f_N(x), f_N(x_0)) \;+\; d(f_N(x_0), f(x_0)) \;<\; \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The proof is finished. $\blacksquare$

For an example of uniform convergence we return to the n–th Bernstein Polynomials

$$B_n^f(x) = \sum_{k=0}^{n} \binom{n}{k} f\Big(\frac{k}{n}\Big) x^k (1-x)^{n-k},$$

which are defined for any $f : [0,1] \to \mathbb{R}$. It will be shown in ch.**??** (The Weierstrass Approximation Theorem), that if f is **any** continuous function on the unitinterval, then $B_n^f(\cdot) \overset{uc}{\to} f(\cdot)$ on $[0,1]$ as $n \to \infty$.

We have done already most of the work to prove this for the three continuous functions $x \to 1$, $x \to x$, and $x \to x^2$.

Proposition 13.7.

> ⋆ *Let $f : [0,1] \to \mathbb{R}$ be one of the functions*
>
> $$1 : x \mapsto 1; \qquad id : x \mapsto x; \qquad id^2 : x \mapsto x^2; \qquad (0 \leq x \leq 1).$$
>
> *Then,*
>
> $$B_n^f(\cdot) \overset{uc}{\to} f(\cdot) \text{ on } [0,1] \text{ as } n \to \infty.$$

PROOF: We derived in prop.**??** on p.**??** the formulas

$$B_n^1(x) \;= 1, \qquad B_n^{id}(x) \;= x, \qquad B_n^{id^2}(x) \;=\; \frac{1}{n}x \;+\; \frac{n-1}{n}x^2 \qquad\qquad (x \in \mathbb{R}).$$

Note that $\big(B_n^1\big)_n$ is the constant function sequence $p_n^1(\cdot) = 1$, and $\big(B_n^{id}\big)_n$ is the constant function sequence $B_n^{id}(\cdot) = id$. We have seen in example 13.4 (Constant sequence of functions) on p.618 that any constant function sequence has itself as uniform limit, thus the proposition is true for the functions 1 and id.

The function $id^2 : x \mapsto x^2$ needs a little more work. Let $\varepsilon > 0$. If $0 \leq x \leq 1$ then

$$\begin{aligned} d\big(B_n^{id^2}(x), id^2(x)\big) \;&= \Big|\Big(\frac{1}{n}x \;+\; \frac{n-1}{n}x^2\Big) \;-\; x^2\Big| \\ &= \Big|\frac{1}{n}x \;-\; \frac{1}{n}x^2\Big| \;=\; \frac{1}{n} \cdot |x| \cdot |1-x| \;\leq\; \frac{1}{n}. \end{aligned}$$

We choose $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\varepsilon}$. This is always possible since the natural numbers are not bounded above in $\mathbb{R}$. Let $n \geq n_0$. Then $\frac{1}{n} \leq \frac{1}{n_0} < \varepsilon$, hence $d\big(B_n^{id^2}(x), id^2(x)\big) < \varepsilon$ for all $x \in [0,1]$. It follows that (13.36) in the definition of uniform convergence is satisfied, hence $B_n^{id^2} \overset{uc}{\to} id$. ■

Proposition 13.8.

> *Let X be a nonempty set, (Y,d) a metric space and let $f_n, f : X \to Y \quad (n \in \mathbb{N})$. Then*
> *f is the uniform limit of the function sequence $(f_n)_n$*
> *$\Leftrightarrow$ there exists a sequence $\delta_n \geq 0$ such that **1)** $\delta_n \to 0$ as $n \to \infty$, and*
> ***2)** $d\big(f_n(x), f(x)\big) \leq \delta_n$ for all $x \in X$ and $n \in \mathbb{N}$.*

PROOF:

(A) First we prove that uniform convergence $f_n \overset{uc}{\to} f$ implies that there are real numbers $\delta_n \geq 0$ that satisfy both **(1)** and **(2)**: It follows from Definition 13.8 on p.617 (Uniform convergence of function sequences) that the numbers $\delta_n := \sup\{d\big(f_n(x), f(x)\big) : x \in X\}$ converge to zero and thus define such a sequence.

(B) We now prove that the existence of a sequence $\delta_n \geq 0$ that satisfies both **(1)** and **(2)** implies $f_n \overset{uc}{\to} f$ on X. Let $\varepsilon > 0$. It follows from $\lim\limits_{k\to\infty} \delta_k = 0$ that there exists $n_0 \in \mathbb{N}$ such that $\delta_k < \varepsilon$ for all $k \geq n_0$. Thus $d\big(f_k(x), f(x)\big) \leq \delta_k < \varepsilon$ for all $x \in X$ and $k \geq n_0$. It follows from Definition 13.8 that $f_n \overset{uc}{\to} f$ on X. ■

13.2.2 Infinite Series

We start by repeating the definition of a sequence given in section **??** on p.**??**: (x_j) is nothing but a family of things x_j which are indexed by a consecutive set of integers, usually the natural numbers or the nonnegative integers. We make throughout this chapter on infinite series the following assumption:

> Unless explicitly stated otherwise, sequences are always indexed $1, 2, 3, \ldots$, i.e., the first index is 1 and, given any index, you obtain the next one by adding 1 to it. ☐

Proposition 13.9 (Convergence criteria for series)**.**

> *A series $s := \sum a_k$ of real numbers converges if and only if for all $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that one of the following is true:*
>
> (13.39a) $$\left|\sum_{k=n}^{\infty} a_k\right| < \varepsilon \qquad \text{for all } n \geq n_0$$
>
> (13.39b) $$\left|\sum_{k=n}^{m} a_k\right| < \varepsilon \qquad \text{for all } m,n \geq n_0$$

PROOF: Write

$$s = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{n} a_k + \sum_{k=n+1}^{\infty} a_k = s_n + \sum_{k=n+1}^{\infty} a_k \tag{13.40}$$

Remember the convergence criteria for real–valued sequences. Convergence of a sequence (s_n) to a real number s means that, for any $\varepsilon > 0$, all but finitely many members s_n will be inside the ε–neighborhood $N_\varepsilon(s)$ of s. Expressed in terms of the distance to s this means there exists a suitable $n_0 \in \mathbb{N}$ such that

$$|s - s_n| < \varepsilon \qquad \text{for all } n \geq n_0$$

(see (**??**) on p.**??**). According to (13.40) we can write that as

$$\Big| \sum_{k=n+1}^{\infty} a_k \Big| < \varepsilon \qquad \text{for all } n \geq n_0,$$

which is the same as (13.39.a) because it does not matter whether we look at the sum of all terms bigger than n or $n+1$.

