Math 330 - Additional Material

Student edition with proofs

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This document contains $\boxed{\text{chapter 16-22}}$ of the Math 330 lecture notes.

Last update: February 17, 2026

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16 Approximation theorems $\boxed{\star}$

Note that this chapter is starred, hence optional,

Introduction 16.1.

Everyone who knows about limits is familiar with the concept of approximations. For example the sequence $\left(\frac{1}{n}\right)_n$ becomes arbitrarily close to the number zero, i.e., it approximates zero since $\lim\limits_{n\to\infty}\frac{1}{n}=0$. Another example is the sequence $\big(s_n(x)\big)_n$ which we define for $x\in\mathbb{R}$ as $s_n(x):=\sum\limits_{j=0}^{n}\frac{x^j}{j!}$. It converges for each x to the number e^x and thus approximates that number. We can rephrase this last example as the approximation of the function $f:x\mapsto e^x$ by the sequence of functions $s_n:x\mapsto\sum\limits_{j=0}^{n}\frac{x^j}{j!}$.

Here is another example about the approximation of functions with sequences of functions. For Let U be an open subset of $\mathbb{R}$ and let $f:U\to\mathbb{R}$ be a function which is differentiable for each $x\in U$. For each $n\in\mathbb{N}$ let

$$\triangle_n^f:U\to\mathbb{R};\qquad x\mapsto\triangle_n^f(x):=\begin{cases}\frac{1}{n}\big(f(x+\frac{1}{n})-f(x)\big) & \text{if}\, x+\frac{1}{n}\in U,\\ 0 & \text{otherwise.}\end{cases}$$

Since U is open x is an interior point of U, thus $x+\frac{1}{n}\in U$ eventually, thus $\lim\limits_{n\to\infty}\triangle_n^f(x)=f'(x)$, the derivative of f at x

The convergence $\lim\limits_{n\to\infty}\sum\limits_{j=0}^{n}\frac{x^j}{j!}\;=\;e^x$ is much better behaved than the convergence $\lim\limits_{n\to\infty}\triangle_n^f(x)=f'(x)$ since one can prove that it is uniform on each interval $[a,b]$ where $a,b\in\mathbb{R}$. Matter of fact the following is true for any "power series" $s(x):=\sum\limits_{j=0}^{\infty}c_j(x-x_0)^j$ where $x_0\in\mathbb{R}$ and the "coefficients" $c_j\in\mathbb{R}$ for all j. If $r,a,b\in\mathbb{R}$ such that $x_0-r<a<b<x_0+r$ (thus $r>0!$) and such that the partial sum functions $s_n(x):=\sum\limits_{j=0}^{n}c_j(x-x_0)^j$ converge pointwise to $s(x)$ for $a\le x\le b$ then this convergence is uniform in x.

On the other hand the convergence of the difference quotient functions $\triangle_n^f(\cdot)$ to the derivative f' will generally not be uniform on such closed and bounded intervals, and how could it? The function f is continuous on U since it is differentiable at each $x\in U$, thus $\triangle_n^f(\cdot)$ is continuous as the scalar multiple of the difference of the two continuous functions $f(\cdot+\frac{1}{n})$ and f. But it follows from thm.**??** on p.**??** that uniform limits of continuous functions are continuous, and there are plenty of differentiable functions which are not continuous at all points.

Here is an example.

$$\text{Let}\;\; f(x)\;:=\begin{cases}-x^2 & \text{if}\, x<0,\\ x^2 & \text{if}\, x\ge 0.\end{cases}\qquad \text{Then}\;\; f'(x)\;=\;\begin{cases}-2x & \text{if}\, x<0,\\ \;\;2x & \text{if}\, x\ge 0\end{cases}\;=\;2|x|$$

is not differentiable at $x=0$. $\square$

So the question arises what kind of functions are the uniform limit of functions of a more specialized nature. Weierstrass found in the late 19th century a very general answer in his approximation theorem:

For any continuous function f on a closed and bounded interval $[a, b]$ one can find a sequence of polynomials $p_n(x)$ such that f is the uniform limit of those polynomials.

We will prove Weierstrass's approximation theorem in this chapter.

16.1 The Positive, Linear Operators $f \mapsto B_n^f$

Introduction 16.2.
We examined in ch.**??** (Bernstein Polynomials) the n–th Bernstein polynomial

$$B_n^f : \mathbb{R} \to \mathbb{R}; \qquad x \mapsto B_n^f(x) \ = \sum_{k=0}^{n} \binom{n}{k} f\Big(\frac{k}{n}\Big) x^k (1-x)^{n-k}$$

which one can associate with any function $f : [0,1] \to \mathbb{R}$, no matter how badly it may behave.

If we think of f as the argument of an assignment $f \mapsto B_n^f$ then this mapping defines a function $B_n : \mathscr{F}([0,1],\mathbb{R}) \to \mathscr{F}([0,1],\mathbb{R})$ from the set $\mathscr{F}([0,1],\mathbb{R})$ of all real–valued functions on the unit interval to itself.

We will see in prop.16.2 below that each of those functions B_n is a linear function from the vector space $\mathscr{F}([0,1],\mathbb{R})$ to itself. This proposition further shows that each B_n is positive in the sense that if the argument f is nonnegative, i.e., $f(x) \geq 0$ for all $0 \leq x \leq 1$ then its image B_n^f also is nonnegative.

The functions B_n^f are continuous since all polynomials are continuous, and this would allow us to shrink the codomain of B_n to the set $\mathscr{C}([0,1],\mathbb{R})$ of all continuous, real–valued functions on the unit interval, or even to the set of all polynomials $p(x)$ where $0 \leq x \leq 1$. We prefer not to do so for the following reason:

It is customary to call a linear function $F : \mathscr{F} \to \mathscr{F}$ which possesses a vector space $\mathscr{F}$ of real–valued functions both as domain and codomain a linear operator on $\mathscr{F}$, and to call a linear operator on $\mathscr{F}$ which assigns nonnegative functions to nonnegative functions a positive linear operator on $\mathscr{F}$. In short, we will see in this chapter that the assignments $B_n : \mathscr{F}([0,1],\mathbb{R}) \to \mathscr{C}([0,1],\mathbb{R})$ are positive linear operators for each $n \in \mathbb{N}$. $\square$

Definition 16.1 (Positive linear operators)**.**

Let (X,d) be a metric space, and let $\mathscr{F}$ be a subspace of the vector space $\mathscr{F}(X,\mathbb{R})$, i.e., with any two functions $f(\cdot), g(\cdot) \in \mathscr{F}$ their sum $f+g$ also belongs to $\mathscr{F}$ and that the function λf ($\lambda \in \mathbb{R}$) also belongs to $\mathscr{F}$.

A ***linear operator*** T on $\mathscr{F}$ is a linear function [1] $T : \mathscr{F} \to \mathscr{F}$

A ***positive linear operator*** T on $\mathscr{F}$ is a linear operator on $\mathscr{F}$ with the following property:

(16.1) $\quad f \geq 0 \ \Rightarrow \ Tf \geq 0, \quad$ i.e., $\quad f(x) \geq 0$ for all $x \in X \ \Rightarrow \ Tf(x) \geq 0$ for all $x \in X$.

Proposition 16.1 (Properties of positive linear operators)**.**

Let T be a positive linear operator on a subspace $\mathscr{F}$ of $\mathscr{F}(X,\mathbb{R})$. Then,

(a) *T is **monotone increasing**, i.e., for any two functions $f,g \in \mathscr{F}$ such that $f \leq g$ it is true that $T(f) \leq T(g)$. In other words,*

$$f(x) \leq g(x) \text{ for all } x \in X \;\Rightarrow\; T(f)(x) \leq T(g)(x) \text{ for all } x \in X. \tag{16.2}$$

(b) *Assume that $T(|f|)$ is defined for a function $f \in \mathscr{F}$. Then $|T(f)| \leq T(|f|)$. In other words,*

$$|T(f)(x)| \;\leq\; T(|f|)(x) \qquad \text{for all } x \in X. \tag{16.3}$$

Proof of **(a)**:

If $f \leq g$ then $g - f \geq 0$, hence $T(g-f) \geq 0$ since T is positive. Linearity of T then yields $T(g) - T(f) = T(g-f) \geq 0$. But $T(g) - T(f) \geq 0$ means the same as $T(f) \leq T(g)$, and we have proven part **(a)**.

Proof of **(b)**:

We have $f \leq |f|$ since $f(x) \leq |f(x)|$ for all $x \in X$. It follows from **(a)** that $T(f) \leq T(|f|)$. We also have $-f \leq |f|$ since $-f(x) \leq |f(x)|$ for all $x \in X$. It now follows from **(a)** that $T(-f) \leq T(|f|)$. But T is linear, so $T(-f) = -T(f)$, thus we have -$T(f) \leq T(|f|)$. In summary we have shown that both

$$T(f)(x) \leq T(|f|)(x) \text{ and } -T(f)(x) \leq T(|f|)(x) \text{ for all } x \in X.$$

It now follows from prop.**??** on p.**??** that $|T(f)(x)| \leq T(|f|)(x)$ for all $x \in X$, i.e., $|T(f)| \leq T(|f|)$. and we have proven part **(a)**. We have proven part **(b)**. ■

Proposition 16.2 (Linearity and positivity of Bernstein polynomial assignments)**.**

(a) *Let $f(\cdot), g(\cdot)$ be two real–valued functions on $[0,1]$ and $\alpha, \beta \in \mathbb{R}$. Let $h : [0,1] \to \mathbb{R}$ be defined as*

$$h \;:=\; \alpha f + \beta g, \text{ i.e., } h(x) \;=\; \alpha f(x) + \beta g(x) \; (0 \leq x \leq 1).$$
$$\text{Then } B_n^h \;=\; \alpha B_n^f + \beta B_n^g, \text{ i.e., } B_n^h(x) \;=\; \alpha B_n^f(x) + \beta B_n^g(x) \; (x \in \mathbb{R}).$$

To express this more succinctly:

$$B_n^{\alpha f + \beta g} \;=\; \alpha B_n^f + \beta B_n^g. \tag{16.4}$$

(b) *Let f be a real–valued function on $[0,1]$ which is nonnegative, i.e., $f(x) \geq 0$ for $0 \leq x \leq 1$. Then $B_n^f(\cdot) \geq 0$ on $[0,1]$ (but not necessarily for $x \notin [0,1]$).*

PROOF of **(a)**: For any $x \in [0,1]$ we have

$$\begin{aligned}
B_n^h(x) &= \sum_{k=0}^n \binom{n}{k} h(\tfrac{k}{n})\ x^k\ (1-x)^{n-k} \\
&= \sum_{k=0}^n \binom{n}{k} \Big(\alpha f(\tfrac{k}{n}) + \beta g(\tfrac{k}{n})\Big)\ x^k\ (1-x)^{n-k} \\
&= \sum_{k=0}^n \binom{n}{k} \alpha f(\tfrac{k}{n})\ x^k\ (1-x)^{n-k}\ +\ \sum_{k=0}^n \binom{n}{k} \beta g(\tfrac{k}{n})\ x^k\ (1-x)^{n-k} \\
&= \alpha \sum_{k=0}^n \binom{n}{k} f(\tfrac{k}{n})\ x^k\ (1-x)^{n-k}\ +\ \beta \sum_{k=0}^n \binom{n}{k} g(\tfrac{k}{n})\ x^k\ (1-x)^{n-k} \\
&= \alpha B_n^f(x)\ +\ \beta B_n^g(x)
\end{aligned}$$

PROOF of **(b)**: For any $x \in [0,1]$ we have both $0 \le x \le 1$ and $0 \le (1-x) \le 1$. It follows that both x^k and $(1-x)^{n-k}$ are nonnegative as products of nonnegative numbers.

Note that $0 \le k \le n$ implies that $\frac{k}{n} \in [0,1]$. We assumed that $f \ge 0$ on $[0,1]$, thus all numbers $f(k/n)$ are nonnegative. Finally, any binomial coefficient is nonnegative because it is defined as $\frac{n!}{k!(n-k)!}$, and the numbers $n!, k!$ and $(n-k)!$ all are nonnegative.

Thus each summand $\binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$ is nonnegative as a product of nonnegative numbers. It follows that $B_n^f(x)$ is nonnegative as the sum of nonnegative items.

To summarize: If restricted to the continuous functions $\mathscr{C}([0,1],\mathbb{R})$, each $B_n(\cdot)$ is a positive linear operator on $\mathscr{C}([0,1],\mathbb{R})$. ■

Corollary 16.1.

> *Let $n \in \mathbb{N}$. Then $B_n(\cdot)$ is a positive linear operator on $\mathscr{C}([0,1],\mathbb{R})$.*

PROOF:

Since $B_n^f\big(\mathscr{C}([0,1],\mathbb{R})\big) \subseteq B_n^f\big(\mathscr{F}([0,1],\mathbb{R})\big) \subseteq B_n^f\big(\mathscr{C}([0,1],\mathbb{R})\big)$ it follows from prop.16.2 above that the restriction of B_n^f to $\mathscr{C}([0,1],\mathbb{R})$ is a linear positive operator on $\mathscr{C}([0,1],\mathbb{R})$.

■

16.2 Korovkin's First Theorem

Introduction 16.3. The main task of this chapter will be the proof of Korovkin's first theorem. This theorem gives a simple condition which guarantees that a sequence T_n of positive linear operators defined on $\mathscr{C}([a,b],\mathbb{R})$ has the property that T_n^f converges uniformly to f for <u>any</u> continuous function f defined on the closed and bounded interval $[a,b]$ of two real numbers a and b. The proof given here follows Bauer [1]. □

Unless stated differently we assume the following for all of this subchapter:
a and b are two real numbers such that $a < b$, and

$$T_n(\cdot) : \mathscr{C}([a,b],\mathbb{R}) \to \mathscr{C}([a,b],\mathbb{R}); \qquad f(\cdot) \mapsto T_n^f(\cdot) = T_n(f)(\cdot)$$

is a sequence of positive linear operators on $\mathscr{C}([a,b],\mathbb{R})$. This means in particular that for each continuous real–valued function $f(\cdot)$ on $[a,b]$ the image

$$T_n^f : x \mapsto T_n^f(x)$$

is itself a continuous, real–valued function on $[a,b]$.

Before we state and prove Korovkin's First Theorem, there are two technical issues which we will prove separately in form of two lemmata so that the proof of the theorem itself does not become to lengthy.

Lemma 16.1. *Let there be real numbers $a < b$, and let f be a continuous, real–valued function on $[a,b]$. Let $\varepsilon > 0$. Then there exists (a potentially very large) number $\alpha \in \mathbb{R}$ such that*

$$|f(x) - f(y)| \ < \ \varepsilon \ + \ \alpha(x-y)^2 \qquad \text{for all } x,y \in [a,b]. \tag{16.5}$$

This number α can be chosen once and for all, independently of x and y.

PROOF: The interval $[a,b]$ is a closed and bounded subset of $\mathbb{R}$ and hence compact. This follows from the Heine–Borel Theorem (**??** on p.**??**).
Since f is continuous it follows from cor.**??** on p.**??** that the set $f([a,b])$ possesses a min and a max, in particular that that this set is bounded: If $\gamma = \max(|a|,|b|)$ then

$$|f(x)| \ < \ \gamma \quad \text{for all } x \in [a,b]. \tag{16.6}$$

It further follows from cor.**??** on p.**??** that the continuous function f is uniformly continuous on $[a,b]$. Thus we can find for any $\varepsilon > 0$, no matter how small, some $\delta^\star > 0$ such that

$$|f(x) - f(y)| \ < \ \varepsilon \quad \text{for all } x,y \in [a,b] \text{ such that } |x-y| < \delta^\star.$$

Let $\delta := {\delta^\star}^2$. The uniform continuity characterization above then reads

$$|f(x) - f(y)| \ < \ \varepsilon \quad \text{for all } x,y \in [a,b] \text{ such that } (x-y)^2 < \delta. \tag{16.7}$$

Let γ be the constant from (16.6), i.e., $|f(z)| < \gamma$ for all $z \in [a,b]$. We will show that

$$\alpha \ := \ \frac{2\gamma}{\delta}$$

satisfies (16.5) by looking separately at the two cases $(x-y)^2 \leq \delta$ and $(x-y)^2 > \delta$.
Case 1: Assume that $(x-y)^2 \leq \delta$.