Alternatively, there was the Cauchy criterion

$$|s_i - s_j| < \delta \text{ for all } i, j \geq n_0$$

(see (**??**) on p.**??**) which ensures convergence to some number s without specifying what it might actually be. Again we use (13.40) and obtain, assuming without loss of generality that $i < j$,

$$\Big| \sum_{k=i+1}^{j} a_k \Big| < \delta \text{ for all } j > i \geq n_0 \quad \blacksquare$$

Corollary 13.1.

If a series $\sum a_j$ *converges then* $\lim\limits_{n\to\infty} a_n = 0.$

PROOF: Let $\varepsilon > 0$. It follows from 13.39b that there is some $n_0 \in \mathbb{N}$ such that $|a_m - 0| = \big| \sum_{k=m}^{m} a_k \big| < \varepsilon$ for all $m \geq n_0$. But this means that the sequence a_n converges to zero. $\blacksquare$

Here is a second corollary. It is a generalization of [1] B/G (Beck/Geoghegan) prop.12.3, p.115.

Corollary 13.2 (Dominance criterion for series)**.**

Let $N \in \mathbb{N}$ *and let* $\sum a_j$ *and* $\sum b_j$ *be two series such that* $|b_k| \leq a_k$ *for all* $k \geq N$*.*
It follows that if $\sum a_k$ *converges, then* $\sum b_k$ *converges.*

Moreover, if $|b_k| \leq a_k$ *for all* $k \in \mathbb{N}$*, then,* $\Big|\sum_{k=1}^{\infty} b_j\Big| \leq \sum_{k=1}^{\infty} a_j$

PROOF: Let $\varepsilon > 0$. It follows from 13.39b that there is some $n_0 \in \mathbb{N}$ such that $\left| \sum\limits_{k=m}^{n} a_k \right| < \varepsilon$ for all $m, n \geq n_0$. Let $M := \max(n_0, N)$. We obtain

$$\left| \sum_{k=i+1}^{j} b_j \right| \leq \sum_{k=i+1}^{j} |b_j| \leq \sum_{k=i+1}^{j} a_j < \varepsilon \quad \text{for all } j > i \geq M.$$

We conclude from (13.39b) that $\sum b_k$ converges.

Now assume that $|b_k| \leq a_k$ for all $k \in \mathbb{N}$. Let

$$s_n := \sum_{k=1}^{n} |a_k|, \quad s := \lim_{n\to\infty} s_n, \qquad t_n := \sum_{k=1}^{n} b_k, \quad t := \lim_{n} t_n\,.$$

It follows from the triangle inequality for real numbers that $|t_n| \leq s_n$ for all $n \in \mathbb{N}$. We apply prop.**??** on p.**??** to deduce that

$$|t| = \lim_{n} |t_n| \leq \lim_{n} s_n = s\,.$$

This completes the proof. ■

Remark 13.13.

It is very important to remember that a series either converges to a finite number or it diverges. If it diverges it may be the case that $\sum\limits_{k=1}^{\infty} a_k = \infty$ or $\sum\limits_{k=1}^{\infty} a_k = -\infty$ or there is no limit at all. As an example for a series which has no limit, look at the oscillating sequence

$$a_0 = 1; \quad a_1 = -1; \quad a_2 = 1; \quad a_3 = -1; \cdots \quad s_n = \sum_{k=0}^{n} (-1)^n \tag{13.41}$$

The above is an example of a series that starts with an index other than 1 (zero). s_n obviously does not have limit $+\infty$ or $-\infty$ because s_n is 1 for all even n and 0 for all odd n. Do not make the mistake of thinking that the limit of the series is zero because you fail to notice the odd indices and only see that $s_0 = s_2 = s_4 = \cdots = s_{2j} = 0$.

Note that for any $j \in \mathbb{N}$ we have $|s_j - s_{j-1}| = 1$ because at each step we either add or subtract 1. This means that no matter what real number a and how big a number $n_0 \in \mathbb{N}$ we choose, it will never be true that $|a - s_j| < 1$ for all $j \in \mathbb{N}$ and a cannot be a limit of the series. [11]

Just so you understand the difference between limits and contact points (see (Definition **??**) on p.**??**): Even though neither $(a_j)_j$ nor $(s_j)_j$ has a limit, the tail sets for both have two contact points each. The ones for $(a_j)_j$ have the contact points $\{1, -1\}$ and the ones for $(s_j)_j$ have the contact points $\{0, 1\}$. □

We now turn our attention to convergence properties of series. We copy from Chapter **??** (Cardinality I: Finite and Countable Sets) the notation $[N] = [1, N]_{\mathbb{N}}$ for $N \in \mathbb{N}$.

---

[11] We could also have concluded as follows: $|s_j - s_{j-1}| = 1$ implies that the Cauchy formulation of the convergence criteria for series (see (13.39a) on p.622) is not satisfied, hence no convergence of the series.

Definition 13.10 (Finite permutations)**.**

☆ Let $N \in \mathbb{N}$. A **permutation** of $[N]$ is a bijection

$$\pi(\cdot) : [N] \to [N]; \qquad j \mapsto \pi(j)\,.$$

As usual

$$\pi^{-1}(\cdot) : [N] \to [N]; \qquad \pi(j) \mapsto \pi^{-1}\pi(j) = j,$$

denotes the inverse function of $\pi(\cdot)$. We recall that it associates with each image $\pi(j)$ the unique argument j, which is mapped by $\pi(\cdot)$ to $\pi(j)$. It is customary to write

$$i_1 \text{ instead of } \pi(1), \quad i_2 \text{ instead of } \pi(2), \quad \dots, \quad i_j \text{ instead of } \pi(j), \quad \dots\,. \ \square$$

We extend the previous definition from $[N]$ to $\mathbb{N}$.

Definition 13.11 (Permutations of $\mathbb{N}$)**.**

A **permutation** of $\mathbb{N}$ is a bijective function

$$\pi(\cdot) : \mathbb{N} \to \mathbb{N}; \qquad j \mapsto \pi(j). \ \square$$

Permutations are the means of describing a **rearrangement** or **reordering** of the members of a finite or infinite sequence or series. Look at any sequence (a_j). Given a permutation $\pi(\cdot)$ of the natural numbers, we can form the sequence $(b_k) := (a_{\pi(k)})$, i.e.,

$$b_1 = a_{\pi(1)}, \quad b_2 = a_{\pi(2)}, \quad \dots, \quad b_k = a_{\pi(k)}, \quad \dots$$

We can use the inverse permutation, $\pi^{-1}(\cdot)$, to regain the a_j from the b_j because

$$b_{\pi^{-1}(k)} \;=\; a_{\pi^{-1}(\pi(k))} \;=\; a_k.$$

Proposition 13.10.

Let (a_n) be a sequence of nonnegative real numbers. Exactly one of the following is true:

(a) *Either the series $\sum a_n$ converges (to a finite number). In that case,*

$$\sum_{n=1}^{\infty} a_n \;=\; \sum_{n=1}^{\infty} a_{\pi(n)} \quad \textit{for } \textbf{\textit{any}} \textit{ permutation } \pi(\cdot) \textit{ of } \mathbb{N}.$$

(b) *Or the series $\sum\limits_{n=1}^{\infty} a_n$ has limit ∞. In that case, it is true for **any** permutation $\pi(\cdot)$ of $\mathbb{N}$, that*

the reordered series $\sum\limits_{n=1}^{\infty} a_{\pi(n)}$ also has limit ∞.

PROOF of **(a)**: Let $b_j := a_{\pi(j)}$ and, hence, $a_k = b_{\pi^{-1}(j)}$. Let $N \in \mathbb{N}$. Let

$$\alpha \ := \ \max\{\pi(j) : j \leq N\} \qquad \text{and} \qquad \beta \ := \ \max\{\pi^{-1}(k) : k \leq N\}. \tag{13.42}$$

Note that $\alpha \geq N$ and $\beta \geq N$. Because all terms a_j, b_k are nonnegative it follows that

$$\sum_{j=1}^{N} b_j \ = \ \sum_{j=1}^{N} a_{\pi(j)} \ \leq \ \sum_{k=1}^{\alpha} a_k \ \leq \ \sum_{k=1}^{\alpha} a_k \ + \ \sum_{k=\alpha+1}^{\infty} a_k \ = \ \sum_{k=1}^{\infty} a_k,$$
$$\sum_{k=1}^{N} a_k \ = \ \sum_{k=1}^{N} b_{\pi^{-1}(k)} \ \leq \ \sum_{j=1}^{\beta} b_j \ \leq \ \sum_{j=1}^{\beta} b_j \ + \ \sum_{j=\beta+1}^{\infty} b_j \ = \ \sum_{j=1}^{\infty} b_j.$$

We take limits as $N \to \infty$ and it follows from prop.**??** on p.**??** that

$$\sum_{j=1}^{\infty} b_j \ \leq \ \sum_{k=1}^{\infty} a_k \quad \text{and} \quad \sum_{k=1}^{\infty} a_k \ \leq \ \sum_{j=1}^{\infty} b_j, \quad \text{hence} \quad \sum_{k=1}^{\infty} a_k \ = \ \sum_{j=1}^{\infty} b_j.$$

This proves part **(a)** of the proposition.

PROOF of **(b)**: Assume that $\sum a_j$ diverges. Because all terms a_j are nonnegative, the sequence s_n of the partial sums is nondecreasing and hence has a limit s. $s \notin \mathbb{R}$ because we assumed that $\sum a_j$ is not convergent and we can rule out $s = -\infty$ because $s \geq a_0 \geq 0$. It follows that $s = \infty$.

Assume to the contrary that there is a rearrangement $\sum b_j := \sum a_{\pi(j)}$ of $\sum a_j$ which converges to a limit $t \in \mathbb{R}$. According to the already proved part **(a)** the rearrangement $\sum a_j = \sum b_{\pi^{-1}(j)}$ converges to the same (finite) limit t. We have reached a contradiction. ■

Definition 13.12 (absolutely convergent series)**.**

A series $\sum a_j$ is **absolutely convergent**, if the corresponding series $\sum |a_j|$ of its absolute values converges. ☐

Proposition 13.11.

Let $\sum a_k$ be an absolutely convergent series. Then $\sum a_k$ converges and

$$\left|\sum_{k=1}^{\infty} a_k\right| \ \leq \ \sum_{k=1}^{\infty} |a_k|. \tag{13.43}$$

PROOF: This follows from the dominance criterion (cor.13.2) ■

It follows from prop.13.10 on p.625 that if a series of nonnegative terms converges then its value is invariant under rearrangements of that series. The next theorem states that any absolutely convergent series also has that property. It was proved by the German mathematician Peter Gustav Lejeune Dirichlet (1805-1859). We will see later [12] that the reverse is also true: Any series whose value is invariant under rearrangements is absolutely convergent.

---

[12] see cor.13.4 on p.635

Theorem 13.7.

Let $\sum a_k$ be an absolutely convergent series. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a permutation of $\mathbb{N}$, i.e., the series $\sum b_k$ with $b_k := a_{\pi(k)}$ is a rearrangement of the series $\sum a_k$. Then $\sum b_k$ converges and has the same limit as $\sum a_k$. (Note that $\sum a_k$ converges according to Proposition 13.11.)

PROOF: As a first step we prove that $\sum b_k$ converges: Since $\sum |b_k|$ is a rearrangement of $\sum |a_k|$, $\sum |b_k|$ converges by Proposition 13.10 on p.625. By Proposition 13.11, $\sum b_k$ converges.

Let $s := \sum\limits_{k=1}^{\infty} a_k$, $t := \sum\limits_{k=1}^{\infty} b_k$. For $n \in \mathbb{N}$ let $s_n := \sum\limits_{k=1}^{n} a_k$ and $t_n := \sum\limits_{k=1}^{n} b_k$.

Let $\varepsilon > 0$. Since $\sum |a_k|$ converges, there exists $n_0 \in \mathbb{N}$ such that

$$\sum_{k=n_0+1}^{n_0+m} |a_k| \ \leq \ \sum_{k=n_0+1}^{\infty} |a_k| \ < \ \varepsilon \quad \text{for all } m \in \mathbb{N}. \tag{13.44}$$

Let $A := \{\pi(j) : 1 \leq j \leq n_0\}$ and $p_0 := \max(A)$. This maximum exists because the set A is finite.

Then $p_0 \geq n_0$. Each of $a_1, a_2, \dots, a_{n_0}$ is a term of s_{n_0}, hence of s_{p_0}.
Moreover each of $b_1 = a_{\pi(1)}, b_2 = a_{\pi(2)}, \dots, b_{p_0} = a_{\pi(p_0)}$ is a term of t_{p_0}.

Let $n, p \geq p_0$. Then each of $a_1, a_2, \dots, a_{n_0}$ is a term of s_n
and each of $b_1 = a_{\pi(1)}, b_2 = a_{\pi(2)}, \dots, b_{p_0} = a_{\pi(p_0)}$ is a term of t_p.

$p_0 = \max(A)$ is so big that each of $a_1, \dots, a_{n_0}$ is one of $b_1, \dots, b_{p_0}$.