Observe that it is always true that
$$\varepsilon \ < \ \varepsilon \ + \ \alpha(x-y)^2$$
since $\alpha(x-y)^2 \geq 0$. The rest is easy. According to (16.7), we have
$$(x-y)^2 \leq \delta \ \Rightarrow \ |f(x)-f(y)| \ < \ \varepsilon \ \leq \ \varepsilon \ + \ \alpha(x-y)^2,$$
and **case 1** is proven.
Case 2: Assume that $(x-y)^2 > \delta$, i.e., $\delta < (x-y)^2$.
γ was chosen such that $|f(x)| < \gamma$ on $[a,b]$ (see (16.6)). Thus we have
$$\begin{aligned} |f(x)-f(y)| \ &\leq |f(x)| + |f(y)| \\ &< \gamma + \gamma \ = \ \alpha\delta \ < \ \alpha(x-y)^2 < \varepsilon \ + \ \alpha(x-y)^2, \end{aligned} \tag{16.8}$$
i.e., **case 2** also holds, and hence α satisfies (16.5). This concludes the proof of the lemma. ■

Lemma 16.2. *If the sequence of positive linear operators T_n on $\mathscr{C}([a,b],\mathbb{R})$ satisfies*
$$\textit{i)}\ T_n^1 \xrightarrow{uc} 1, \qquad \textit{ii)}\ T_n^{id} \xrightarrow{uc} id, \qquad \textit{iii)}\ T_n^{id^2} \xrightarrow{uc} id^2.$$
Then $\lim\limits_{n\to\infty} \|T_n^{id^2} - 2idT_n^{id} + id^2T_n^1\|_\infty \ = \ 0.$

PROOF: Let $A_n := T_n^{id^2} - 2idT_n^{id} + id^2T_n^1$. Then
$$\begin{aligned} A_n \ &= \ T_n^{id^2} - id^2T_n^1 + 2id^2T_n^1 - 2idT_n^{id} \\ &= \ (T_n^{id^2} - id^2T_n^1) \ + \ 2(id^2T_n^1 - idT_n^{id}) \ = \ B_n + 2C_n \end{aligned}$$
where we define $B_n := T_n^{id^2} - id^2T_n^1$ and $C_n := id^2T_n^1 - idT_n^{id}$. Since
$$0 \ \leq \ \|B_n + 2C_n\|_\infty \ \leq \ \|B_n\|_\infty \ + \ 2\|C_n\|_\infty$$
it suffices to prove that $\lim\limits_{n\to\infty} \|B_n\|_\infty \ = \ \lim\limits_{n\to\infty} \|C_n\|_\infty = 0$.
Proof that $\lim\limits_{n\to\infty} \|B_n\|_\infty \ = \ 0$:
We rewrite $B_n \ = \ (T_n^{id^2} - id^2) \ + \ id^2(1 - T_n^1)$. The continuous function $id^2 : x \mapsto x^2$ assumes a maximum on $[a,b]$ which we call γ. Thus $\|B_n\|_\infty \leq \|T_n^{id^2} - id^2)\|_\infty \ + \ \gamma\|1 - T_n^1\|_\infty$. Uniform convergence $T_n^1 \xrightarrow{uc} 1$ and $T_n^{id^2} \xrightarrow{uc} id^2$ yields
$$\lim_{n\to\infty} \|T_n^{id^2} - id^2)\|_\infty = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \gamma\|1 - T_n^1\|_\infty = \gamma \lim_{n\to\infty} \|1 - T_n^1\|_\infty = 0,$$
hence $\lim\limits_{n\to\infty} \|B_n\|_\infty = 0$
Proof that $\lim\limits_{n\to\infty} \|C_n\|_\infty \ = \ 0$:
$$\begin{aligned} C_n \ &= id^2T_n^1 - idT_n^{id} = \ (id^2T_n^1 - id^2 + id^2) - (idT_n^{id} + id^2 - id^2) \\ &= \ id^2(T_n^1 - 1) + id^2 \ - \ id(T_n^{id} - id) - id^2 \\ &= \ id^2(T_n^1 - 1) \ - \ id(T_n^{id} - id) \end{aligned}$$

hence

$$\|C_n\|_\infty \;\le\; \|id^2(T_n^1 - 1)\|_\infty \;+\; \|id(T_n^{id} - id)\|_\infty.$$

But $|id(x)| = |x| \le \gamma' := \max(|a|, |b|)$, and we saw that $|id^2(x)| = x^2 \le \gamma$ for all $a \le x \le b$. Thus

$$\|C_n\|_\infty \;\le\; \gamma\|(T_n^1 - 1)\|_\infty \;+\; \gamma'\|(T_n^{id} - id)\|_\infty.$$

It now follows from $T_n^1 \xrightarrow{uc} 1$ and $T_n^{id} \xrightarrow{uc} id$ that

$$\lim_{n\to\infty} \gamma\|T_n^1 - 1)\|_\infty = 0 = \gamma \lim_{n\to\infty} \|T_n^1 - 1)\|_\infty \;\; = 0 \quad\text{and}\quad \lim_{n\to\infty} \gamma'\|T_n^{id} - id\|_\infty = \gamma' \lim_{n\to\infty} T_n^{id} - id\|_\infty = 0,$$

hence $\lim\limits_{n\to\infty} \|C_n\|_\infty = 0.$

We noted earlier that the convergence of $\|B_n\|_\infty$ and $\|C_n\|_\infty$ was sufficient to prove the lemma. ■

We have assembled everything to formulate and prove Korovkin's First Theorem. Note that uniform convergence [2] plays a very prominent role.

Theorem 16.1 (Korovkin's First Theorem)**.**

Assume that we have uniform convergence

$$T_n^f(\cdot) \;\xrightarrow{uc}\; f(\cdot),$$

for the following three elements f of $\mathscr{C}([a,b],\mathbb{R})$:

$$\begin{aligned} 1(\cdot) &: x \mapsto 1 && \textit{the constant function } 1,\\ id(\cdot) &: x \mapsto x && \textit{the identity on } [a,b],\\ id^2(\cdot) &: x \mapsto x^2. \end{aligned}$$

Then $T_n^f \xrightarrow{uc} f$ for all $f \in \mathscr{C}([a,b],\mathbb{R})$.

PROOF: Let $\varepsilon > 0$. According to lemma 16.1 on p.708 there exists $\alpha > 0$ such that

(16.9) $$|f(x) - f(y)| \;<\; \varepsilon \;+\; \alpha(x-y)^2 \qquad \text{for all } x, y \in [a,b].$$

We interpret the above as an inequality where x acts as a variable (a function argument) and y is "fixed but arbitrary". We thus obtain for each y the inequality of functions,

(16.10) $$|f(\cdot) - f(y)| \;<\; \varepsilon \;+\; \alpha(id(\cdot) - y)^2 \qquad \text{for all } y \in [a,b].$$

According to (16.3) on p.706, $|T_n^h(\cdot)|$ does not exceed $T_n|h|$ for any $h \in \mathscr{C}([a,b],\mathbb{R})$. We thus obtain for $h : x \mapsto f(x) - f(y)$

(16.11) $$\begin{aligned} |T_n f(\cdot) - f(y)T_n 1(\cdot)| &= \big|T_n\big(f(\cdot) - f(y)1(\cdot)\big)\big|\\ &\le T_n(|f - f(y)1|)\\ &= T_n(|f - f(y)|) \end{aligned}$$

[2] see (**??**) on p.**??**

which is, according to (16.9), smaller than

$$T_n(\varepsilon \ + \ \alpha(id-y)^2 \ = T_n\varepsilon \ + \ \alpha T_n(id - y\cdot 1)^2 \tag{16.12}$$
$$= \varepsilon T_n(1) \ + \ \alpha\Big(T_n(id^2) - 2yT_n(id) + y^2T_n(1)\Big).$$

We combine (16.11) and (16.12) and obtain

$$|T_nf - f(y)T_n1| \ \le \ \varepsilon T_n1 \ + \ \alpha\Big(T_nid^2 - 2yT_nid + y^2T_n1\Big).$$

For the next step we recall that each of the expressions $T_nf, -T_n1, T_nid^2, T_nid$ is a function on the interval $[a,b]$. We evaluate them at the argument y and obtain

$$|T_nf(y) - f(y)T_n1(y)| \ \le \ \varepsilon T_n1(y) \ + \ \alpha\Big(T_nid^2(y) - 2yT_nid(y) + y^2T_n1(y)\Big). \tag{16.13}$$

This last inequality we can in turn interpret as one for the functions

$$y \mapsto |T_nf(y) - f(y)T_n1(y)|$$
$$y \mapsto \varepsilon T_n1(y) \ + \ \alpha\Big(T_nid^2(y) - 2id(y)T_nid(y) + id(y)^2T_n1(y)\Big).$$

It then follows from (16.13) that

$$|T_nf - fT_n1| \ \le \ \varepsilon T_n1 \ + \ \alpha\Big(T_nid^2 - 2idT_nid + id^2T_n1\Big). \tag{16.14}$$

The function f is continuous and hence bounded on the closed interval $[a,b]$. Thus there exists $\gamma > 0$ such that $|f(x)| < \gamma$ for all $x \in [a,b]$. (See Corollary **??** on p.**??** of this document>)
We use the triangle inequality twice in the following:

$$\begin{aligned} |T_nf - f| \ &= |T_nf - fT_n1 + fT_n1 - f| \\ &= |T_nf - fT_n1 + f(T_n1 - 1)| \\ &\le |T_nf - fT_n1| \ + \ |f(T_n1 - 1)| \\ &\le |T_nf - fT_n1| \ + \ \gamma|(T_n1 - 1)|. \end{aligned} \tag{16.15}$$

We combine (16.15) and (16.14) and obtain

$$\begin{aligned} |T_nf - f| \ &\le |T_nf - fT_n1| \ + \ \gamma|(T_n1 - 1)| \\ &\le \varepsilon T_n1 \ + \ \alpha\Big(T_nid^2 - 2idT_nid + id^2T_n1\Big) \ + \ \gamma|(T_n1 - 1)| \\ &\le \varepsilon\|T_n1\|_\infty \ + \ \alpha\|T_nid^2 - 2idT_nid + id^2T_n1\|_\infty \ + \ \gamma\|(T_n1 - 1)\|_\infty. \end{aligned}$$

This inequality is true for each argument $a \le x \le b$, thus

$$\sup_{a\le x\le b} |T_nf(x) - f(x)| \le \varepsilon\|T_n1\|_\infty \ + \ \alpha\|T_nid^2 - 2idT_nid + id^2T_n1\|_\infty \ + \ \gamma\|(T_n1 - 1)\|_\infty,$$

i.e.,

$$\|T_nf - f\|_\infty \le \varepsilon\|T_n1\|_\infty \ + \ \alpha\|T_nid^2 - 2idT_nid + id^2T_n1\|_\infty \ + \ \gamma\|(T_n1 - 1)\|_\infty. \tag{16.16}$$

It follows from $T_n 1 \xrightarrow{uc} 1$ that there exists $N_1 \in \mathbb{N}$ such that $\|T_n 1 - 1\|_\infty \leq 1$ for all $n \geq N_1$, thus

$$\|\varepsilon T_n 1\|_\infty = \|\varepsilon \ + \ \varepsilon(T_n 1 - 1)\|_\infty \leq \|\varepsilon\|_\infty \ + \ \varepsilon\|(T_n 1 - 1)\|_\infty \leq 2\varepsilon \ \text{ for all } n \geq N_1. \tag{16.17}$$

It also follows from $T_n 1 \xrightarrow{uc} 1$ that $\lim\limits_{n\to\infty} \|(T_n 1 - 1)\|_\infty = 0$, hence

$$\lim_{n\to\infty} \gamma|(T_n 1 - 1)| \leq \gamma \lim_{n\to\infty} \|(T_n 1 - 1)\|_\infty = 0.$$

Thus there exists $N_2 \in \mathbb{N}$ such that

$$\gamma|(T_n 1 - 1)| \ \leq \ \varepsilon \ \text{ for all } n \geq N_2. \tag{16.18}$$

It moreover follows from lemma 16.2 above that

$$\lim_{n\to\infty} \alpha\|T_n^{id^2} - 2id T_n^{id} + id^2 T_n^1\|_\infty = \ \alpha \lim_{n\to\infty} \|T_n^{id^2} - 2id T_n^{id} + id^2 T_n^1\|_\infty \ = \ 0.$$

Thus there exists $N_3 \in \mathbb{N}$ such that

$$\alpha\|T_n^{id^2} - 2id T_n^{id} + id^2 T_n^1\|_\infty \ \leq \ \varepsilon \ \text{ for all } n \geq N_3. \tag{16.19}$$

Let $N := \max(N_1, N_2, N_3)$. It follows from (16.16), (16.17), (16.18) and (16.19) that

$$\|T_n f - f\|_\infty \ \leq \ 4\varepsilon \ \text{ for all } n \geq N_3.$$

Of course the choice of N will depend on the choice of ε, but the fact of importance is that such $N = N(\varepsilon)$ can be found for any $\varepsilon > 0$, no matter how small. It follows that

$$\lim_{n\to\infty} \|T_n f - f\|_\infty = 0, \qquad \text{i.e.,} \quad T_n^f \xrightarrow{uc} f.$$

Since $f \in \mathscr{C}([a,b],\mathbb{R})$ was arbitrarily chosen we are finished with the proof of Korovkin's First Theorem. ■

16.3 The Weierstrass Approximation Theorem

Introduction 16.4. We have seen earlier that the mappings

$$B_n : \mathscr{C}([0,1],\mathbb{R}) \to \mathscr{C}([0,1],\mathbb{R}); \qquad f(\cdot) \mapsto B_n^f(\cdot)$$

which assign to a function $f : [0,1] \to \mathbb{R}$ it's n^{th} Bernstein polynomial

$$x \ \mapsto \ B_n^f(x) \ = \ \sum_{k=0}^n \binom{n}{k} f(\frac{k}{n}) \ id(\cdot)^k \ (1 - id(\cdot))^{n-k}$$

have the following properties:

(a) They are linear positive operators on the space $\mathscr{C}([0,1],\mathbb{R})$ of all continuous real–valued functions on the unit interval $[0,1]$. (see prop.16.2 on p.706).

(b) We have uniform convergence $B_n^f(\cdot) \overset{uc}{\to} f(\cdot)$ for the three functions

$$1 : x \mapsto 1; \qquad id(\cdot) : x \mapsto x; \qquad id^2(\cdot) : x \mapsto x^2; \qquad (0 \leq x \leq 1).$$

(see proposition (**??**) on p.**??**). We will obtain as an easy consequence of Korovkin's First Theorem that any continuous real–valued function defined on the unit interval is the uniform limit of a sequence of polynomials, and we then generalize this result to any closed and bounded interval $[a,b]$. This result was first proven (by entirely different means by Weierstrass in the 1880s. □

Proposition 16.3 (Weierstrass Approximation Theorem on $[0,1]$)**.**

Any continuous real–valued function on the unit interval $[0,1]$ can be uniformly approximated by a sequence of polynomials.

PROOF: We have seen in cor.16.1 on p.707 that the Bernstein polynomial assignments $B_n(\cdot)$ form a sequence of positive linear operators on $\mathscr{C}([0,1],\mathbb{R})$. We also have seen that if f is one of the three continuous functions $1 : x \mapsto 1$, $id : x \mapsto x$, $id^2 : x \mapsto x^2$ then $B_n^f(\cdot) \xrightarrow{uc} f(\cdot)$ It follows from Korovkin's First Theorem that this uniform convergence extends to all continuous functions on the unit interval. This proves the theorem since all functions $x \mapsto B_n^f(x)$ are Bernstein polynomials and hence polynomials. ■

The trick that we will use to generalize this last result from $[0,1]$ to $[a,b]$ consists of using the fact that those intervals can be bijected by functions of the form $\psi(x) = mx + b$, and that uniform convergence $f_n \xrightarrow{uc} f$ implies that of $f_n \circ \varphi \xrightarrow{uc} f \circ \varphi$ and vice versa.

Lemma 16.3.

Let $n \in [0,\infty[_{\mathbb{Z}}$, $\alpha_j, m, b \in \mathbb{R}$, $\alpha_n \neq 0$.

Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial $p(x) = \sum\limits_{j=0}^{n} \alpha_j x^j$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as $\varphi(x) = mx + b$.

Then $p \circ \varphi : \mathbb{R} \to \mathbb{R}$; $x \mapsto \sum\limits_{j=0}^{n} \alpha_j (mx+b)^j$ is a polynomial.

The proof is left as exercise 16.1 (see p.714). ■

Proposition 16.4.