It follows from all this that each of $a_1, \dots, a_{n_0}$ is a term both of s_n and t_p, hence none of those terms appears in the difference $s_n - t_p$. We obtain from (13.44) for big enough $m \in \mathbb{N}$ (the bigger of $\max(\{\pi(j) : 1 \leq j \leq n\})$ and p) that

$$|s_n - t_p| \leq \sum_{k=n_0+1}^{n_0+m} |a_k| \ < \ \varepsilon.$$

This implies

$$|s - t_p| \leq |s - s_n| + |s_n - t_p| \ \leq \ |s - s_n| \ + \sum_{k=n_0+1}^{n_0+m} |a_k| \ < \ |s - s_n| + \varepsilon.$$

We had chosen $n \geq n_0$ and it follows from (13.44) that $|s - s_n| < \varepsilon$, hence $|s - t_p| < 2\varepsilon$.

But p could be any integer $\geq p_0$, and p_0 only depends (via n_0) on ε.

To summarize: for all $\varepsilon > 0$ there exists p_0 such that $p \geq p_0$ implies $|s - t_p| < 2\varepsilon$. But then $\lim\limits_{p \to \infty} t_p = s$. On the other hand, $\lim\limits_{p \to \infty} t_p = t = \sum\limits_{p \to \infty} b_k$.

This concludes the proof that $\sum\limits_{p \to \infty} a_k = \sum\limits_{p \to \infty} b_k$. ■

Proposition 13.12.

Let $\sum a_n$ be an absolutely convergent series. Let $(a_{n_k})_k$ be a subsequence of $(a_n)_n$. Then, $\sum a_{n_k}$ converges absolutely.

PROOF: The proof is left as exercise 13.10. ■

The last proposition allows us to use the following simplified summation notation for absolutely convergent series.

Remark 13.14.

Assume that $\sum a_n$ is absolutely convergent. Let $n_1 < n_2 < \dots$ be a subsequence of all natural numbers and let $J := \{n_j : j \in \mathbb{N}\}$.

- Then we write $\displaystyle\sum_{j\in J} a_{n_j} := \sum_{j=1}^{\infty} a_{n_j}$.
- In particular, we write $\displaystyle\sum_{j\in\mathbb{N}} a_j := \sum_{j=1}^{\infty} a_j$, for the full sequence $n_j = j$ of indices. □

There are series which are convergent but not absolutely convergent. They are given a special name:

Definition 13.13 (conditionally convergent series)**.**

A series $\sum a_j$ is called **conditionally convergent**, if it is convergent but not absolutely convergent. □

We introduce alternating series to give a simple example of a conditionally convergent series.

Definition 13.14 (Alternating Series)**.**

⋆

A series $\sum a_j$ is called an **alternating series** if it is of the form $\sum(-1)^j a_j$ with either all terms a_j being strictly positive or all of them being strictly negative. □

Proposition 13.13 (Leibniz Test for Alternating Series)**.**

Let $a_1 \geq a_2 \geq \dots \downarrow 0$ be a nonincreasing sequence which decreases to zero. Then, the alternating series $\sum(-1)^k a_k$ converges.

PROOF: For each $n \in \mathbb{N}$ we have

$$\begin{aligned} s_{2n+1} &= s_{2n-1} + \big(s_{2n} - s_{2n+1}\big) \ \geq s_{2n-1}, \\ s_{2n+2} &= s_{2n} - \big(s_{2n+1} - s_{2n+2}\big) \ \leq s_{2n}, \\ s_{2n-1} &\leq s_{2n+1} = \big(s_{2n} - a2n + 1\big) \ \leq s_{2n}. \end{aligned}$$

Hence, if $k, n \in \mathbb{N}$ such that $k \geq n$ then

$$s_{2n+1} \leq s_{2k+1} \leq s_{2k} \leq s_{2n}, \qquad |s_{2n} - s_{2n+1}| = s_{2n} - s_{2n+1} = a_{2n+1}. \tag{13.45}$$
(13.46)

Let $\varepsilon > 0$. It follows from $\lim\limits_{n \to \infty} a_n = 0$ that there exists $n_0 \in \mathbb{N}$ such that $a_j < \varepsilon$ for all $j \geq n_0$. Let $N := 2n_0 + 1$. Let $i, j \in \mathbb{N}$ such that $i \geq N$. Then either $i = 2k$ or $i = 2k + 1$ for some suitable natural number $k \geq n_0$. Likewise, either $j = 2'$ or $j = 2k' + 1$ for some suitable natural number $k' \geq n_0$.

It follows from (13.45) that $s_{2n_0+1} \leq s_i, s_j \leq s_{2n_0}$. Because $|s_{2n_0} - s_{2n_0+1}| = a_{2n_0+1} < \varepsilon$, we have proven that the sequence s_n is Cauchy, hence converges because $\mathbb{R}$ is complete. ■

Example 13.6 (Alternating series)**.**

The series $\sum (-1)^n$ and the **alternating harmonic series** $\sum (-1)^n / n$ are examples of alternating series.

It is known from calculus that the **harmonic series** $\sum 1/n$ is divergent: $\sum\limits_{j=1}^{\infty} \dfrac{1}{n} = \infty$. On the other hand, according to the Leibniz test, $\sum (-1)^n / n$ converges. It follows that the alternating harmonic series is convergent but not absolutely convergent, i.e., it is conditionally convergent. ☐

We are going to prove Riemann's Rearrangement Theorem, from which it can be easily deduced that if $\sum a_j$ is conditionally convergent and $x \in \mathbb{R}$, a rearrangement $\sum a_{\pi_j}$ can be found which converges to x. In preparation we will prove the following lemma.

Lemma 13.2.

⋆ *Let $\sum a_k$ be a series. We split it into two series $\sum p_k$ and $\sum q_k$ as follows.*

- *p_j denotes the jth strictly positive member of the sequence $(a_k)_k$.*
- *q_j denotes the jth strictly negative member of that sequence.*

Then, the following is true:

(a) *If $\sum a_k$ is absolutely convergent, then both $\sum p_k$ and $\sum q_k$ are (absolutely) convergent.*

(b) *If $\sum a_k$ is conditionally convergent, then $\sum p_k$ has limit ∞ and $\sum q_k$ has limit $-\infty$.*

PROOF of **(a)**: Let $\alpha := \sum_{i=1}^{\infty} |a_i|$ and let $j \in \mathbb{N}$.

Let m be the index such that a_m is the jth (not mth!) strictly positive member of the sequence $(a_k)_k$. Then each p_i for $i \leq j$ is some $|a_k|$ for a suitable $k \leq m$. It follows from $m \geq j$ that

$$\sum_{i=1}^{j} p_i \ \leq \ \sum_{i=1}^{m} |a_i| \ \leq \ \sum_{i=1}^{\infty} |a_i| \ < \ \infty.$$

The above is true for all $j \in \mathbb{N}$ and it follows that $\sum\limits_{i=1}^{\infty} p_i \ < \ \infty$. The proof that $\sum q_k$ has a finite limit is similar.

PROOF of **(b)**: The proof will be done in three parts. In part 1 we will show that not both $\sum p_k$ and $\sum q_k$ can converge. In part 2 we will show that $\sum p_k = \infty$ and $\sum q_k \in \mathbb{R}$ leads to a contradiction. In part 3 we will show that $\sum q_k = -\infty$ and $\sum p_k \in \mathbb{R}$ leads to a contradiction.

Part 1: Let us assume that $\sum p_k < \infty$ and $\sum q_k > -\infty$.
For any $n \in \mathbb{N}$ we have

$$\sum_{k=1}^{n} |a_k| \ \leq \ \sum_{k=1}^{n} p_k \ + \ \sum_{k=1}^{n} (-q_k).$$

This is true because each one of $a_1, \dots, a_n$ is one of the first n strictly positive numbers $p_1, \dots, p_n$ or one of the strictly positive numbers $-q_1, \dots, -q_n$ or it is zero, in which case it contributes nothing to the series. Both series $\sum a_k$ and $\sum (-q_k)$ are nondecreasing, hence for each fixed n,

$$\sum_{k=1}^{n} |a_k| \ \leq \ \sum_{k=1}^{\infty} p_k \ - \ \sum_{k=1}^{\infty} q_k.$$

It follows that if both $\sum p_k$ and $\sum q_k$ are convergent then so is $\sum |a_k|$, i.e., this series is absolutely convergent. We have a contradiction.

Part 2: Let us assume that $\sum p_k = \infty$ and $\sum q_k \in \mathbb{R}$.
We fix $n \in \mathbb{N}$. Let M_n be the index of p_n, i.e., M_n is the smallest index j such that $a_j = p_n$. Note that

$$a_{M_n} \ = \ p_n \quad (\star) \qquad\qquad \text{and} \quad M_n \geq n. \quad (\star\star)$$

Let

$$I_n \ := \ \{ i \leq M_n : a_i > 0 \}, \qquad J_n \ := \ \{ j \leq M_n : a_i < 0 \}.$$

Then

$$\sum_{k=1}^{M_n} a_k \ = \ \sum_{i \in I_n} a_k \ + \ \sum_{j \in J_n} a_k \ \overset{(\star)}{=} \ \sum_{i=1}^{n} p_i \ + \ \sum_{j \in J_n} a_k \ \geq \ \sum_{i=1}^{n} p_i \ + \ \sum q_k. \tag{13.47}$$

It follows from $(\star\star)$ that if $n \to \infty$ then $M_n \to \infty$. Hence, from (13.47),

$$\sum a_k \ = \ \lim_{n\to\infty} \sum_{k=1}^{n} a_k \ = \ \lim_{n\to\infty} \sum_{k=1}^{M_n} a_k \ \geq \ \lim_{n\to\infty} \left(\sum_{i=1}^{n} p_i \ + \ \sum q_k \right) \ = \ \infty,$$

contrary to the assumption that $\sum a_k$ converges. We have reached a contradiction.

Part 3: Let us assume that $\sum q_k = \infty$ and $\sum p_k \in \mathbb{R}$.
We obtain a contradiction by applying part 2 to the series $\sum(-a_k)$. ■

The next theorem is due to the German mathematician Bernhard Riemann (1826-1866).

Theorem 13.8 (Riemann's Rearrangement Theorem)**.**

Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$. and let the series $\sum a_k$ be conditionally convergent. Then a rearrangement $\sum b_k$ of $\sum a_k$ exists such that

$$\liminf_{n\to\infty} \sum_{k=1}^{n} b_k \ = \ \alpha \qquad \text{and} \qquad \limsup_{n\to\infty} \sum_{k=1}^{n} b_k \ = \ \beta\,.$$

PROOF: ★

We may assume that $a_j \neq 0$ for all $j \in \mathbb{N}$ because terms of value zero do not contribute anything to the partial sums, hence omitting them leaves the limit of the series and any rearrangement unchanged.

We split $\sum a_j$ into the series $\sum p_j$ of its positive members and $\sum q_j$ of its negative members in the same way as was done in lemma 13.2:

p_j is the jth strictly positive member of the sequence $(a_k)_k$;
q_j is the jth strictly negative member of $(a_k)_k$.

It was proved in lemma 13.2 that $\displaystyle\sum_{k=1}^{\infty} p_k \ = \ \infty$ and $\displaystyle\sum_{k=1}^{\infty} q_k \ = \ -\infty$.

case 1: $\beta \geq 0$.

Let $U_1 := \{k \in \mathbb{N} : p_1 + p_2 + \cdots + p_k > \beta\}$. U_1 is not empty because $\sum p_j$ has limit ∞, hence $u_1 := \min(U_1)$ exists. We call the list $p_1, p_2, \dots, p_{u_1}$ the **first upcrossing** of the (unfinished) series $\sum b_k$.

We now construct the first piece of the desired rearrangement $\sum b_k$. Let

$$n_1 := u_1; \quad b_1 := p_1,\ b_2 := p_2,\ \dots,\ b_{n_1} := p_{u_1}; \quad \sigma_1 \ := \ \sum_{j=1}^{n_1} b_j.$$

Note that n_1 is the first (and so far, only) index n of the series $\sum b_k$ for which $\sum_{k=1}^{n} b_k$ exceeds β.

Let $L_1 := \{k \in \mathbb{N} : \sigma_1 + \sum_{j=1}^{k} q_j < \alpha\}$. L_1 is not empty because $\sum q_j$ has limit $-\infty$, hence $l_1 := \min(L_1)$ exists. We call the list $q_1, q_2, \dots, q_{l_1}$ the **first downcrossing** of $\sum b_k$.

We add more terms to $b_1, b_2, \dots, b_{n_1}$.

$$n_2 := n_1 + l_1; \quad b_{n_1+1} := q_1, b_{n_1+2} := q_2, \dots, b_{n_2} := q_{l_1}; \quad \sigma_2 \ := \ \sum_{j=1}^{n_2} b_j.$$

Note that n_2 is the first index n of $\sum b_k$ for which $\sum_{k=1}^{n} b_k$ drops below α.