Let $A \subseteq \mathbb{R}$, $\varphi : \mathbb{R} \to \mathbb{R}$ defined as $\varphi(x) = mx + b$ $(m, b \in \mathbb{R})$. Further, let $f_n, f \in \mathscr{C}(\varphi(A), \mathbb{R})$, $n \in \mathbb{N}$. (Thus, f_n and f are continuous functions on $\varphi(A) = \{\varphi(x) : x \in A\}$.)
Assume further that $f_n \xrightarrow{uc} f$ on $\varphi(A)$. Then $f_n \circ \varphi \xrightarrow{uc} f \circ \varphi$, on A.

PROOF:
For any $a \in A$ we have $\varphi(a) \in \varphi(A)$, thus

$$|f_n \circ \varphi(a) - f \circ \varphi(a)| \;\leq\; \sup_{y \in \varphi(A)} |f_n(y) - f(y)| \;=\; \|f_n - f\|_\infty.$$

But $a \in A$ was arbitrary, thus $\|f_n \circ \varphi - f \circ \varphi\|_\infty \leq \|f_n - f\|_\infty$ It follows from $f_n \xrightarrow{uc} f$ on $\varphi(A)$ that $\lim\limits_{n\to\infty} \|f_n - f\|_\infty = 0$, hence $\lim\limits_{n\to\infty} \|f_n \circ \varphi - f \circ \varphi\|_\infty = 0$, hence $f_n \circ \varphi \xrightarrow{uc} f \circ \varphi$ on A. ■

Corollary 16.2.

Let $a, b \in \mathbb{R}$ such that $a < b$. Let $\varphi : [a,b] \to [0,1]$; $\varphi(x) := \dfrac{x-a}{b-a}$. Then

(a) *φ is a bijection $[a,b] \xrightarrow{\sim} [0,1]$.*
(b) *If $h_n, h \in \mathscr{C}([0,1],\mathbb{R})$ $(n \in \mathbb{N})$ such that $h_n \xrightarrow{uc} h$ on $[0,1]$, then $h_n \circ \varphi \xrightarrow{uc} h \circ \varphi$ on $[a,b]$.*

PROOF of **(a)**: It is easily verified that $\psi : [0,1] \to [a,b]$ defined as $\psi(t) := a + t(b-a)$ defines the inverse of φ.

PROOF of **(b)**: This follows from prop.16.4 on p.713 setting $A := [a,b]$ since then $\varphi(x) = \frac{1}{b-a} \cdot x - \frac{a}{b-a}$ is of the form $mx + b$, and since it follows from **(a)** that $\varphi(A) = \varphi([a,b]) = [0,1]$. ■

The Weierstrass approximation theorem for continuous functions on the unit interval (prop.16.3 on p.713) is now easily generalized to continuous functions on an arbitrary closed and bounded interval.

Theorem 16.2 (Weierstrass Approximation Theorem)**.**

Let $a, b \in \mathbb{R}$ such that $a < b$. Then any continuous real–valued function on $[a,b]$ can be uniformly approximated by a sequence of polynomials.

PROOF: Let $f \in \mathscr{C}([a,b],\mathbb{R}$, and let $\varphi : [a,b] \xrightarrow{\sim} [0,1]$ be defined as $\varphi(x) := \frac{x-a}{b-a}$. We recall from cor.16.2 on p.714 that φ is bijective. Let $h := f \circ \varphi^{-1}$. Then h is continuous on the unit interval and thus, according to prop.16.3, f is the uniform limit of a sequence of polynomials h_n (we might choose, e.g., the Bernstein polynomials $h_n := B_n^{f \circ \varphi^{-1}}$). It follows from prop.16.4 on p.713 that $h_n \circ \varphi \xrightarrow{uc} h \circ \varphi$ on $\varphi^{-1}[0,1]$. But $h \circ \varphi = f \circ \varphi^{-1} \circ \varphi = f$ and $\varphi^{-1}[0,1] = [a,b]$, thus $h_n \circ \varphi \xrightarrow{uc} f$ on $[a,b]$.

We finally note that it follows from lemma 16.3 on p.713 that the functions $p_n := h_n \circ \varphi$ are polynomials since the h_n are polynomials. Since $p_n \xrightarrow{uc} f$ on $[a,b]$ we have proven the theorem. ■

16.4 Exercises for Ch.16

Exercise 16.1.

Prove lemma 16.3 on p.713 of this document: Let $n \in [0,\infty[_{\mathbb{Z}}$, $\alpha_j, m, b \in \mathbb{R}$, $\alpha_n \neq 0$.

Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial $p(x) = \sum\limits_{j=0}^{n} \alpha_j x^j$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined as $\varphi(x) = mx + b$.

Then $p \circ \varphi : \mathbb{R} \to \mathbb{R}$; $x \mapsto \sum\limits_{j=0}^{n} \alpha_j (mx+b)^j$ is a polynomial.

Hint: Apply the binomial theorem to $(mx+b)^j$. □

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17 Construction of the Number Systems $\boxed{\star}$

17.1 The Peano Axioms

Note that this chapter is starred, hence optional.

Definition 17.1 (Set of nonnegative integers)**.**

We define the set $\mathbb{N}_0$ (the nonnegative integers) axiomatically as follows:

Ax.1 There is an element "0" contained in $\mathbb{N}_0$.

Ax.2 There is a function $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ such that

Ax.2.1 σ is injective,

Ax.2.2 $0 \notin \sigma(\mathbb{N}_0)$ (range of σ),

Ax.2.3 Induction axiom: Let $U \subseteq \sigma(\mathbb{N}_0)$ such that **(a)** $0 \in U$, **(b)** If $n \in U$ then $\sigma(n) \in U$. It then follows that $U = \mathbb{N}_0$.

We define $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$. $\square$

Definition 17.2 (Iterative function composition)**.**

Let $X \neq \emptyset$ and $f : X \to X$.

We use the induction axiom above to define f^n for an arbitrary function $f : X \to X$:

(a) $f^0 := \mathrm{id}_X : x \mapsto x$, **(b)** $f^1 := f$, **(c)** $f^2 := f \circ f$ (function composition), **(c)** $f^{\sigma(n)} := f \circ f^n$. $\square$

Proposition 17.1.

f^n is defined for all $n \in \mathbb{N}_0$.

PROOF: Let $U := \{k \in \mathbb{N}_0 : f^k \text{ is defined }\}$. Then $0 \in U$ as $f^0 = id_A$ and if $k \in U$, i.e., f^k is defined then $f^{\sigma(k)} = f \circ f^k$ also is defined, i.e., $\sigma(k) \in U$. It follows from Ax.2.3 that $U = \mathbb{N}_0$. ■

Remark 17.1 ($\sigma(\cdot)$ as successor function)**.**

Of course the meaning of $\sigma(n)$ will be that of $n+1$:

$$0 \overset{\sigma}{\mapsto} 1 \overset{\sigma}{\mapsto} 2 \overset{\sigma}{\mapsto} 3 \overset{\sigma}{\mapsto} \dots \quad \square$$

Definition 17.3 (Addition and multiplication on $\mathbb{N}_0$)**.**

Let $m, n \in \mathbb{N}_0$. Let

$$m + n := \sigma^n(m), \tag{17.1}$$
$$m \cdot n := (\sigma^m)^n(0). \quad \square \tag{17.2}$$

Note for the above that we know the meaning of $(\sigma^m)^n$: $f := \sigma^m$ is a function $A \to A$ and we have established in prop.17.1 the meaning of f^n, i.e., $(\sigma^m)^n$.

Proposition 17.2.

Addition and multiplication satisfy all commonly known rules of arithmetic, such as

$m + n$	$= n + m$	*commutativity of addition*
$k + (m + n)$	$= (k + m) + n$	*associativity of addition*
$m \cdot n$	$= n \cdot m$	*commutativity of multiplication*
$k \cdot (m \cdot n)$	$= (k \cdot m) \cdot n$	*associativity of multiplication*
$k \cdot (m + n)$	$= k \cdot m + k \cdot n$	*distributivity of addition*
$n \cdot 1$	$= 1 \cdot n = n$	*neutral element for multiplication*

Here, 1 is defined as $1 = \sigma(0)$.

PROOF: Drudge work. ■

Definition 17.4 (Order relation $m < n$ on $\mathbb{N}_0$)**.**

Let $m, n \in \mathbb{N}_0$.

(a) We say m is less than n and we write $m < n$, if there exists $x \in \mathbb{N}$ such that $n = m + x$.
(b) We say m is less or equal than n and we write $m \leq n$, if $m < n$ or $m = n$.
(c) We say that m is greater than n and we write $m > n$, if $n < m$.
We say m is greater or equal than n and we write $m \geq n$, if $n \leq m$. □

Proposition 17.3.

"$<$" and "$\leq$" satisfy all commonly known rules, such as

- *Trichotomy of the order relation: Let $m, n \in \mathbb{N}_0$. Then exactly one of the following is true:*

$$m < n, \qquad m = n, \qquad m > n.$$

PROOF: Drudge work. ■

17.2 Constructing the Integers from $\mathbb{N}_0$

For the following, review B/G project 6.9 in ch.6.1 and B/G prop.6.25 in ch.6.3.

Definition 17.5 (Integers as equivalence classes)**.**

We define the following equivalence relation
$(m_1, n_1) \sim (m_2, n_2)$ on the cartesian product $\mathbb{N}_0 \times \mathbb{N}_0$:

$$(m_1, n_1) \sim (m_2, n_2) \;\Leftrightarrow\; m_1 + n_2 = n_1 + m_2 \tag{17.3}$$

We write $\mathbb{Z} := \{[(m, n)] : m, n \in \mathbb{N}_0\}$. In other words, $\mathbb{Z}$ is the set of all equivalence classes with respect to the equivalence relation (17.3).
We "embed" $\mathbb{N}_0$ into $\mathbb{Z}$ with the following injective function $e : \mathbb{N}_0 \to \mathbb{Z}$: e(m) := [(m,0)].
From this point forward we do not distinguish between $\mathbb{N}_0$ and its image $e(\mathbb{N}_0) \subseteq \mathbb{Z}$ and we do not distinguish between $\mathbb{N}$ and its image $e(\mathbb{N}) \subseteq \mathbb{Z}$. In particular we do not distinguish between the two zeros 0 and $[(0, 0)]$ and between the two ones 1 and $[(1, 0)]$.
Finally we write $-n$ for the integer $[(0, n)]$. □

With those abbreviation we obtain

Proposition 17.4 (Trichotomy of the integers)**.**

Let $z \in \mathbb{Z}$. Then exactly one of the following is true:
- ***(a)*** $z \in \mathbb{N}$, *i.e.,* $z = [(m, 0)]$ *for some* $m \in \mathbb{N}$
- ***(b)*** $-z \in \mathbb{N}$, *i.e.,* $z = [(0, n)]$ *for some* $n \in \mathbb{N}$
- ***(c)*** $z = 0$. □

PROOF: Drudge work. ■

Remark 17.2.

(a) The intuition that guided the above definition is that
the pairs $(4, 0), (7, 3), (130, 126)$ all define the same integer 4
and the pairs $(0, 4), (3, 7), (126, 130)$ all define the same integer -4.
(b) If it had been possible to define subtraction $m - n$ for all $m, n \in \mathbb{N}_0$ then (17.3) could be stated

$$(m_1, n_1) \sim (m_2, n_2) \;\Leftrightarrow\; m_1 - n_1 = m_2 - n_2.$$

Looking at the equivalent pairs $(4, 0), (7, 3), (130, 126)$ we get $4 - 0 \;=\; 7 - 3 \;=\; 130 - 126 \;=\; 4$
For $(0, 4), (3, 7), (126, 130)$ we obtain $0 - 4 \;=\; 3 - 7 \;=\; 126 - 130 \;=\; -4$. □

Definition 17.6 (Addition, multiplication and subtraction on $\mathbb{Z}$)**.**

Let $[(m_1,n_1)]$ and $[(m_2,n_2)] \in \mathbb{Z}$. We define

$$-[(m_1,n_1)] := [n_1,m_1], \tag{17.4}$$
$$[(m_1,n_1)] + [(m_2,n_2)] := [(m_1+m_2, n_1+n_2)] \tag{17.5}$$
$$[(m_1,n_1)] \cdot [(m_2,n_2)] := [(m_1m_2+n_1n_2, m_1n_2+n_1m_2)] \tag{17.6}$$

We write $[(m_1,n_1)] - [(m_2,n_2)]$ ("$[(m_1,n_1)]$ minus $[(m_2,n_2)]$")
as an abbreviation for $[(m_1,n_1)] + \big(-[(m_2,n_2)]\big)$.

We write $[(m_1,n_1)] < [(m_2,n_2)]$ if $[(m_2,n_2)] - [(m_1,n_1)] \in \mathbb{N}$, i.e., if there is $k \in \mathbb{N}$ such that $[(m_2,n_2)] - [(m_1,n_1)] = [(k,0)]$. We then say that $[(m_1,n_1)]$ is less than $[(m_2,n_2)]$.

We write $[(m_1,n_1)] \leq [(m_2,n_2)]$ if $[(m_1,n_1)] < [(m_2,n_2)]$ or if $[(m_1,n_1)] = [(m_2,n_2)]$ and we then say that $[(m_1,n_1)]$ is less than or equal to $[(m_2,n_2)]$.

We write $[(m_1,n_1)] > [(m_2,n_2)]$ if $[(m_2,n_2)] < [(m_1,n_1)]$ and we then say that $[(m_1,n_1)]$ is greater than $[(m_2,n_2)]$.

We write $[(m_1,n_1)] \geq [(m_2,n_2)]$ if $[(m_2,n_2)] \leq [(m_1,n_1)]$ and we then say that $[(m_1,n_1)]$ is greater than or equal to $[(m_2,n_2)]$.

We write $\mathbb{Z}_{\geq 0}$ for the set of all integers z such that $z \geq 0$ and $\mathbb{Z}_{\neq 0}$ for the set of all integers z such that $z \neq 0$. You should convince yourself that $\mathbb{Z}_{\geq 0} = \mathbb{N}_0$. $\square$

It turns out that all three operations are "well defined" in the sense that the resulting equivalence classes on the right of each of the three equations above do not depend on the choice of representatives in the classes on the left. Further we have

Proposition 17.5.

Let $m,n \in \mathbb{N}_0$. Then

$$[(m,n)] + [(0,0)] = [(0,0)] + [(m,n)] = [(m,n)], \tag{17.7}$$
$$\big(-[(m,n)]\big) + [(m,n)] = [(m,n)] + \big(-[(m,n)]\big) = [0,0] \tag{17.8}$$
$$[(m,n)] \cdot [(1,0)] = [(1,0)] \cdot [(m,n)] = [(m,n)], \tag{17.9}$$

i.e., $[(0,0)]$ becomes the neutral element with respect to addition, $[(1,0)]$ becomes the neutral element with respect to multiplication and $-[(m,n)]$ becomes the additive inverse of $[(m,n)]$.

PROOF: Drudge work. ■

Remark 17.3.

Again, if it had been possible to define subtraction $m-n$ for all $m,n \in \mathbb{N}_0$ then it would be easier to see why addition and multiplication have been defined as you see it in Definition 17.6:

Addition is defined such that $(m_1-n_1)+(m_2-n_2) = (m_1+m_2)-(n_1+n_2)$
and multiplication: $(m_1-n_1)\cdot(m_2-n_2) = \big(m_1m_2 + (-n_1)(-n_2)\big) - \big(m_1n_2+n_1m_2\big)$. $\square$

17.3 Constructing the Rational Numbers from $\mathbb{Z}$

Again, we suggest that you review B/G project 6.9 in ch.6.1 and B/G prop.6.25 in ch.6.3.

Definition 17.7 (Fractions as equivalence classes)**.**

We define the following equivalence relation $(p,q) \sim (r,s)$ on the cartesian product $\mathbb{Z} \times \mathbb{Z}_{\neq 0}$:

$$(p,q) \sim (r,s) \;\Leftrightarrow\; p \cdot s \;=\; q \cdot r \tag{17.10}$$

We write $\mathbb{Q} := \{[(p,q)] : p,q \in \mathbb{Z} \text{ and } q \neq 0\}$. In other words, $\mathbb{Q}$ is the set of all equivalence classes with respect to the equivalence relation (17.10).

We "embed" $\mathbb{Z}$ into $\mathbb{Q}$ with the injective function $e : \mathbb{Z} \to \mathbb{Q}$ defined as $e(z) := [(z,1)]$. $\square$

Remark 17.4.

(a) The intuition that guided the above definition is that the pairs $(12,4), (-21,-7), (105,35)$ all define the same fraction $3/1$ and the pairs $(4,-12), (-7,21), (-35,105$ all define the same fraction $-1/3$.