Let $U_2 := \Big\{ k \in \mathbb{N} : k > u_1 \text{ and } \sigma_2 + \sum\limits_{j=u_1+1}^{u_1+k} p_j > \beta \Big\}$. U_2 is not empty because $\sum\limits_{j=u_1+1}^{\infty} p_j$ has limit ∞, hence $u_2 := \min(U_2)$ exists. We call $p_{u_1+1},\ p_{u_1+2}, \cdots,\ p_{u_2}$ the **second upcrossing** of $\sum b_k$.
We add more terms to $b_1, b_2, \dots, b_{n_2}$.

$$n_3 := n_2 + u_2; \quad b_{n_2+1} := p_{u_1+1},\ b_{n_2+2} := p_{u_1+2}, \dots,\ b_{n_3} := p_{u_1+u_2}; \quad \sigma_3 \ := \sum_{j=1}^{n_3} b_j.$$

Note that n_3 is the second index n of the series $\sum b_k$ for which $\sum_{k=1}^{n} b_k$ exceeds β.

Let $L_2 := \Big\{ k \in \mathbb{N} : k > l_1 \text{ and } \sigma_3 + \sum\limits_{j=l_1+1}^{l_1+k} q_j < \alpha \Big\}$. L_2 is not empty because $\sum\limits_{j=l_1+1}^{\infty} q_j$ has limit $-\infty$, hence $l_2 := \min(L_2)$ exists. We call $q_{l_1+1},\ q_{l_1+2}, \cdots,\ q_{l_2}$ the **second downcrossing** of $\sum b_k$.
We add more terms to $b_1, b_2, \dots, b_{n_3}$.

$$n_4 := n_3 + l_2; \quad b_{n_3+1} := q_{l_1+1},\ b_{n_3+2} := q_{l_1+2}, \dots,\ b_{n_4} := q_{l_1+l_2}; \quad \sigma_4 \ := \sum_{j=1}^{n_4} b_j.$$

Note that n_4 is the second index n of the series $\sum b_k$ for which $\sum_{k=1}^{n} b_k$ drops below α.
It should be clear how we proceed. Let us assume that we have constructed the Nth upcrossing $p_{u_{N-1}+1},\ p_{u_{N-1}+2}, \cdots,\ p_{u_N}$ and from it

$$\begin{aligned} n_{(2N-1)} &:= n_{(2N-2)} + u_N; \\ b_{(n_{(2N-2)}+1)} &:= p_{(u_{(N-1)}+1)},\ b_{(n_{(2N-2)}+2)} := p_{(u_{(N-1)}+2)},\ \dots,\ b_{(n_{(2N-1)})} := p_{u_N}, \\ \sigma_{(2N-1)} \ &:= \sum_{j=1}^{n_{(2N-1)}} b_j. \end{aligned}$$

Let us further assume that we have constructed the Nth downcrossing
$q_{l_{N-1}+1},\ q_{l_{N-1}+2}, \cdots,\ q_{l_N}$ and from it

$$\begin{aligned} n_{(2N)} &:= n_{(2N-1)} + l_N; \\ b_{(n_{(2N-1)}+1)} &:= q_{(l_{(N-1)}+1)},\ b_{(n_{(2N-1)}+2)} := q_{(l_{(N-1)}+2)},\ \dots,\ b_{n_{(2N)}} := q_{l_N}, \\ \sigma_{2N} \ &:= \sum_{j=1}^{n_{(2N)}} b_j. \end{aligned}$$

We proceed to construct the $(N+1)$th upcrossing and the $(N+1)$th downcrossing as follows.

Let $U_{N+1} := \Big\{ k \in \mathbb{N} : k > u_N \text{ and } \sigma_{2N} + \sum\limits_{j=u_N+1}^{u_N+k} p_j > \beta \Big\}$. U_{N+1} is not empty because $\sum\limits_{j=u_N+1}^{\infty} p_j$ has limit ∞, hence $u_{N+1} := \min(U_{N+1})$ exists. We call $p_{(u_N+1)},\ p_{(u_N+2)}, \cdots,\ p_{u_{(N+1)}}$ the $(N+1)$th upcrossing of $\sum b_k$.

We add more terms to $b_1, b_2, \dots, b_{n_{2N}}$.

$$\begin{aligned} n_{(2N+1)} &:= n_{(2N)} + u_{(N+1)};\\ b_{(n_{(2N)}+1)} &:= p_{(u_N+1)},\ b_{(n_{(2N)}+2)} := p_{(u_N+2)},\ \dots,\ b_{(n_{(2N+1)})} := p_{u_{(N+1)}},\\ \sigma_{(2N+1)} &:= \sum_{j=1}^{n_{(2N+1)}} b_j. \end{aligned}$$

Let $L_{N+1} := \Big\{k \in \mathbb{N} : k > l_N \text{ and } \sigma_{2N+1} + \sum\limits_{j=l_N+1}^{l_N+k} q_j < \alpha\Big\}$. L_{N+1} is not empty because $\sum\limits_{j=l_N+1}^{\infty} q_j$ has limit ∞, hence $l_{N+1} := \min(L_{N+1})$ exists. We call $q_{(l_N+1)},\ q_{(l_N+2)}, \cdots,\ q_{l_{(N+1)}}$ the $(N+1)$th downcrossing of $\sum b_k$.

We add more terms to $b_1, b_2, \dots, b_{n_{(2N+1)}}$.

$$\begin{aligned} n_{(2(N+1))} &:= n_{(2N+1)} + l_{(N+1)};\\ b_{(n_{(2N+1)}+1)} &:= q_{(l_N+1)},\ b_{(n_{(2N+1)}+2)} := q_{(l_N+2)},\ \dots,\ b_{(n_{2(N+1)})} := q_{l_{(N+1)}},\\ \sigma_{2(N+1)} &:= \sum_{j=1}^{n_{2(N+1)}} b_j. \end{aligned}$$

We have defined by recursion $\sum\limits_{k=1}^{n_m} b_k$ for all $m \in \mathbb{N}$

We now show that the increasing sequence $(n_m)_{m\in\mathbb{N}}$ is not bounded above. We observe that $n_{(2m)}$ is the number of terms that belong to the first m upcrossings plus the first m downcrossings. Each upcrossing and each downcrossing must have at least one term because at least one term p_j is needed to move a partial sum from below α to above β and at least one term q_j is needed to move a partial sum from above β to below α. Hence $n_{2m} \geq 2m$ and this proves that the sequence $(n_m)_{m\in\mathbb{N}}$ is indeed not bounded above.

It follows that $\sum b_k$ has infinitely many terms.

We note that all positive terms p_j and all negative terms q_j are being used in sequence, starting with the first one. This shows that each one of the terms of $\sum a_k$ has become part of $\sum b_k$ and it follows that $\sum b_k$ is indeed a rearrangement of $\sum a_k$.

Let $s_n := \sum_{j=1}^{n} b_j$. $n_1, n_3, n_5, \dots$ are (precisely the) integers n for which $s_n > \beta$ and $n_2, n_4, n_6, \dots$ are (precisely the) integers n for which $s_n < \alpha$. There are infinitely many of each and it follows from thm.**??** (Characterization of limsup and liminf) on p.**??** that

$$\liminf_{n\to\inf} s_n \leq \alpha \quad \text{and} \quad \limsup_{n\to\inf} s_n \geq \beta. \tag{13.48}$$

We now prove that for any $\varepsilon > 0$

$$\liminf_{n\to\inf} s_n \geq \alpha - \varepsilon \quad \text{and} \quad \limsup_{n\to\inf} s_n \leq \beta + \varepsilon. \tag{13.49}$$

Let $\varepsilon > 0$. The terms $(a_n)_n$ of the original series $\sum a_k$ converge to zero because $\sum a_k$ converges (see cor.13.1 on p.623). It follows that there exists $n_0 \in \mathbb{N}$ such that $|a_j| < \varepsilon$ for all $j \geq n_0$. We show next that

$$|p_j| \ = \ p_j < \varepsilon \text{ and } |q_j| \ = \ -q_j < \varepsilon \text{ for all } j \geq n_0. \tag{13.50}$$

$|p_j| = p_j < \varepsilon$ is true whenever $j \geq n_0$ because p_j is the jth positive member of $(a_n)_n$, hence $p_j = a_i$ for some $i \geq j \geq n_0$. Likewise, $|q_j| = -q_j < \varepsilon$ whenever $j \geq n_0$ because q_j is the jth negative member of $(a_n)_n$, hence $q_j = a_i$ for some $i \geq j \geq n_0$. We have proved (13.50).

We recall that $n_1, n_3, n_5, \dots$ are precisely the integers n for which $s_n > \beta$, so

$$s_{(n_1-1)} \leq \beta,\ s_{(n_3-1)} \leq \beta,\ \dots,\ s_{(n_{(2j-1)}-1)} \leq \beta,\ \dots.$$

But then $s_{(n_{(2j-1)})} \leq \beta + \varepsilon$ because less than ε was added to the previous term (which is no bigger than β) for any j so big that the last item in the jth upcrossing is less than ε

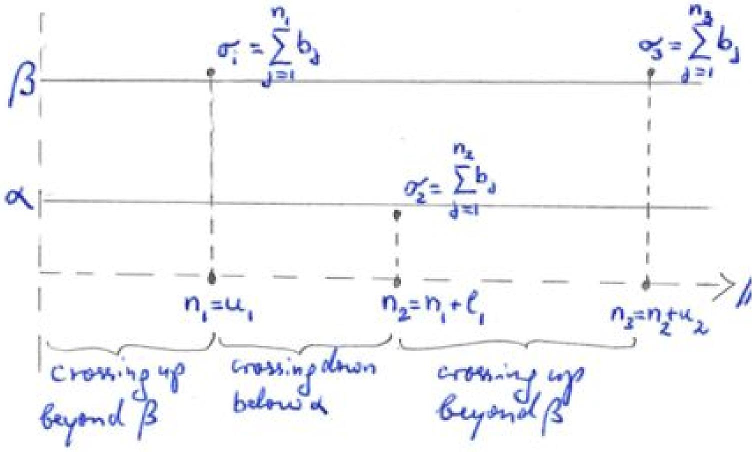
It follows from (13.50) that j is certainly big enough if $j \geq n_0$ because each upcrossing has size of at least 1. This shows that there are at most finitely many indices n such that $s_n > \beta + \varepsilon$ and we conclude that $\limsup_n s_n \leq \beta + \varepsilon$. A similar reasoning allows us to conclude that $\liminf_n s_n \geq \alpha - \varepsilon$.

We have proved (13.49) and this implies, together with (13.48), that

$$\liminf_{n\to\inf} s_n = \alpha \quad \text{and} \quad \limsup_{n\to\inf} s_n = \beta.$$

The picture to the right illustrates how the partial sums $\sigma_n = \sum_{j=1}^{n} b_j$ alternatingly rise above β and fall below α.
$|p_j| = p_j$ and $|q_j| = -q_j \Rightarrow$ both $|p_j|$ and $|q_j|$ drop below ε eventually.
Thus $\beta \leq \limsup_n \sigma_n \leq \varepsilon + \beta$ and $\beta - \varepsilon \leq \liminf_n \sigma_n \leq \beta$ eventually, thus

$$\begin{cases} \limsup_n \sigma_n = & \beta, \\ \liminf_n \sigma_n = & \alpha. \end{cases}$$



We have proved the theorem for case 1: $\beta \geq 0$

case 2: $\beta < 0$. We proceed exactly as in case 1. The only difference is that we start with a downcrossing that gets us below α rather than an upcrossing to obtain a rearrangement $\sum c_k$ for which a partial sum $\sum_{j=1}^{n} a_j$ exceeds α when n is the last term of an upcrossing and it drops below β when n is the last term of a downcrossing.

Because a_j converges to zero there will again only be finitely many upcrossings and downcrossings with terms that exceed ε. For all others the partial sums cannot exceed β or drop below α by more than ε and we conclude as before that

$$\liminf_{n\to\inf} \sum_{k=1}^{n} c_k \;=\; \alpha \quad \text{and} \quad \limsup_{n\to\inf} \sum_{k=1}^{n} c_k \;=\; \beta. \ \blacksquare$$

Corollary 13.3.

Let the series $\sum a_k$ *be conditionally convergent and let* $\alpha \in \mathbb{R}$.
Then, a rearrangement $\sum b_k$ *of* $\sum a_k$ *exists such that*

$$\lim_{n\to\infty} \sum_{k=1}^{n} b_k \ = \ \alpha.$$

PROOF: We apply Riemann's Rearrangement Theorem to the special case $\beta = \alpha$: There is a rearrangement $\sum b_j$ of $\sum a_j$ such that

$$\liminf_{n\to\infty} \sum_{k=1}^{n} b_k \ = \ \alpha \quad \text{and} \quad \limsup_{n\to\infty} \sum_{k=1}^{n} b_k \ = \ \alpha.$$

It follows now from thm.**??** on p.**??** that $\sum b_j$ converges to α. ■

We have seen that if a series is absolutely convergent then it is convergent and each rearrangement converges to the same limit. Here is the reverse.

Corollary 13.4.