(b) If it had been possible to define division p/q for all $p,q \in \mathbb{Z}$ for which $q \neq 0$ then (17.10) could be rewritten as

$$(p,q) \sim (r,s) \;\Leftrightarrow\; p/q \;=\; r/s$$

Looking at the equivalent pairs $(12,4), (-21,-7), (105,35)$ we get $12/4 \;=\; (-21)/(-7) \;=\; 105/35 \;=\; 3$ and for $(4,-12), (-7,21), (-35,105)$ we get $4/(-12) \;=\; (-7)/21 \;=\; (-35)/105 \;=\; -1/3$.

(c) It is easy to see that $(p,q) \sim (r,s)$ if and only if there is (rational) $\alpha \neq 0$ such that $r = \alpha p$ and $s = \alpha q$. A formal proof is just drudgework. $\square$

Definition 17.8 (Addition, multiplication, subtraction and division in $\mathbb{Q}$)**.**

Let $[(p_1,q_1)]$ and $[(p_2,q_2)] \in \mathbb{Q}$. We define

$$-[(p_1,q_1)] \;:=\; [(-p_1,q_1)], \tag{17.11}$$
$$[(p_1,q_1)] + [(p_2,q_2)] \;:=\; [(p_1q_2 + q_1p_2, n_1n_2)] \tag{17.12}$$
$$[(p_1,q_1)] - [(p_2,q_2)] \;:=\; [(p_1,q_1)] + \big(-[(p_2,q_2)]\big) \tag{17.13}$$
$$[(p_1,q_1)] \cdot [(p_2,q_2)] \;:=\; [(p_1p_2, q_1q_2)] \tag{17.14}$$
$$[(p_1,q_1)]^{-1} \;:=\; [(1,1)]/[(p_1,q_1)] \;:=\; [(q_1,p_1)] \text{ (if } p_1 \neq 0), \tag{17.15}$$
$$[(p_1,q_1)]/[(p_2,q_2)] \;:=\; [(p_1q_2, q_1p_2)] \;=\; [(p_1,q_1)] \cdot [(p_2,q_2)]^{-1} \text{ (if } p_2 \neq 0) \tag{17.16}$$

$\square$

It turns out that operations above are "well defined" in the sense that the resulting equivalence classes on the right of each of the three equations above do not depend on the choice of representatives in the classes on the left. [3]

Further, we have

Proposition 17.6 (Trichotomy of the rationals).

Let $x \in \mathbb{Q}$. Then exactly one of the following is true:
- ***(a)*** *Either **(a)** $x > 0$, i.e., $x = [(p, q)]$ for some $p, q \in \mathbb{N}$,*
- ***(b)*** *$-x > 0$, i.e., $x = [(-p, q)]$ for some $p, q \in \mathbb{N}$,*
- ***(c)*** *$x = 0$.* □

PROOF: Drudge work. ■

17.4 Constructing the Real Numbers via Dedekind Cuts

The material presented here, including the notation, follows [4] Rudin, Walter: Principles of Mathematical Analysis.

Note that in this section small greek letters denote **sets** of rational numbers!

The idea behind real numbers as intervals of rational numbers with no lower bounds, called Dedekind cuts, is as follows:

Given a real number x, one can associate with it the set

$$\{q \in \mathbb{Q} : q < x\}.$$

This set will be called the cut aka Dedekind cut associated with x. The mapping

$$\Phi : x \mapsto \Phi(x) := \{q \in \mathbb{Q} : q < x\} \tag{17.17}$$

is injective because, if $x, y \in \mathbb{R}$ such that $x \neq y$, say, $x < y$, then we have

$$\{q \in \mathbb{Q} : q < x\} \subsetneq \{q \in \mathbb{Q} : q < y\},$$

because there are (infinitely many) rational numbers in the open interval $]x, y[$.

We obtain surjectivity of Φ for free, if we take as codomain the set of all cuts.

Because Φ is bijective we can "identify" any real number with its cut. We now go in reverse: we start with a definition of cuts which does not reference the real number x, Rather., we define them just in terms of rational numbers and define addition, multiplication and the other usual operations on those cuts and show that those cuts have all properties of the real numbers as they were axiomatically defined in B/G ch.8, including the **completeness axiom** which states that each subset A of $\mathbb{R}$ with upper bounds has a least upper bound $\sup(A)$, i.e., a minimum in the set of all its upper bounds.

Definition 17.9 (Dedekind cuts).

[3] This was shown for multiplication $[(p_1, q_1)] \cdot [(p_2, q_2)] = [(p_1 p_2, q_1 q_2)]$ in exercise **??** on p.**??**.

(Rudin, def.1.4)

We call a subset $\alpha \subseteq \mathbb{Q}$ a **cut** or **Dedekind cut** if it satisfies the following:

(a) $\alpha \neq \emptyset$ and $\alpha^{\complement} \neq \emptyset$

(b) Let $p, q \in \mathbb{Q}$ such that $p \in \alpha$ and $q < p$. Then $q \in \alpha$.

(c) α does not have a max: $\forall p \in \alpha\ \exists q \in \alpha$ such that $p < q$.

Given a cut α, let $p \in \alpha$ and $q \in \alpha^{\complement}$. We call p a **lower number** of the cut α and we call q an **upper number** of α. $\square$

Theorem 17.1.

(Rudin thm.1.5)

Let $\alpha \subseteq \mathbb{Q}$ be a cut. Let $p \in \alpha, q \in \alpha^{\complement}$. Then $p < q$.

Assume to the contrary that $q \leq p$. Then we either have $p = q$ which means that either both p, q belong to α or both belong to its complement, a contradiction to our assumption. Or we have $q < p$. It then follows from $p \in \alpha$ and Definition 17.9.b that $q \in \alpha$, contrary to our assumption. $\blacksquare$

Theorem 17.2.

(Rudin thm.1.6)

Let $r \in \mathbb{Q}$. Let $r^\star := \{p \in \mathbb{Q} : p < r\}$. Then $r^\star$ is a cut, and $r = \min\left((r^\star)^{\complement}\right)$.

PROOF: In the following let $p, q, r \in \mathbb{Q}$.

PROOF of Definition 17.9.a: $r - 1 < r \Rightarrow r - 1 \in r^\star \Rightarrow r^\star \neq \emptyset$. Further, $r \in (r^\star)^{\complement} \Rightarrow (r^\star)^{\complement} \neq \emptyset$.

PROOF of Definition 17.9.b: Let $q < p$ and $p \in r^\star$. Then also $q \in r^\star = \{p' \in \mathbb{Q} : p' < r\}$.

PROOF of Definition 17.9.c: Let $p \in r^\star$. Then $p < (p+r)/2 < r$, hence $(p+r)/2 \in r^\star$ and r cannot be the max of $r^\star$. $\blacksquare$

Definition 17.10 (Rational cuts)**.**

Let $r \in \mathbb{Q}$. The cut

$$r^\star = \{p \in \mathbb{Q} :\ p < r\}$$

from the previous theorem is called the **rational cut** associated with r. $\square$

Remark 17.5.
If we define intervals in $\mathbb{Q}$ in the usual way for $p, q \in \mathbb{Q}$:

$$]p,q[\;:=\; \{r \in \mathbb{Q} : p < r < q\}, \quad [p,q] \;:=\; \{r \in \mathbb{Q} : p \le r \le q\}, \;\text{ etc.}$$

then rational cuts $r^\star (r \in \mathbb{Q})$ are those for which $r^\star =]-\infty, r[$ and $(r^\star)^\complement = [r, \infty[$ whereas for non-rational cuts α we cannot specify the "thingy" that should take the role of r. It would be the $\sup(\alpha)$ if we already had defined the set of all real numbers and we could understand α as a subset of those real numbers. $\square$

Definition 17.11 (Ordering Dedekind cuts)**.**

(Rudin def.1.9) Let α, β be two cuts.
We say $\alpha < \beta$ if $\alpha \subsetneq \beta$ (strict subset) and we say $\alpha \le \beta$ if $\alpha < \beta$ or $\alpha = \beta$, i.e., $\alpha \subseteq \beta$. $\square$

Proposition 17.7 (Trichotomy of the cuts)**.**

(Rudin thm.1.10)
Let α, β be two cuts. Then either $\alpha < \beta$ or $\alpha > \beta$ or $\alpha = \beta$.

PROOF: We only need to show that if $\alpha \nsubseteq \beta$ then $\beta \subsetneq \alpha$.
So let $\alpha \nsubseteq \beta$. Then $\alpha \setminus \beta$ is not empty and there exists $q \in \alpha \setminus \beta$.
But then $q > b$ for all $b \in \beta$. Also, if $a \in \mathbb{Q}$ and $a \le q$ then $a \in \alpha$ (we applied Definition 17.9.b twice.)
As $b < q$ for all $b \in \beta$ it follows that $\beta \subseteq \alpha$. We saw earlier that $\alpha \setminus \beta \ne \emptyset$ and this proves that $\beta \ne \alpha$, i.e., $\beta \subsetneq \alpha$. $\blacksquare$

Theorem 17.3 (Addition of two cuts)**.**

(Rudin thm.1.12) Let α, β be two cuts and let

$$\alpha + \beta \;:= \{a + b : a \in \alpha, b \in \beta\}.$$

Then the set of all cuts is an abelian group with this operation. In other words, + is commutative and associative with a neutral element (which turns out to be $0^\star$, the rational cut corresponding to $0 \in \mathbb{Q}$) and a suitably defined cut $-\alpha$ for a given cut α which satisfies $\alpha + (-\alpha) = (-\alpha) + \alpha = 0^\star$
Having defined negatives $-\alpha$ for all cuts we then also can define their absolute values

$$|\alpha| \;:=\; \begin{cases} \alpha & \text{if} \, \alpha \ge 0^\star, \\ -\alpha & \text{if} \, \alpha < 0^\star. \end{cases}$$

PROOF: Not given here. $\blacksquare$

Theorem 17.4 (Multiplication of two cuts)**.**

Let $\alpha \geq 0^\star, \beta \geq 0^\star$ be two nonnegative cuts. Let

$$\alpha \cdot \beta := \begin{cases} \{q \in \mathbb{Q} : q < 0\} \cup \{ab : a \in \alpha, b \in \beta\} & \text{if } \alpha \geq 0^\star, \beta \geq 0^\star, \\ -|\alpha| \cdot |\beta| & \text{if } \alpha < 0^\star, \beta \geq 0^\star \text{ or } \alpha \geq 0^\star, \beta < 0^\star, \\ |\alpha| \cdot |\beta| & \text{if } \alpha < 0^\star, \beta < 0^\star. \end{cases}$$

Then the set $\alpha \cdot \beta$ is a cut, called the product of α and β.

It can be proved that for each cut $\alpha \neq 0^\star$ there is a cut α^{-1} uniquely defined by the equation $\alpha \cdot \alpha^{-1} = 1^\star$.

Theorem 17.5 (The set of all cuts forms a field)**.**

Let Let $\mathbb{R}$ be the set of all cuts. Then $\mathbb{R}$ satisfies axioms 8.1 - 8.5 of B/G:

Addition and multiplication are both commutative and associative and the law of distributivity $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$ holds.

The cut $0^\star$ is the neutral element for addtition and the cut $1^\star$ is the neutral element for multiplication.

$-\alpha$ is the additive inverse of any cut α and α^{-1} is the multiplicative inverse of $\alpha \neq 0^\star$.

Further the set $\mathbb{R}_{>0} := \{\alpha \in \mathbb{R} : \alpha > 0^\star\}$ satisfies B/G axiom 8.26.

PROOF: It follows from prop.17.7 on p.723 that $\mathbb{R}_{>0}$ satisfies B/G axiom 8.26. Proofs of the other properties of $\mathbb{R}$ are not given here. ■

In the remainder of this section we will see that the completeness axiom B/G ax.8.52 (every subset of $\mathbb{R}$ with upper bounds has a supremum) is a consequence from the properties of cuts and there is no need to state it as an axiom.

Theorem 17.6.

(Rudin thm.1.29)

Let $\alpha, \beta \in \mathbb{R}$ and let $\alpha < \beta$. Then there exists $q \in \mathbb{Q}$ such that $\alpha < q^\star < \beta$

PROOF: Any $q \in \beta \setminus \alpha$ will do. ■

Theorem 17.7.

(Rudin thm.1.30) Let $\alpha \in \mathbb{R}, p \in \mathbb{Q}$. Then $p \in \alpha \iff p^\star < \alpha$, i.e., $p^\star \subsetneq \alpha$

PROOF of $\Leftarrow$): Let $p \in \alpha$. it follows for any $q \in p^\star$ that $q < p \in \alpha$, hence $q \in \alpha$, hence $p^\star \subseteq \alpha$. As $p \notin p^\star = \{p' \in \mathbb{Q} : p' < p\}$ but $p \in \alpha$ we have strict inclusion $p^\star \subsetneq \alpha$.

PROOF of $\Rightarrow$): As $p^\star \subsetneq \alpha$ there exists $q \in \alpha \setminus p^\star$. As $q \geq p$ and $q \in \alpha$ we obtain $p \in \alpha$ from Definition 17.9.b. ■

Theorem 17.8 (Dedekind's Theorem)**.**

(Rudin thm.1.32)
Let $\mathbb{R} = A \biguplus B$ a partitioning of $\mathbb{R}$ such that

(a) $A \neq \emptyset$ *and* $B \neq \emptyset$
(b) $\alpha \in A, \beta \in B \ \Rightarrow \ \alpha < \beta$ *(i.e.,* $\alpha \subsetneq \beta$*).*

Then there exists a unique cut $\gamma \in \mathbb{R}$ such that if $\alpha \in A$ then $\alpha \leq \gamma$ and if $\beta \in B$ then $\gamma \leq \beta$.

PROOF: We first prove uniqueness and afterwards the existence of γ.

PROOF of uniqueness: Assume there is $\gamma'' \in \mathbb{R}$ which satisfies $\alpha \leq \gamma''$ for all $\alpha \in A$ and $\gamma'' \leq \beta$ for all $\beta \in B$.

We may assume that $\gamma < \gamma''$. It follows from thm.17.6 on p.724 that there is $\gamma' \in \mathbb{R}$ (matter of fact, a rational cut) such that $\gamma < \gamma' < \gamma''$. But $\gamma < \gamma'$ implies that $\gamma' \in B$ and $\gamma' < \gamma''$ implies that $\gamma' \in A = B^\complement$. We have reached a contradiction and conclude that γ must be unique.

PROOF of existence of γ: Let $\gamma \ := \ \bigcup \big[\alpha : \alpha \in A\big]$.

Step 1: We now show that γ is a cut.

We first show that Definition 17.9.a is satisfied. As $B \neq \emptyset$ there is some $\beta \in B$. As $\beta^\complement \neq \emptyset$ there is some $q \in \beta^\complement$. It follows from $\alpha \subseteq \beta$ for all $\alpha \subseteq \gamma = \ \bigcup \big[\alpha : \alpha \in A\big]$ that $\gamma \subseteq \beta$, hence $\gamma^\complement \supseteq \beta^\complement$. It follows from $q \in \beta^\complement$ that $q \in \gamma^\complement$, hence $\gamma^\complement \neq \emptyset$. Further, it follows from $A \neq \emptyset$ that $\gamma \neq \emptyset$. We conclude that Definition 17.9.a is satisfied.

Next we show the validity of Definition 17.9.b. Let $p \in \gamma$, i.e., $p \in \alpha_0$ for some $\alpha_0 \in A$. Let $q < p$. Then $q \in \alpha_0 \subseteq \ \bigcup \big[\alpha : \alpha \in A\big]$, i.e., $q \in \gamma$. We conclude that Definition 17.9.b is satisfied.

Now we show the validity of Definition 17.9.c. Let $p \in \gamma$, i.e., $p \in \alpha_0$ for some $\alpha_0 \in A$. As the cut α_0 does not have a maximum there exists some $q \in \alpha_0$ such that $q > p$. As $\alpha_0 \subseteq \gamma$, hence $q \in \gamma$ We have seen that any $p \in \gamma$ is strictly dominated by some $q \in \gamma$. It follows that γ does not have a max and this shows that Definition 17.9.c is satisfied. We conclude that γ is a cut and step 1 of the proof for existence is completed.

Step 2: It remains to show that $\alpha \leq \gamma \leq \beta$ for all $\alpha \in A$ and $\beta \in B$. It is trivial that $\alpha \leq \gamma$ for all $\alpha \in A$ because $\gamma \ := \ \bigcup \big[\alpha : \alpha \in A\big]$.

To show that $\gamma \leq \beta$ for all $\beta \in B$ we prove that the opposite statement that

$$\gamma > \beta, \ \text{ i.e., } \gamma \setminus \beta \neq \emptyset \ \text{ for some cut } \beta \in B \tag{17.18}$$

will lead to a contradiction. As $q \in \gamma$ there is some $\alpha_0 \in A$ such that $q \in \alpha_0$. Actually, $q \in \alpha_0 \setminus \beta$ because $q \notin \beta$. But then $\alpha_0 \nless \beta$ even though $\alpha_0 \in A$ and $\beta \in B$, contrary to the assumptions about the partitioning $A \biguplus B$ of $\mathbb{R}$. ■

Corollary 17.1.

Let $\mathbb{R} = A \uplus B$ be a partitioning of $\mathbb{R}$ such that

$$\textbf{\textit{(a)}}\ A \neq \emptyset \text{ and } B \neq \emptyset$$
$$\textbf{\textit{(b)}}\ \alpha \in A, \beta \in B \ \Rightarrow\ \alpha < \beta \ \ (i.e.,\ \alpha \subsetneq \beta).$$

Then either $\max(A)(= l.u.b.(A))$ *exists or* $\min(B)(= g.l.b.(B))$ *exists.*

PROOF: According to thm.17.8 there exists $\gamma \in \mathbb{R}$ such that if $\alpha \in A$ then $\alpha \leq \gamma$ and if $\beta \in B$ then $\gamma \leq \beta$. Clearly γ is an upper bound of α and a lower bound of β. It follows that if $\gamma \in A$ then $\max(A) = \gamma$ and if $\gamma \notin A$, i.e., $\gamma \in B$, then $\min(B) = \gamma$. ■

Theorem 17.9 (Completeness theorem for $\mathbb{R}$)**.**

(Rudin thm.1.36)
Let $\emptyset \neq E \subset \mathbb{R}$ and assume that E is bounded above. Then E has a least upper bound.
It is denoted by $\sup(E)$, *also* $l.u.b.(E)$.

PROOF: Let B be the set of all upper bounds for E, i.e., $b \in B$ if and only if $b \geq x$ for all $x \in E$. Then B is not empty by assumption. Let $A := B^{\complement} = \{\alpha \in \mathbb{R} : \alpha < x \text{ i.e., } \alpha \subsetneq x \text{ for some } x \in E\}$. In other words, $\alpha \in A$ if and only if α is not an upper bound of E.

A is not empty either: As $E \neq \emptyset$ there is some $x \in E$. Let $\alpha := x - 1$. Cleary $x \leq \alpha$ is not true for all $x \subseteq E$. It follows that α is not an upper bound of E, hence $\alpha \in A$, hence A is not empty.

Moreover we have $\alpha < \beta$ for all $\alpha \in A$ and $\beta \in B$. Because for any $\alpha \in A$ there is some $x \in E$ such that $\alpha < x$ and we have $x \leq \beta$ for all upper bounds β, i.e., for all $\beta \in B$.

It follows that the sets A and B form a partition which satisfies the requirements of Dedekind's Theorem (thm.17.8). Hence there exists $\gamma \in \mathbb{R}$ such that $\alpha \leq \gamma \leq \beta$ for all $\alpha \in A$ and $\beta \in B$.

We now show that the assumption $\gamma \in A$ leads to a contradiction. As γ is not an upper bound of A there exists $x \in E$ such that $\gamma < x$. According to thm.17.6 on p. 724 there exists $\gamma' \in \mathbb{R}$ such that $\gamma < \gamma' < x$. It follows that $\gamma' \notin B$, i.e., $\gamma' \in A$, in contradiction to the fact that $\gamma \geq a$ for all $a \in A$.

It follows that $\gamma \notin A$, i.e., $\gamma \in B$ and we conclude from cor.17.1 that $\gamma = \min(B)$, i.e., $\gamma = \sup(E)$. ■

17.5 Constructing the Real Numbers via Cauchy Sequences

This chapter was created after discussions with Nguyen-Phan Tam about teaching the Math 330 course: she plans to construct the real numbers from the rationals by means of equivalence classes of Cauchy sequences in $\mathbb{Q}$.

In the following we always assume that

$$i, j, k, m, n \in \mathbb{N},\ \varepsilon, p, q, r, s, p_n, p_{i,j}, \dots \in \mathbb{Q},\ x, y, z, x_n, x_{i,j}, \dots \in \mathbb{R}\,.$$

The construction of the real numbers from the rationals is done according to the following steps:

(a) def. convergence in $\mathbb{Q}$: $\lim\limits_{n\to\infty} q_n = q \iff \forall$ pos. $\varepsilon \in \mathbb{Q}\ \exists\ N \in \mathbb{Q}$ such that if $n \geq N$ then $|q_n - q| < \varepsilon$.

(b) def. Cauchy seqs. in $\mathbb{Q}$: $(q_n)_n$ is Cauchy $\iff \forall$ pos. $\varepsilon \in \mathbb{Q}\ \exists\ N \in \mathbb{Q}$ such that if $i, j \geq N$ then $|q_i - q_j| < \varepsilon$.

(c) Let $\mathscr{C} := \{$ all Cauchy sequences in $\mathbb{Q}\}$. For $(q_n)_n, (r_n)_n$ we define $(q_n)_n \sim (r_n)_n$ iff $\lim\limits_{n\to\infty}(r_n - q_n) = 0$.

(d) Let $q \in \mathbb{Q}$ and $q_n := q\ \forall\ n$. Write q for $[(q_n)_n]$.

(e) Let $\mathbb{R} := \mathscr{C}_{/\sim}$. Show that for $[(p_n)], [(q_n)] \in \mathscr{C}$ the operations $([(p_n)_n], [(q_n)_n]) \mapsto [(p_n + q_n)_n]$ and $([(p_n)_n], [(q_n)_n]) \mapsto [(p_n \cdot q_n)_n]$ are well defined (do not depend on the particular members chosen from the equivalence classes).

(f) Let $[(p_n)_n] \neq 0$ (i.e., $\lim\limits_n p_n \neq 0$), i.e., we may assume $p_n \neq 0$ for all n. Show $-[(q_n)_n] := [(-q_n)_n]$ and $[(p_n)_n]^{-1} := [(1/p_n)_n]$ are additive and multiplicative inverses

g1. Define $[(p_n)_n] < [(q_n)_n]$ iff $\exists\ \varepsilon > 0$ and $N \in \mathbb{N}$ such that $q_n - p_n \geq \varepsilon\ \forall\ n \geq N$.

g2. Define $[(p_n)_n] \leq [(q_n)_n]$ iff $\forall\ \varepsilon > 0$ *exists* $N \in \mathbb{N}$ such that $q_n - p_n \geq -\varepsilon\ \forall\ n \geq N$.

g3. show that $[(p_n)_n] < [(q_n)_n]$ iff $[(p_n)_n] \leq [(q_n)_n]$ and $[(p_n)_n] \neq [(q_n)_n]$.

(h) Show that $(\mathbb{R}, +, \cdot, <)$ satisfies the axioms of B/G ch.8 with the exception of the completeness axiom.
Easy to see this specific item: If $[(p_n)_n] > 0$ then there is $[(q_n)_n] > 0$ such that $[(q_n)_n] < [(p_n)_n]$: choose $\varepsilon > 0$ as in **g1** (remember: $\varepsilon \in \mathbb{Q}$) and set $q_n := \varepsilon/2$.

(i) Embed $\mathbb{Q}$ into $\mathbb{R}$: $q \mapsto \bar{q} := [(q, q, q, \dots)]$.

(j) Define limits and Cauchy sequences in $\mathbb{R}$ just as in **(a)** and **(b)**.

k. Let $(q_n)_n$ be Cauchy in $\mathbb{Q}$. Prove that $\bar{q_n} \to [(q_j)_j]$

l. Let $x_n \in \mathbb{R}$ such that $(x_n)_n$ is Cauchy in $\mathbb{R}$. With a density argument we find $q_n \in \mathbb{Q}$ such that $x_n \leq \bar{q}_n \leq x_n + 1/n$. Now show that **(1)** $(q_n)_n$ is Cauchy and then **(2)** $\lim\limits_n x_n = [(q_n)_n]$.

m. Prove completeness according to B/G: If nonempty $A \subseteq \mathbb{R}$ is bounded above then its set of upper bounds U has a min: Let $Q_n := \{i/j : i, j \in \mathbb{Z}$ and $j \leq n\}$. Let $U_n := U \cap Q_n$. Let $u_n := \min(U_n)$ (exists because $n \cdot U_n \subset Z$ is bounded below and has a min. Easy to see that u_n is Cauchy (in $\mathbb{Q}$ and, because distance$(u_n, A) \leq 1/n$, $[(u_n)_n]$ is the least upper bound of A.

Proofs for **(k)** and **l** in particular and an entire section on constructing $\mathbb{R}$ from $\mathbb{Q}$ by means of equivalence classes of Cauchy sequences can be found in [3] Haaser/Sullivan: Real Analysis.

18 Measure Theory ⋆

Note that this entire section is starred, hence optional.
Introduction:
The following are the best known examples of measures ($a_j, b_j \in \mathbb{R}$):

$$\begin{aligned}
&\text{Length :}\quad \lambda^1([a_1,b_1]) \;:=\; b_1-a_1,\\
&\text{Area :}\quad \lambda^2([a_1,b_1]\times[a_2,b_2]) \;:=\; (b_1-a_1)(b_2-a_2),\\
&\text{Volume :}\quad \lambda^3([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]) \;:=\; (b_1-a_1)(b_2-a_2)(b_3-a_3).
\end{aligned}$$

Then there also are probability measures: $P\{$ a die shows a 1 or a 6$\} = 1/3$.
We will explore in this chapter some of the basic properties of measures.

18.1 Basic Definitions

Definition 18.1 (Extended real–valued functions)**.**

$$\overline{\mathbb{R}}_+ \;:=\; \mathbb{R}_+ \;\cup\; \{+\infty\} \;=\; \{x\in\mathbb{R} : x\geq 0\} \;\cup\; \{+\infty\}$$

is the set of all nonnegative real numbers augmented by the elements ∞ and $-\infty$.
A function $F : X \to Y$ whose codomain Y is a subset of

$$\overline{\mathbb{R}} \;:=\; \mathbb{R} \;\cup\; \{\infty\} \;\cup\; \{-\infty\}$$

is called an **extended real–valued function**.

There are many issues with functions that allow some arguments to have infinite value (hint: if $F(x) = \infty$ and $F(y) = \infty$, what is $F(x) - F(y)$?)
We only list the following rule which might come unexpected to you:

$$0\cdot\pm\infty \;=\; \pm\infty\cdot 0 \;=\; 0. \;\square \tag{18.1}$$

This convention is very convenient, but it comes at a price: it is no longer true that all sequences $(a_n)_n$ and $(b_n)_n$ of real numbers that have limits $a = \lim\limits_{n\to\infty} a_n$, $b = \lim\limits_{n\to\infty} b_n$, satisfy $\lim\limits_{n\to\infty} a_n b_n = ab$.
Counterexample: $a_n = n, b_n = \frac{1}{n}$.

The definition of a σ–algebra was given previously in Definition **??** (Rings, algebras, and σ–Algebras of Sets) on p.**??**. We repeat it here in equivalent terms for your convenience.

Definition 18.2 (σ–algebras)**.**

Let Ω be a nonempty set and let $\mathfrak{F}$ be a set that contains some, but not necessarily all, subsets of Ω.
$\mathfrak{F}$ is called a **σ–algebra** or **σ–field** for Ω if it satisfies the following:

$$\emptyset \in \mathfrak{F}, \tag{18.2a}$$

$$A \in \mathfrak{F} \quad \Rightarrow \quad A^{\complement} \in \mathfrak{F} \tag{18.2b}$$

$$(A_n)_{n\in\mathbb{N}} \in \mathfrak{F} \quad \Rightarrow \quad \bigcup_{n\in\mathbb{N}} A_n \in \mathfrak{F} \tag{18.2c}$$

- The pair $(\Omega, \mathfrak{F})$ is called a **measurable space**.
- The elements of $\mathfrak{F}$ (they are set!) are called **measurable sets**. $\square$

Remark 18.1.
If$\mathfrak{F}$ is a σ–algebra then

$$\emptyset \in \mathfrak{F} \quad \text{and} \quad \Omega \in \mathfrak{F} \tag{18.3a}$$

$$A \in \mathfrak{F} \quad \Rightarrow \quad A^{\complement} \in \mathfrak{F} \tag{18.3b}$$

$$(A_n)_{n\in\mathbb{N}} \in \mathfrak{F} \quad \Rightarrow \quad \text{and} \quad \bigcap_{n\in\mathbb{N}} A_n \in \mathfrak{F} \tag{18.3c}$$

The last assertion is a consequence of De Morgan's laws (Theorem **??** on p.**??**).

If $\mathfrak{F}$ is a σ–algebra then If countably many (i.e., a finite or infinite sequence of) operations are performed that involve
• unions, • intersections, • complements, • set differences, • symmetric differences of elements of $\mathfrak{F}$ then the resulting set also belongs to $\mathfrak{F}$. $\square$

Proposition 18.1 (Minimal sigma–algebras)**.** *Let Ω be a nonempty set.*
A*: The intersection of arbitrarily many σ–algebras is a σ–algebra.*
B*: Let $\mathfrak{E}$ be a set which contains subsets of Ω. It is not assumed that $\mathfrak{E}$ is a σ–algebra. Then there exists a σ–algebra which contains $\mathfrak{E}$ and is minimal in the sense that it is contained in any other σ–algebra that also contains $\mathfrak{E}$. We name this σ–algebra $\sigma(\mathfrak{E})$ because it clearly is uniquely determined by $\mathfrak{E}$. It is constructed as follows:*

$$\sigma(\mathfrak{E}) \;=\; \bigcap \{\mathfrak{F} : \mathfrak{F} \supseteq \mathfrak{E} \textit{ and } \mathfrak{F} \textit{ is a } \sigma\textit{–algebra for } \Omega\}. \tag{18.4}$$

PROOF of ***A***:
We must prove (18.2a), (18.2b) and (18.2c). Let $(\mathfrak{F}_\alpha)_\alpha$ be an arbitrary family of σ–algebras for Ω.

Let
$$\mathfrak{F} := \bigcap_{\alpha} \mathfrak{F}_\alpha .$$
$\emptyset$ and Ω belong to each σ–algebra according to (18.2a). It follows that they both belong to the intersection $\bigcap_\alpha \mathfrak{F}_\alpha$, i.e., $\mathfrak{F}$ satisfies (18.2a). Let $A \in \mathfrak{F}$. Then $A \in \mathfrak{F}_\alpha$ for each α. $\complement A$ belongs to each σ–algebra according to (18.2b). It follows that $\complement A \in \bigcap_\alpha \mathfrak{F}_\alpha$, i.e., $\mathfrak{F}$ satisfies (18.2b). Finally, let $A_n \in \mathfrak{F}$ for all $n \in \mathbb{N}$. Then $A_n \in \mathfrak{F}_\alpha$ for all $n \in \mathbb{N}$ and for each α, $\cup_{n\in\mathbb{N}} A_n$ and $\bigcap_{n\in\mathbb{N}} A_n$ both belong to each σ–algebra according to (18.2c). It follows that they both belong to the intersection $\bigcap_\alpha \mathfrak{F}_\alpha$, i.e., $\mathfrak{F}$ satisfies (18.2c). It follows that $\mathfrak{F}$ is a σ–algebra.
PROOF of ***B***:
First of all, we know that $\sigma(\mathfrak{E})$ is an intersection of σ–algebras and, according to part ***A*** of this proposition, really is a σ–algebra. We now prove that $\sigma(\mathfrak{E})$ contains $\mathfrak{E}$ and is the minimal σ–algebra with that property. First let us prove that $\sigma(\mathfrak{E}) \supseteq \mathfrak{E}$. But that is obvious because it is the intersection of sets all of which contain $\mathfrak{E}$. On the other hand, $\sigma(\mathfrak{E})$ is the intersection of **all** σ–algebras that contain $\mathfrak{E}$, so it is impossible for any other σ–algebra to both be a strict subset of $\sigma(\mathfrak{E})$ and also contain $\mathfrak{E}$. ■

Definition 18.3 (Abstract measures)**.**

Let $(\Omega, \mathfrak{F})$ be a measurable space.
A **measure** on $\mathfrak{F}$ is an extended real–valued function

$$\mu(\cdot) : \mathfrak{F} \to \overline{\mathbb{R}}_+; \quad A \mapsto \mu(A) \qquad \text{such that}$$

$$\mu(\emptyset) = 0 \qquad \text{(positivity)} \tag{18.5}$$
$$A, B \in \mathfrak{F} \text{ and } A \subseteq B \quad \Rightarrow \quad \mu(A) \leq \mu(B) \qquad \text{(monotony)} \tag{18.6}$$
$$(A_n)_{n\in\mathbb{N}} \in \mathfrak{F} \text{ disjoint} \quad \Rightarrow \quad \mu\Big(\biguplus_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \mu(A_n) \qquad (\sigma\text{–additivity}) \tag{18.7}$$

- The triplet $(\Omega, \mathfrak{F}, \mu)$ is called a **measure space**
- We call μ a **finite measure** on $\mathfrak{F}$ if $\mu(\Omega) < \infty$.
- If $\mu(\Omega) = 1$ then $\mu(\cdot)$ is called a **probability measure**. □

Disjointness in (18.7) means that $A_i \cap A_j = 0$ for any $i, j \in \mathbb{N}$ such that $i \neq j$ (see Definition **??** on p.**??**).