Let $\sum a_k$ *be a convergent series with limit* $\alpha \in \mathbb{R}$ *such that* $\sum b_k = \alpha$*, for each rearrangement. Then* $\sum a_k$ *converge absolutely.*

PROOF: We assume to the contrary that the series $\sum a_k$ is not absolutely convergent, i.e., $\sum a_k$ is conditionally convergent. We apply Riemann's Rearrangement Theorem and find that there is a rearrangement of $\sum a_j$ which converges to a different real number, contrary to our assumption. ■

Corollary 13.5 (Dichotomy for convergent series)**.**

Let series $\sum a_k$ *be a convergent series. Then*
(a) *either all rearrangements of* $\sum a_k$ *converge to the same limit,*
(b) *or, for any* $\alpha \in \mathbb{R}$*, there is a rearrangement of* $\sum a_k$ *which converges to* α.

PROOF: Either $\sum a_k$ is absolutely convergent and **(a)** is true according to Riemann's Rearrangement Theorem or the series it is conditionally convergent and **(b)** is true according to cor.13.3. ■

13.3 Exercises for Ch.13

13.3.1 Exercises for Ch.13.1

Exercise 13.1.
Prove prop.**??** (Opposite of continuity) on p.**??**:

A sequence $(x_k)_k$ with values in (X, d) does not have $L \in X$ as its limit if and only if there exists some $\varepsilon > 0$ and $n_1 < n_2 < n_3 < \cdots \in \mathbb{N}$ such that $d(x_{n_j}, L) \geq \varepsilon$ for **all** j. ☐

Exercise 13.2.

Prove that $f(x) := \frac{1}{x}$ is **not** uniformly continuous on $]0,1]$. See example 13.3 on p.613.

Hint: Examine the sequence $x_n := \frac{1}{n}$. $\square$

Exercise 13.3. In Example 13.5 on p.618 the functions $f_n(\cdot)$ were defined as follows on the closed unit interval $[\,0,1\,]$:

$$f_n(x) := \begin{cases} n^2 x & \text{for } 0 \leq x \leq \frac{1}{n} \\ \frac{1}{x} & \text{for } \frac{1}{n} \leq x \leq 1 \end{cases}$$

Prove that f_n is continuous for all $n \in \mathbb{N}$. $\square$

Exercise 13.4.

Prove prop.13.2 on p.610 of this document: Let $(X,\mathfrak{U})$, $(Y,\mathfrak{V})$ and $(Z,\mathfrak{W})$ be topological spaces. Let $f : X \to Y$ be continuous at $x_0 \in X$ and $g : Y \to Z$ continuous at $f(x_0)$. Then the composition $g \circ f : X \to Y$ is continuous at x_0. $\square$

Exercise 13.5.

Give alternate proofs of exercise 13.4 above in the special case of metric spaces by using the sequence continuity definition (Definition 13.1 on p.603): Let (X,d), (Y,d') and (Z,d'') be metric spaces. Let $f : X \to Y$ be continuous at $x_0 \in X$ and $g : Y \to Z$ continuous at $f(x_0)$. Then the composition $g \circ f : X \to Y$ is continuous at x_0. $\square$

Exercise 13.6.

Prove prop.13.5 on p.611 of this document: Let d be the standard Euclidean metric and let d' be the discrete metric on the set $\mathbb{R}$ of all real numbers. Let

$$f : (\mathbb{R},') \to (\mathbb{R},d); \quad x \mapsto x \qquad \text{and} \qquad g : (\mathbb{R},d) \to (\mathbb{R},d'); \quad x \mapsto x$$

both be the identity function on $\mathbb{R}$. Then f is continuous at every point of $\mathbb{R}$, but g is not continuous anywhere on $\mathbb{R}$. $\square$

Exercise 13.7.

Let $X := [1,\infty[$ equipped with the standard Euclidean metric $d(x,x') = |x - x'|$.

Let $f_n : X \to \mathbb{R}$; $\quad x \mapsto \frac{nx+5}{(nx+3)^2}$. Prove that $f_n(\cdot) \overset{uc}{\to} 0$ on X. $\square$

Exercise 13.8.

Let $X := \mathbb{R}$, equipped with the Euclidean metric $d(x,x') = |x - x'|$. Let

$$f_n : \mathbb{R} \to \mathbb{R}; \qquad x \mapsto \frac{\sin(n^2 x)}{n}.$$

- **(a)** Prove that $f_n(\cdot) \overset{uc}{\to} 0$ on $\mathbb{R}$.
- **(b)** Prove that there is $x_0 \in \mathbb{R}$ such that the sequence $f'n(x_0)$ does not converge (pointwise). $\square$

Exercise 13.9.

Let $X := \mathbb{R}$ equipped with the standard Euclidean metric $d(x, x') = |x - x'|$.

Let $f_n : \mathbb{R} \to \mathbb{R}$ be the following sequence of functions:

$$f_n(x) \ := \begin{cases} 0 & \text{if } |x| > \frac{1}{n}, \\ nx + 1 & \text{if } \frac{-1}{n} \leq x \leq 0, \\ -nx + 1 & \text{if } 0 \leq x \leq \frac{1}{n}, \end{cases}$$

i.e., the point $\big(x, f_n(x)\big)$ is on the straight line between $(-\frac{1}{n}, 0)$ and $(0, 1)$ for $\frac{-1}{n} \leq x \leq 0$, it is on the straight line between $(0, 1)$ and $(\frac{1}{n}, 0)$ for $0 \leq x \leq \frac{-1}{n}$, and it is on the x–axis for all other x. Draw a picture! Let $f(x) := 0$ for $x \neq 0$ and $f(0) := 1$.

- **(a)** Prove that f_n converges pointwise to f on $\mathbb{R}$.
- **(b)** Prove that f_n does not converge uniformly to f on $\mathbb{R}$. $\square$

You may use without proof that each of the functions f_n is continuous on $\mathbb{R}$.

13.3.2 Exercises for Ch.13.2

Exercise 13.10.

Prove prop.13.12 on p.627: Let $\sum a_n$ be an absolutely convergent series. Let $(a_{n_k})_k$ be a subsequence of $(a_n)_n$. Then $\sum a_{n_k}$ converges absolutely. $\square$

References

[1] Matthias Beck and Ross Geoghegan. The Art of Proof. Springer, 1st edition, 2010.

List of Symbols

$\mathscr{C}(X, \mathbb{R})$ – continuous real–valued functions on X
, 607

$f_n(\cdot) \to f(\cdot)$ – pointwise convergence , 617
$f_n(\cdot) \overset{uc}{\to} f(\cdot)$ – uniform convergence , 617
$\lim\limits_{x \to x_0} f(x)$ – continuous at x_0 , 603
$\|f\|$ – norm of linear f , 614
$\mathscr{C}_{\mathscr{B}}(X, \mathbb{R})$, 608

Index