A measure space can support many different measures.

Traditionally, mathematicians write $P(A)$ rather than $\mu(A)$ for probability measures and the elements of $\mathfrak{F}$ (the measurable subsets) are thought of as **events** for which $P(A)$ is interpreted as the probability with which the event A might happen.

Example 18.1 (Lebesgue measure)**.**

The most important measures we encounter in real life are those that measure the length of sets in one dimension, the area of sets in two dimensions and the volume of sets in three dimensions. Given intervals $[a,b] \in \mathbb{R}$, rectangles $[a_1,b_1]\times[a_2,b_2] \in \mathbb{R}^2$, boxes or quads $[a_1,b_1]\times[a_2,b_2]\times[a_3,b_3] \in \mathbb{R}^3$ and n**-dimensional parallelepipeds** $[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n] \in \mathbb{R}^n$, we define

$$\begin{aligned}
\lambda^1([a,b]) &:= b-a,\\
\lambda^2([a_1,b_1]\times[a_2,b_2]) &:= (b_1-a_1)(b_2-a_2),\\
\lambda^3([a_1,b_1]\times[a_2,b_2]\times[a_3,b_3]) &:= (b_1-a_1)(b_2-a_2)(b_3-a_3),\\
\lambda^n([a_1,b_1]\times\cdots\times[a_n,b_n]) &:= (b_1-a_1)(b_2-a_2)\dots(b_n-a_n)
\end{aligned} \tag{18.8}$$

It can be shown that any measure that is defined on all parallelepipeds in $\mathbb{R}^n$ can be uniquely extended to a measure on the σ-algebra $\mathcal{B}^n$ generated by those parallelepipeds [4] λ^n is called n**-dimensional Lebesgue measure**

Note that Lebesgue measure is not finite. $\square$

Example 18.2.

You can easily verify that the following set function defines a measure on an arbitrary nonempty set Ω with an arbitrary σ-field $\mathfrak{F}$.

$$\mu(\emptyset) := 0; \qquad \mu(A) := \infty \text{ if } A \neq \emptyset$$

Keep this example in mind if you contemplate infinity of measures. $\square$

Remark 18.2 (Finite disjoint unions)**.** The σ–additivity of measures is what makes working with them such a pleasure in many ways. You can now express it as follows: Given any mutually disjoint sequence of measurable sets, the measure of the disjoint union is the sum of the measures. The last property (18.2c) for σ–algebras is required for exactly that reason: you cannot take advantage of the σ–additivity of a measure μ if its domain does not contain countable unions and intersections of all its constituents.

Note that if we have only finitely many sets then "σ–additivity" which stands for "additivity of countably many" becomes simple additivity. We obtain the following by setting $A_{N+1} = A_{N+2} = \ldots = 0$:

$$\begin{aligned}
&A_1, A_2, \dots, A_N \in \mathfrak{F} \text{ mutually disjoint}\\
&\qquad \Rightarrow\ \mu(A_1 \uplus A_2 \uplus \ldots \uplus A_N) \;=\; \mu(A_1) + \mu(A_2) + \ldots + \mu(A_N) \quad \text{(additivity)}.
\end{aligned} \tag{18.9}$$

In the case of only two disjoint measurable sets A and B the above simply becomes

$$\mu(A \uplus B) \;=\; \mu(A) + \mu(B). \ \square$$

In many circumstances you have a set function on a σ–algebra which behaves like a measure but you can only prove that it is additive instead of σ–additive. You should not be surprised that there is a special name for those "generalized measures":

[4] This is not entirely correct: we must demand that the measure is σ-finite, i.e., there are measurable sets with finite measure whose union is the entire space. Such is the case for Lebesgue measure: Let $A_k := [-k,k]^n$. The union of those sets is $\mathbb{R}^k$ and $\lambda^n(A_K) = (2k)^n < \infty$.

Definition 18.4 (Contents as additive measures)**.** Let Ω be a nonempty set and let $\mathfrak{F}$ be a σ–algebra for Ω.

A **content** on $\mathfrak{F}$ is a real–valued function $m(\cdot) : \mathfrak{F} \to \mathbb{R}$, $A \mapsto m(A)$ which satisfies

(18.10a) $m(\emptyset) = 0$ (positivity)

(18.10b) $A, B \in \mathfrak{F}$ and $A \subseteq B \Rightarrow m(A) \leq m(B)$ (monotony)

(18.10c) $A_1, A_2, \dots, A_N \in \mathfrak{F}$ mutually disjoint $\Rightarrow m\Big(\biguplus_{n=1}^{N} A_n\Big) = \sum_{n=1}^{N} m(A_n)$ (additivity). $\square$

Note that $\mu(\Omega) < \infty$ for a content μ. After this digression on contents let us go back to measures.

Proposition 18.2 (Simple properties of measures)**.** *Let $A, B, \in \mathfrak{F}$ and let μ be a measure on $\mathfrak{F}$. Then*

(18.11a) $\mu(A) \geq 0$ *for all* $A \in \mathfrak{F}$,

(18.11b) $A \subseteq B \Rightarrow \mu(B) = \mu(A) + \mu(B \setminus A)$,

(18.11c) $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$.

If μ is finite then also

(18.12a) $A \subseteq B \Rightarrow \mu(B \setminus A) = \mu(B) - \mu(A)$,

(18.12b) $\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$.

PROOF: The first property follows from the fact that $\mu(\emptyset) = 0$, $\emptyset \subseteq A$ for all $A \in \mathfrak{F}$ and (18.6.
To prove the second property, observe that $B = A \uplus (B \setminus A)$.
Proving the third property is more complicated because neither A nor B may be a subset of the other. We first note that because $A \setminus B \subseteq A$, $B \setminus A \subseteq A$ and $A \cap B \subseteq A$, $\mu(A \cup B) = \infty$ can only be true if $\mu(A) = \infty$ or $\mu(B) = \infty$. In this case (18.11c) is obviously true. Hence we may assume that $\mu(A \cup B) < \infty$. We have

(18.13a) $A \cup B = (A \cap B) \uplus (B \setminus A) \uplus (A \setminus B)$

(18.13b) $A \cup B = A \uplus (B \setminus A) = B \uplus (A \setminus B)$

It follows from (18.13a) that

(18.14) $\mu(A \cup B) = \mu(A \cap B) + \mu(B \setminus A) + \mu(A \setminus B)$

It follows from (18.13b) that

(18.15) $2 \cdot \mu(A \cup B) = \mu(A) + \mu(B \setminus A) + \mu(B) + \mu(A \setminus B)$

We subtract the left and right sides of (18.14) from those of (18.15) and obtain

$$\begin{aligned}\mu(A \cup B) &= \mu(A) + \mu(B \setminus A) + \mu(B) + \mu(A \setminus B) - \mu(A \cap B) - \mu(B \setminus A) - \mu(A \setminus B)\\ &= \mu(A) + \mu(B) - \mu(A \cap B)\end{aligned}$$

and the third property is proved. ■

18.2 Sequences of Sets – limsup and liminf

Assumption 18.1 (Existence of a universal set)**.** We assume the existence of a set X which contains all sets A_n, B_n, C_n that are used here in sequences. $\square$

Definition 18.5 (Monotone set sequences)**.** A sequence A_k of arbitrary subsets of X is called

(18.16a) **nondecreasing** if $A_1 \subseteq A_2 \subseteq \dots$

(18.16b) **nonincreasing** if $A_1 \supseteq A_2 \supseteq \dots$

(18.16c) **strictly increasing** if $A_1 \subsetneq A_2 \subsetneq \dots$

(18.16d) **strictly decreasing** if $A_1 \supsetneq A_2 \supsetneq \dots$

Each one of those sequences is called a **monotone set sequence**. $\square$

Might as well define limits of monotone sequences of sets. It's certainly intuitive enough:

Definition 18.6 (Limits of monotone set sequences)**.** Given are sets $A_n, B_n \subseteq X$ $(n \in \mathbb{N})$. Assume that

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots \qquad \text{and let } A := \bigcup_{k \in \mathbb{N}} A_k$$
$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots \qquad \text{and let } B := \bigcap_{k \in \mathbb{N}} B_k$$

We say that A is the limit of the sequence $(A_j)_{j \in \mathbb{N}}$ and B is the limit of the sequence $(B_j)_{j \in \mathbb{N}}$ and we write

$$A = \lim_{n \to \infty} A_n \quad \text{or} \quad A_n \uparrow A \ \text{ for } n \to \infty, \tag{18.17a}$$
$$B = \lim_{n \to \infty} B_n \quad \text{or} \quad B_n \downarrow B \ \text{ for } n \to \infty. \ \square \tag{18.17b}$$

The above are not terribly useful definitions. What does it matter whether we write $A = \lim\limits_{n \to \infty} A_n$ or $A = \cup_{k \in \mathbb{N}} A_k$? Things would be very different if we went further and defined limits of sequences of sets. Doing so is at the very beginning of a branch of Mathematics called Measure Theory and its (slightly) more applied version, Abstract Probability Theory.

Definition 18.7 (lim inf and lim sup of set sequences)**.** Given are sets $A_n, B_n \subseteq X$ $(n \in \mathbb{N})$. Let

$$\liminf_{n \to \infty} A_n := \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k \quad (\textbf{limit inferior}) \tag{18.18}$$
$$\limsup_{n \to \infty} A_n := \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} A_k \quad (\textbf{limit superior}) \tag{18.19}$$

In general those two will not coincide. But if they do then we define

$$\lim_{n\to\infty} A_n \ := \ \liminf_{n\to\infty} A_n \ = \ \limsup_{n\to\infty} A_n \tag{18.20}$$

We call $\lim\limits_{n\to\infty} A_n$ the **limit** of the sequence (A_n) and we write

$$A_n \to A \quad \text{for } n \to \infty. \ \square$$

The following comments should make matters easier to understand if you abbreviate

Lemma 18.1 (lim inf and lim sup as monotone limits)**.** *Given are sets* $A_n, B_n \subseteq X$ $(n \in \mathbb{N})$*. Let*

$$A_{\star n} \ := \bigcap_{k \geq n} A_k \qquad \textit{Then} \quad A_{\star n} \uparrow \liminf_{n\to\infty} A_n \tag{18.21}$$

$$A^{\star}{}_n \ := \bigcup_{k \geq n} A_k \qquad \textit{Then} \quad A^{\star}{}_n \downarrow \limsup_{n\to\infty} A_n \tag{18.22}$$

PROOF: Let $m, n \in \mathbb{N}$ such that $m < n$. Then

$$\begin{aligned} A_{\star m} &= \bigcap_{k=m}^{n-1} A_k \cap \bigcap_{k \geq n} A_k &= \bigcap_{k=m}^{n-1} A_k \cap A_{\star n} &\subseteq A_{\star n} \\ A^{\star}{}_m &= \bigcup_{k=m}^{n-1} A_k \cup \bigcup_{k \geq n} A_k &= \bigcup_{k=m}^{n-1} A_k \cup A^{\star}{}_n &\supseteq A^{\star}{}_n \end{aligned}$$

This proves that $A_{\star n}$ is nondecreasing and $A * \star_n$ is nonincreasing. By the very definition of the limit of a monotone sequence of sets it is true that

$$\begin{aligned} \lim_{n\to\infty} A_{\star n} &= \bigcup_{n \in \mathbb{N}} A_{\star n} = \liminf_{n\to\infty} A_n \\ \lim_{n\to\infty} A^{\star}{}_n &= \bigcap_{n \in \mathbb{N}} A^{\star}{}_n = \limsup_{n\to\infty} A_n \end{aligned}$$

■

18.3 Conditional Expectations as Generalized Averages

EMPTY!!
EMPTY!!
EMPTY!!

19 Algebraic Structures $\boxed{\star}$

This chapter is at its very beginnings. It has been created because it is mentioned in one of the first lectures that the axiomatically defined set $\mathbb{Z}$ of the first chapter of [2] B/G forms a group.

Note that this chapter is starred and hence optional.

19.1 More on Groups ($\star$)

This chapter adds to the material of ch.**??** (Semigroups and Groups) on p.**??**.

Example 19.1.

Being a group is a lot more specific than just being a semigroup or monoid. Not all types of numbers form groups for addition and/or multiplication:

Natural numbers:	Neither $(\mathbb{N},+)$ nor $(\mathbb{N},\cdot)$ are groups: $(\mathbb{N},+)$ does not even have a neutral element, $(\mathbb{N},\cdot)$ has 1 as a neutral element but there is no multiplicative inverse for, say, 5 because $1/5 \notin \mathbb{N}$.
Integers:	We have seen in example **??** that $(\mathbb{Z},+)$ is an abelian group but $(\mathbb{Z},+)$ is not a group.
Rational numbers:	$(\mathbb{Q},+)$ is an abelian group but $(\mathbb{Q},\cdot)$ is **not** a group because the number 0 does not have a multiplicative inverse: There is no number x such that $0 \cdot x = 1$. But note that the set $\mathbb{Q}^\star$ of all non-zero rational numbers is an abelian group.
Real numbers:	$(\mathbb{R},+)$ is an abelian group but $(\mathbb{R},\cdot)$ is **not** a group for the same reason as $(\mathbb{Q},\cdot)$. Again, the set $\mathbb{R}^\star$ of all non-zero real numbers is an abelian group.
Complex numbers:	$(\mathbb{C},+)$ is an abelian group but $(\mathbb{C},\cdot)$ is **not** a group for the same reason as $(\mathbb{Q},\cdot)$. However the set $\mathbb{C}^\star$ of all non-zero complex numbers is an abelian group. $\square$

We now turn our attention to functions which map from a group to another group in such a way that they are, in a sense, compatible with the binary operations on their domain and codomain.

Example 19.2.

Let $(G,\diamond)$ and $(H,\bullet)$ be defined as follows:

$$G := \{g \in \mathbb{R} : g = e^x \text{ for some } x \in \mathbb{R}\}, \quad e^x \diamond e^y := e^x \cdot e^y = e^x + y, \tag{19.1}$$

$$H := \{h \in \mathbb{R} : h = \ln(x) \text{ for some } u \in]0,\infty\}, \quad \ln u \bullet \ln v := \ln u + \ln v = \ln(xy). \tag{19.2}$$

(a) Both $(G,\diamond)$ and $(H,\bullet)$ are abelian groups G has neutral element 1 and H has neutral element 0. (Exercise: Prove it. What are the inverses?)

(b) Let the functions φ and ψ be defined as follows:

$$\varphi : (G,\diamond) \to (H,\bullet), \quad \varphi(g) := \ln g, \tag{19.3}$$

$$\psi : (H,\bullet) \to (G,\diamond), \quad \psi(h) := e^h. \tag{19.4}$$

Then φ and ψ satisfy the following:

$$\varphi(g_1 \diamond g_2) = \varphi(g_1) \bullet \varphi(g_2), \quad \varphi(1) = 0, \quad \varphi(g^{-1}) = \varphi(g)^{-1}, \tag{19.5}$$

$$\psi(h_1 \bullet h_2) = \psi(h_1) \diamond \psi(h_2), \quad \psi(0) = 1, \quad \psi(h^{-1}) = \psi(h)^{-1}. \tag{19.6}$$

Further, the functions φ and ψ are inverse to each other, i.e.,

$$\psi\big(\varphi(g)\big) = g \quad \text{and} \quad \varphi\big(\psi(h)\big) = h \tag{19.7}$$

for all $g \in G$ and $h \in H$. □

If you talk about φ and ψ as "the functions" and $\diamond$ and $\bullet$ as "the operations" you might state the results (19.5) and (19.6) as follows:

The functions are structure compatible with the group operations on their domains and codomains: See also Example **??** on p.**??**.

19.2 More on Commutative Rings and Integral Domains

This chapter adds to the material of ch.**??** (Commutative Rings and Integral Domains) on p.**??**.

The following is an example of a commutative ring with unit which contains zero divisor. This is not a document on algebra and we only give an outline. See, e.g., [2] Beck/Geoghegan ch.6 for details.

Theorem 19.1 (Division Algorithm for Integers)**.** *See [2] Beck/Geoghegan, thm.6.13.*
*Let $n \in \mathbb{N}$ and $m \in \mathbb{Z}$. There exist two integers q (the **quotient**) and r (the **remainder**) such that*

$$m = qn + r \tag{19.8}$$

$$0 \leq r < n \tag{19.9}$$

q and r are uniquely determined by m and n.

PROOF: Will not be given. ■

Example 19.3.

The following examples illustrate the division algorithm.

- **(a)** $m = 63, n = 10$: Then $q = 6$ and $r = 3$ because $63 = 6 \cdot 10 + 3$ and $0 \leq 3 < 10$.
- **(b)** $m = 63, n = 13$: Then $q = 4$ and $r = 11$ because $63 = 4 \cdot 13 + 11$ and $0 \leq 11 < 13$.
- **(c)** $m = -63, n = 13$: Then $q = -5$ and $r = 2$ because $-63 = (-5) \cdot 13 + 2$ and $0 \leq 2 < 13$.
- **(d)** $m = 63, n = 7$: Then $q = 9$ and $r = 0$ because $63 = 9 \cdot 7 + 0$ and $0 \leq 0 < 7$. □

20 Appendix: Addenda to Beck/Geoghegan's "The Art of Proof"

This chapter contains extensions of material found in [2] Beck/Geoghegan, the book which is meant to be read in conjunction with these lecture notes. Some of this material is referenced in earlier chapters.

Notation 20.1.

Even though the subject matter of this chapter is primarily the book [2] Beck/Geoghegan: The Art of Proof, a reference such as prop.9.7 will refer to proposition 9.7 of this document. Proposition 9.7 of the Beck Geoghegan book will be referenced as "B/G prop.9.7" or "[2] B/G prop.9.7" or something similar. ☐

20.1 AoP Ch.1: Integers

Note that B/G ch.1, axioms 1.1 – 1.5 for the set $\mathbb{Z}$ of the integers match the definition of an integral domain which was given in Definition **??** on p.**??** of this document. Does that mean that "integral domain" is just a fancy name for the set $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$? The answer is No! We have seen in prop.**??** of ch.**??** (The Axiomatic Method) that not only the integers, but also the rational numbers and the real numbers with the binary operations of addition and multiplication are integral domains.

So what is going on here? The answer: B/G chose to specify the set $\mathbb{Z}$ in stages. The first set of axioms, the one just mentioned, specifies the algebraic properties of addition and multiplication. B/G axioms 2.1(i) – 2.1(iv) are added in their second chapter to tell us that the integral domain $\mathbb{Z}$ is an ordered integral domain with positive cone $\mathbb{N}$. See Definition **??** on p.**??**. Finally, B/G ax.2.15, the induction axiom, is added. Only at this point is $\mathbb{Z}$ completely specified. We chose in this document to define the integers "in one shot" rather than piecemeal. See axiom **??** on p.**??**. We duplicated the material of B/G ch.1 in ch.refssec:arithm-integral-domains (Arithmetic in Integral Domains) on p.**??** and the material of B/G ch.2.1 and 2.2 in ch.**??** (Order Relations in Integral Domains) on p.**??**.

20.1.1 Ch.1.1 – Axioms

There are no addenda at this point in time.

20.2 AoP Ch.2: Natural Numbers and Induction

The following remark is also part of rem.**??** on p.**??**.

Remark 20.1. B/G ch.1, axioms 1.1 – 1.5 plus 2.1(i) – 2.1(iv) for the set $\mathbb{Z}$ of the integers state that $\mathbb{Z}$ is an ordered integral domain with positive cone $\mathbb{N}$. See Definition **??** on p.**??**. ☐

Remark 20.2.

It follows from the last remark that all of the material in B/G ch.2.1 and 2.2, i.e., all of the propositions and corollaries and definitions inbetween B/G prop.2.1 and prop.2.13, extend to any ordered integral domain $(R, \oplus, \odot, P)$. An example would be prop.**??** on p.**??** which corresponds to B/G ch.1 prop.2.2. $\square$

20.2.1 AoP Ch.2.2 (Ordering the Integers)

The material given here complements ch.2.2 (Ordering the Integers) of [2] Beck/Geoghegan.

Remark 20.3.
B/G Axioms 1.1 – 1.5 together with axiom 2.1 do not suffice to characterize the integers; a symbol different from $\mathbb{Z}$ might have been more appropriate. These axioms are also satisfied by the set $\mathbb{Q}$ of all rational numbers (fractions), provided one interprets the set $\mathbb{N}$ described by axiom 2.1 as the set of all (strictly) positive fractions. They are also satisfied by the set $\mathbb{R}$ of all real numbers (decimals), provided one interprets $\mathbb{N}$ as the set of all (strictly) positive decimals. Only addition of the induction axiom (B/G axiom 2.15) excludes $\mathbb{Q}$ and $\mathbb{R}$. $\square$

Accordingly, all propositions and theorems of the B/G text before the induction axiom apply not only to integers but to rational and real numbers as well.

20.2.2 AoP Ch.2.3 (Induction)

There are no addenda at this point in time.

20.2.3 Bounded Sets in $\mathbb{Z}$

There are no addenda at this point in time.

20.2.4 Exercises for Ch.20.2

There are no exercises at this point in time.

20.3 AoP Ch.3: Some Points of Logic

Here are some references for items discussed in ch.3 of the B/G text to where they appear in ch.**??** (Logic) on p.**??** of this document.

(a) Universal quantifier $\forall$, existental quantifier $\exists$, unique existental quantifier $\exists!$: ch.**??** on p.**??**.

(b) "$(\forall\ x)(\forall\ y)$ has the same meaning as $(\forall\ x$ **and** $y)$": part a of Definition **??** (Doubly quantified expressions) in ch.**??** on p.**??**.

We recommend that you look at prop.**??** and the note which precedes it. The latter is reproduced here:

(1)	The order in which the qualifiers are applied is important. $\forall x \exists y$ generally does not mean the same as $\exists y \forall x$.
(2)	Interchanging variable names in the qualifiers is not OK. $\forall x \exists y$ generally does not mean the same as $\forall y \exists x$.

(c) **Skip part (c) if you have no knowledge of logic beyond what is in ch.3 of B/G.** If you have had some training in logic you may have learned to express "if P (is true) then Q (is true)" as "$P \to Q$" rather than "$P \Rightarrow Q$". There is a difference: Proving a statement of the form "if P then Q" means to show that the statements P and Q are related in a fashion that makes it "logically impossible", [5] for P to be true and Q to be false. "$P \Rightarrow Q$" has the combination $\big($ P is true, Q is false $\big)$ marked as irrelevant (logically impossible); the remaining three combinations give the same outcome as for $P \to Q$, i.e., they all evaluate to **true**.

Example: Let us assume that x is an integer. Let P be the statement P : "$x = 100''$, and let Q be the statement Q : "$x > 10''$. Then it is correct to write $P \Rightarrow Q$ because, no matter the value of x, it is not possible that P is true and Q is false.

(d) The equivalent forms of "if P then Q" such as "Q whenever P" are listed in ch.**??** (Arrow and Implication Operators) on p.**??**. There you also find the definition of the **converse** and the **contrapositive** of an implication.

Here are some notes about B/G ch.3.3: Negations.

(a) The negation of "A **and** B" is "(**not** A) **or** (**not** B)", and the negation of "A **or** B" is "(**not** A) **and** (**not** B)". This is known as "De Morgan's law" for statements (see thm.**??** on p.**??**). Do you see the connection to De Morgan's law for sets? (see thm.**??** on p.**??**).

(b) **Skip part (b) if you have no knowledge of logic beyond what is in ch.3 of B/G.** B/G states that the negation of "$P \Rightarrow Q$" is "P **and not** Q". This should be stated more appropriately as follows: The negation of "$P \Rightarrow Q$" is "P **and** (**not** Q)". The reason for this equivalence is that "$P \Rightarrow Q$" is defined as "**not** (P **or** Q)": we can apply De Morgan's laws to obtain the negation.

(c) We chose to write parentheses in the above expression to avoid ambiguity. But note that there is a "binding power" or preference for the logical operators: The negation "**not** P" of a statement P has higher preference than "P **and** Q" and "P **or** Q":
The meaning of "**not** P **or** Q" is "(**not** P) **or** Q", not "**not** (P **or** Q)."

20.4 AoP Ch.4: Recursion

There are no addenda at this point in time.

20.5 AoP Ch.5: Underlying Notions in Set Theory

There are no addenda at this point in time.

[5] See ch.**??**, Definition **??** on p.**??**.

20.6 AoP Ch.6: Equivalence Relations and Modular Arithmetic

20.6.1 Equivalence Relations

Remark 20.4.
B/G defines in this chapter the absolute value for integers as usual:

$$|m| = \begin{cases} m & \text{if } m \geq 0, \\ -m & \text{if } m < 0. \end{cases}$$

See Definition **??** on p.**??**. Note that this document has generalized this concept to integral domains. See Definition **??** on p.**??**. □

20.6.2 The Division Algorithm

There are no addenda at this point in time.

20.6.3 The Integers Modulo n

There are no addenda at this point in time.

20.6.4 Prime Numbers

There are no addenda at this point in time.

20.6.5 Exercises for Ch.20.6

There are no exercises at this point in time.

20.7 AoP Ch.7: Arithmetic in Base Ten

20.7.1 Base–Ten Representation of Integers

You will find the material of this B/G chapter represented somewhat differently in ch.**??** (The Base–β Representation of the Integers) of this document.

20.8 AoP Ch.8: Real Numbers

B/G ch.8 axiomatically defines the set $\mathbb{R}$ of all real numbers. It was stated in ch.**??** (Numbers) on p.**??** that real numbers are the same as decimal numbers. A proof of this will be given in B/G ch.12 (Decimal Expansions).

20.8.1 Axioms

Remark 20.5.

Note that B/G axioms 8.1 – 8.5 and axiom 8.26 do not suffice to determine what we think of as the real numbers because the set $\mathbb{Q}$ also satisfies each one of them. Only the addition of axiom 8.52 (completeness axiom) will accomplish this.

We encountered a similar situation in the first two chapters of B/G with the set $\mathbb{Z}$ of all integers where axioms 1.1 – 1.5 and 2.1 also are valid for $\mathbb{Q}$ and axiom 2.15 (induction axiom) was needed to completely determine the set $\mathbb{Z}$. ☐

Remark 20.6.

For the following see also rem20.2 on p. 737.

Note that B/G axioms 8.1 – 8.5 in ch.8.1 for the set $\mathbb{R}$ of the real numbers together with B/G prop.8.7 (the cancellation rule holds in $\mathbb{R}$) imply that $\mathbb{R}$ is an integral domain, but one with the additional property that $(\mathbb{R}_{\neq 0}, \cdot)$ is an abelian group [6] (see Definition **??** on p.**??**). B/G prop.8.8 – prop.8.24 only depend on the integral domain properties of $\mathbb{R}$, just as the corresponding propositions of B/G ch.1 only depend on the integral domain properties of $\mathbb{Z}$,

This explains the remark preceding B/G prop.8.8 in which it is reasoned that the proofs of B/G prop.8.8 through B/G prop.8.24 in B/G ch.8.1 are literally the same as those for the corresponding propositions in B/G ch.1. S ☐

20.9 AoP Ch.9: Embedding $\mathbb{Z}$ in $\mathbb{R}$

Remark 20.7. Note that B/G prop.9.10 and prop.9.12 are covered by thm.**??** on p.**??**. ☐

20.10 AoP Ch.10: Limits and Other Consequences of Completeness

There are no addenda at this point in time.

20.11 AoP Ch.11: Rational and Irrational Numbers

There are no addenda at this point in time.

20.12 AoP Ch.12: Decimal Expansions

There are no addenda at this point in time.

20.13 AoP Ch.13: Cardinality

There are no addenda at this point in time.

[6] such an integral domain is called a **field**.

20.14 Exercises for Ch.20

All exercises appear at the end of the individual subchapters.

21 Exam Preparation

Last update: February 23, 2018.

Most of this chapter features problems which are typical for what you might find on one of my Math 330 exams. You will also find here a list of definitions which you **need NOT** to learn by heart. Be aware though that those definitions may be referenced later in the text.

I plan to add to those lists in the future and also include Sample Problems for more topics.

Note that solutions have been written here for many but not all problems!

21.1 Sample Problems for Induction

Problem 21.1. (**Induction**). Let $x_1 = 1, x_2 = 1 + \frac{1}{2}, \dots, x_k = \sum_{j=1}^{k} \frac{1}{j}$ $(k \in \mathbb{N})$.
Prove by induction that $\sum_{k=1}^{n} x_k \ = \ (n+1)x_n - n$ $(n \in \mathbb{N})$.

Solution to #21.1: See Grimaldi Discrete Math 4ED, Exercise 4.1, # 2c, p.176 ■

Problem 21.2. (**Induction**). Prove by induction that $\sum_{j=1}^{n} j(j!) \ = \ (n+1)! - 1$ $(n \in \mathbb{N})$.

Solution to #21.2:

Base case $n = 1$: LS $= 1 \cdot (1!) = \cdot 1 = 1 = 2 - 1 = (2!) - 1 =$ RS.

Induction assumption $(\star)$: $\sum_{j=1}^{n} j(j!) \ = \ (n+1)! - 1$ for all $j \in \mathbb{Z}$ such that $0 \leq j < n$.

Need to show $(\star\star)$: $\sum_{j=1}^{n+1} j(j!) \ = \ (n+2)! - 1$.

$$\begin{aligned} \text{LS of } (\star\star) \ &= \sum_{j=1}^{n} j(j!) + (n+1)(n+1)! \ \overset{(\star)}{=} \ (n+1)! - 1 + (n+1)(n+1)! \\ &= (1)(n+1)! + (n+1)(n+1)! - 1 \ = \ (n+2)(n+1)! - 1 \ = \ \text{RS of } (\star\star). \end{aligned}$$

Thus $(\star\star)$ is valid. This finishes the proof by induction. ■

Problem 21.3. (**Strong Induction**). Let $x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ \dots,\ x_n = x_{n-1} + x_{n-2} + x_{n-3}$ $(n \in \mathbb{N}, n \geq 3)$. Prove by strong induction that $x_n \leq 3^n$ for all $n \in \mathbb{Z}_{\geq 0}$.

Solution to #21.3:

Base cases are $n = 0, n = 1, n = 2$. (We need three base cases because the recursion formula) $x_n = x_{n-1} + x_{n-2} + x_{n-3}$ requires the knowledge of three predecessors.)

$n = 0$ is valid since $x_0 = 1 = 3^0$.
$n = 1$ is valid since $x_1 = 2 < 3 = 3^1$.
$n = 2$ is valid since $x_2 = 3 < 9 = 3^2$.

Induction assumption $(\star)$: $x_j \le 3^j$ for all $j \in \mathbb{Z}$ such that $0 \le j < n$.

Need to show $(\star\star)$: $x_n \le 3^n$ for all $n \ge 3$.

It follows from $(\star)$ that $x_{n-3} \le 3^{n-3}$, $x_{n-2} \le 3^{n-2}$, $x_{n-1} \le 3^{n-1}$.

Thus **(a)** $x_{n-1} + x_{n-2} + x_{n-3} \le 3^{n-3} + 3^{n-2} + 3^{n-1} \le 3 \cdot 3^{n-1}$.

Since $n \ge 3$, LS of $(\star\star)$ $= x^n = x_{n-1} + x_{n-2} + x_{n-3} \overset{\textbf{(a)}}{\le} 3 \cdot 3^{n-1} = 3^n =$ RS of $(\star\star)$.

Thus $(\star\star)$ is valid. This finishes the proof by induction. ■

Problem 21.4. (**Strong Induction**).

Let $x_0 = 2$, $x_1 = 4$, $x_{n+1} = 3x_n - 2x_{n-1}$ for $n \in \mathbb{N}$. Prove by strong induction that $x_n = 2^{n+1}$ for every integer $n \ge 0$. Hint: Is one number enough for the base case?

Solution to #21.4:

Base cases: $n = 0, 1$: $x_0 = 2 = 2^{0+1}$. Further, $x_1 = 4 = 2^{1+1}$. This proves the base cases.

Induction assumption $(\star)$: Let $n \in \mathbb{N}$. Assume that $x_j = 2^{j+1}$ for **all** $0 \le j \le n$.

Need to show $(\star\star)$: $x_{n+1} = 2^{n+2}$.

$$\begin{aligned} \text{LS of } (\star\star) &= x_{n+1} = 3x_n - 2x_{n-1} \quad \text{(the recursive definition)} \\ &= 3(2^{n+1}) - 2(2^n) \quad ((\star) \text{ was applied both to } j = n \text{ and } j = n-1\ ) \\ &= 6 \cdot 2^n - 2 \cdot 2^n = 4 \cdot 2^n = 2^{n+2} = \text{RS of } (\star\star). \end{aligned}$$

Thus $(\star\star)$ is valid. This finishes the proof by induction. ■

Problem 21.5. (**Strong Induction**).

Let $x_0 = 1$, $x_1 = 3$, $x_{n+1} = 2x_n + 3x_{n-1}$ for $n \in \mathbb{N}$. Prove by strong induction that $x_n = 3^n$ for every integer $n \ge 0$. Hint: Is one index enough for the base case?

Solution to #21.5:

Base cases: $n = 0, 1$: $x_0 = 1 = 3^0$. Further, $x_1 = 3 = 3^1$. This proves the base cases.

Induction assumption $(\star)$: Let $n \in \mathbb{N}$. Assume that $x_j = 3^j$ for **all** $0 \le j \le n$.

Need to show $(\star\star)$: $x_{n+1} = 3^{n+1}$..

$$\begin{aligned} \text{LS of } (\star\star) &= x_{n+1} = 2x_n + 3x_{n-1} \quad \text{(the recursive definition)} \\ &= 2(3^n) + 3(3^{n-1}) \quad ((\star) \text{ was applied both to } j = n \text{ and } j = n-1\ ) \\ &= 2 \cdot 3^n + 1 \cdot 3^n = 3 \cdot 3^n = 3^{n+1} = \text{RS of } (\star\star). \end{aligned}$$

Thus $(\star\star)$ is valid. This finishes the proof by induction. ■

Problem 21.6. (**Recursion**).

Let $x_1 = 3$, $x_{n+1} = x_n + 2n + 3$ ($n \in \mathbb{N}$. Prove by induction that $x_n = n(n+2)$ ($n \in \mathbb{N}$).

Solution to #21.6:

Base case $n = 1$: LS $= 3 = 1(1+2) = 1 =$ RS.
Induction assumption $(\star)$: $x_n = n(n+2)$.
Need to show $(\star\star)$: $x_{n+1} = (n+1)(n+3)$

$$\begin{aligned}\text{LS of } (\star\star) &= x_{n+1} \overset{\text{def.}}{=} x_n + 2n + 3 \\ &\overset{(\star)}{=} n(n+2) + 2n + 3 \;=\; n^2 + 4n + 3 \;=\; (n+1)(n+3) \;=\; \text{RS of } (\star\star).\end{aligned}$$

Thus $(\star\star)$ is valid. This finishes the proof by induction. ■

21.2 Sample Problems for Functions and Relations

Problem 21.7. (**Partial order relations**). Remark **??.D** on p.**??** states that if $(X, \preceq)$ is a POset and if $A \subseteq X$ then the relation $\preceq_A$ on A defined as $x \preceq_A y$ if and only if $x \preceq y$ $(x, y \in A)$ is a partial ordering on A. Prove it.

Solution to #21.7:

We must prove reflexivity, antisymmetry, and transitivity.

(a) Proof of reflexivity: Let $a \in A$. We must prove that $a \preceq_A a$. Since $x \in X$ and "$\preceq$" is reflexive as a partial ordering on X we obtain $a \preceq a$, i.e., $a \preceq_A a$.

(b) Proof of antisymmetry: Let $a, b \in A$ such that $a \preceq_A b$ and $b \preceq_A a$. We must prove that $a = b$. Since $a, b \in X$, "$\preceq$" is antisymmetric as a partial ordering on X, $a \preceq_A b \Rightarrow a \preceq b$, and $b \preceq_A a \Rightarrow b \preceq a$, we obtain $a = b$.

(c) Proof of transitivity: Let $a, b, c \in A$ such that $a \preceq_A b$ and $b \preceq_A c$. We must prove that $a \preceq_A c$. Since $a, b, c \in X$, "$\preceq$" is transitive as a partial ordering on X. Moreover $a \preceq_A b \Rightarrow a \preceq b$, and $b \preceq_A c \Rightarrow b \preceq c$, thus $a \preceq c$, i.e., $a \preceq_A c$. ■

Problem 21.8. (**Functions**). Given is a function $f : A \longrightarrow B$ $(A, B \neq \emptyset)$. Give the definitions of each of the following:

(a) f is injective.
(b) f is surjective.
(c) f is bijective.
(d) f has a left-inverse g.
(e) f has a right-inverse h.

For **(d)** and **(e)**, give the "arrow diagram" which show domain and codomain for each function involved. In both cases it will like the one to the left. Each symbol **S** denotes a (possibly different) set and each symbol φ denotes a (possibly different) function.

$$\begin{tikzcd} \mathbf{S} \arrow[r, "\varphi"] \arrow[dr, "\varphi"'] & \mathbf{S} \arrow[d, "\varphi"] \\ & \mathbf{S} \end{tikzcd}$$

Solution to #21.8:

Solution to problems a,b,c:

Injective means one-one: If $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ then $a_1 = a_2$.

Surjective means onto: If $b \in B$ then there is $a \in A$ such that $f(a) = b$.

Bijective means both injective and surjective.

Solution to problem d: If this diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\mathbf{1}_A"'] & B \arrow[d, "g"] \\ & A \end{tikzcd}$$

i.e., $g \circ f = \mathbf{1}_A$, then we call g a **left inverse** of f ("to the left of the reference object f"), i.e., f has a left inverse (namely the function g).

Solution to problem e: If this diagram commutes:

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[dr, "1_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

i.e., $f \circ h = 1_B$, then we call h a **right inverse** of f ("to the right of the reference object f)", i.e., f has a right inverse (namely the function h). ■

Problem 21.9. (**Equivalence relations and partial order relations**).

(a) Let $a, b \in \mathbb{Z}$. State as precisely as possible the definition of $a \mid b$.
(b) Is the relation $a \sim b \iff a \mid b$ **reflexive**? **symmetric**? **antisymmetric**? **transitive**? If true, prove it. If false, give a counterexample.

Solution to #21.9:

(a) There is $z \in \mathbb{Z}$ such that $az = b$.
(b) Is the relation $a \sim b \iff a \mid b$

- **reflexive**: TRUE: If $a \in \mathbb{Z}$ then $a = 1 \cdot a$, hence $a \mid a$, hence $a \sim a$.
- **symmetric**: FALSE: Counterexample: $5 \mid 10$ but $10 \nmid 5$.
- **antisymmetric**: FALSE: Counterexample: $-1 \mid 1$ and $1 \mid -1$, but $1 \neq -1$.
- **transitive**: TRUE: Let $a, b, c \in \mathbb{Z}$ such that $a \sim b$ and $b \sim c$.
  Then $a \mid b$, i.e., $b = ma$, and $b \mid c$, i.e., $c = nb$ for suitable $m, n \in \mathbb{Z}$.
  Thus $c = (mn)a$. Since $(mn) \in \mathbb{Z}$, $a \mid c$, hence $a \sim c$. ■

Problem 21.10. (**Functions and equivalence relations**).
Let $f : X \to Y (X, Y \neq \emptyset)$. Prove that $a \sim b \iff f(a) = f(b)$ is an equivalence relation on X.

The solution to this problem is not given here. ■

21.3 Sample Problems for Convergence and Uniform Convergence

Problem 21.11. (**Convergence in Metric Spaces**).
Prove closed book prop.**??** on p.**??**.

Problem 21.12. (**Convergence in Metric Spaces**).
Prove closed book thm.**??** on p.**??**.

Problem 21.13. (**Convergence in Metric Spaces**).
Prove closed book prop.**??** on p.**??**.

Problem 21.14. (**Convergence in Metric Spaces**).
Prove closed book prop.**??** on p.**??**.

Problem 21.15. (**Uniform convergence**).
Let $(X,d) := ([0,1], d_{|\cdot|})$ be the unit interval, equipped with the standard Euclidean metric $d(x,x') = |x-x'|$, and let $(Y,d') := ([0,1], d_{\|\cdot\|_{L^2}})$ be the same set, but equipped with the metric derived from the L^2–norm $\|f\|_{L^2} = \sqrt{\int_a^b f(x)^2 dx}$. (See Definition **??** on p.**??**).
For $n \in \mathbb{N}$ let $f_n(x) := (-1)^n$ if $0 \le x \le \frac{1}{n}$ and 0 if $\frac{1}{n} < x \le 1$.

(a) Prove that $f_n(\cdot) \to 0$ on (Y,d').
(b) Prove that $f_n(x)$ does not converge pointwise on (X,d). Hint: What exactly must you show?

Solution to #21.15:

PROOF of **(a)**: Let $\delta > 0$. Then $d'(f_n,0) = \sqrt{\int_0^{1/n} 1^2 dt + \int_{1/n}^1 0^2 dt} = \sqrt{\frac{1}{n}}$, and this expression is less than δ whenever $n > \frac{1}{\delta^2}$. Let $N := \min\{j \in \mathbb{N} : j > \frac{1}{\delta^2}\}$. It follows that $d'(f_n,0) < \delta$ for all $n \ge N$. This proves that $f_n(\cdot) \to 0$ on (Y,d').

PROOF of **(b)**: We have $f_n(0) = (-1)^n$ for all $n \in \mathbb{N}$. This sequence is not Cauchy in (X,d) since $d(f_j(0), f_{j+1}(0) = 2$ for all $j \in \mathbb{N}$ and hence not convergent (see thm.**??** (Convergent sequences are Cauchy) on p.**??**). ■

21.4 Other Topics

Problem 21.16. (**Logic**). Given a function $f : X \to Y$, negate the following statements:

(a) There exists $x \in X$ and $y \in Y$ such that $f(x) = y$,
(b) For all $x \in X$ there exists $y \in Y$ such that $f(x) = y$,
(c) $\exists x \in X$ such that $\forall y \in Y$ such that $f(x) \ne y$.
(d) $\forall x_1, x_2 \in X$: if $x_1 \ne x_2$ then $f(x_1) \ne f(x_2)$.

Solution to #21.16:

(a) $\forall x \in X, \forall y \in Y : f(x) \ne y$,
(b) $\exists x \in X$ such that $\forall y \in Y : f(x) \ne y$,
(c) $\forall x \in X \; \exists y \in Y$ such that $f(x) = y$,
(d) $\exists x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$. ■

Problem 21.17. (**Sets**). Prove AND REMEMBER the set identities **2.a** through **2.g** of prop.**??** on p.**??**:

(b) $A \triangle \emptyset = \emptyset \triangle A = A$ (neutral element $\emptyset$ for $\triangle$)
(c) $A \triangle A = \emptyset$ (inverse element $\emptyset$ for $\triangle$) [7]
(d) $A \triangle B = B \triangle A$ (commutativity of $\triangle$)
(e) $(A \cap B) \cap C = A \cap (B \cap C)$ (associativity of $\cap$)
(f) $A \cap \Omega = \Omega \cap A = A$ (neutral element Ω for $\cap$)
(g) $A \cap B = B \cap A$ (commutativity of $\cap$)

Solution to #21.17:

[7] The inverse element for A in the sense of Definition **??** on p.**??**. is A itself!

Trivial (we omitted the tough one, **2.h**). ■

Problem 21.18. (**Set functions**). Given is an arbitrary collection of sets $(A_j)_{j\in J}$. Determine for each assertion below whether it is true or false. If it is true, prove it. If it is false, give a counterexample.

(a) $f(\bigcup_{j\in J} A_j) \subseteq \bigcup_{j\in J} f(A_j)$; **(b)** $\bigcup_{j\in J} f(A_j) \subseteq f(\bigcup_{j\in J} A_j)$;

(c) $f(\bigcap_{j\in J} A_j) \subseteq \bigcap_{j\in J} f(A_j)$; **(d)** $\bigcap_{j\in J} f(A_j) \subseteq f(\bigcap_{j\in J} A_j)$;

You may use the fact that the direct image is increasing with its argument: $A \subseteq B \Rightarrow f(A) \subseteq f(B)$.

Solution to #21.18:

Solution to a:

$$\begin{aligned} y \in f(\bigcup_{j\in J} A_j) &\Rightarrow \exists x \in \bigcup_{j\in J} A_j \text{ such that } f(x) = y \quad \text{(def. direct image)}\\ &\Rightarrow \exists j_0 \in J \text{ such that } x \in A_{j_0} \quad \text{(def. union)}\\ &\Rightarrow y = f(x) \in f(A_{j_0}) \quad \text{(def. direct image)}\\ &\Rightarrow y = f(x) \in \bigcup_{j\in J} f(A_{j_0}) \quad \text{(def. union)} \end{aligned}$$

Solution to b: As the direct image is increasing with its argument and $A_i \subseteq \bigcup_{j\in J} A_j$ for all $i \in J$ it follows that $f(A_i) \subseteq f\big(\bigcup_{j\in J} A_j\big)$ for all $i \in J$. Hence

$$\bigcup_{i\in J} f(A_i) \subseteq \bigcup_{i\in J} \Big(f(\bigcup_{j\in J} A_j)\Big) = f(\bigcup_{j\in J} A_j)$$

The equality on the right hand side results from the fact that the set $f\big(\bigcup_{j\in J} A_j\big)$ does not depend on the index variable $i \in J$ with respect to which the "outer" union takes place. ■

Problem 21.19. (**Cardinality**).

Give an alternate proof of thm.**??** (The real numbers are uncountable) on p.**??** which is based on the fact that the cardinality of a set is less than that of its power set (thm.**??** on p.**??**). Hint: Find bijections $f : \{x \in \mathbb{R} : x = \sum_{j=1}^{\infty} d_j 10^{-j} \text{ and } d_j = 3 \text{ or } 4\ \forall j\} \xrightarrow{\sim} \{3,4\}^{\mathbb{N}}$ and $g : \{3,4\}^{\mathbb{N}} \xrightarrow{\sim} 2^{\mathbb{N}}$.

Solution to #21.19:

Let $\Gamma := \{x \in \mathbb{R} : x = \sum_{j=1}^{\infty} d_j 10^{-j} \text{ and } d_j = 3 \text{ or } 4\ \forall j\}$.

Both $f : \Gamma \xrightarrow{\sim} \{3,4\}^{\mathbb{N}};\ \sum_{j=1}^{\infty} d_j 10^{-j} \mapsto (d_j)_{j\in\mathbb{N}}$ and $g : \{3,4\}^{\mathbb{N}} \xrightarrow{\sim} 2^{\mathbb{N}};\ (d_j)_{j\in\mathbb{N}} \mapsto \{j \in \mathbb{N} : d_j = 4\}$

are bijections, thus $g \circ f$ is a bijection from $\Gamma \subseteq \mathbb{R}$ to $2^{\mathbb{N}}$. According to thm.**??** the cardinality of the latter exceeds that of $\mathbb{N}$. Since $\mathbb{N}$ is countably infinite we obtain that Γ and hence its superset $\mathbb{R}$ is uncountable. ■

Problem 21.20. (**Continuity**). Let $a, b, c, d \in \mathbb{R}$ such that $a < b$ and $c < d$. Let $f :]a, b[\to]c, d[$ be bijective and strictly monotone, i.e., strictly increasing or decreasing. Prove that both f and f^{-1} are continuous.

Hint: Use ε–δ continuity.

The solution to this problem is not given here. ■

Problem 21.21. (**Continuity**).

Let $(X, \mathfrak{U})$ and $(Y, \mathfrak{V})$ be topological spaces and $f : X \to Y$. Prove that f is continuous if and only if the preimage $f^{-1}(F)$ of any closed $F \subseteq Y$ is closed in X.

Solution to #21.21:

PROOF: We utilize for both directions prop.**??** on p.**??**: f is continuous if and only if the preimage $f^{-1}(V)$ of any open $V \subseteq Y$ is open in X, and also the fact that

$$\text{if } B \in Y \text{ then } \left(f^{-1}(B^{\complement})\right)^{\complement} \;=\; f^{-1}(B) \qquad .(\star)$$

This equation follows from **??** on p.**??**: the complement of the inverse image is the inverse image of the complement.

(a) PROOF of f is continuous $\Rightarrow$ f^{-1} (closed) = closed:

Let $F \in Y$ be closed. We must prove that $f^{-1}(F)$ is closed. Complements of closed sets are open and vice versa, hence $F^{\complement}$ is open, hence continuity of f implies that $f^{-1}(F^{\complement})$ is open. Thus $\left(f^{-1}(F^{\complement})\right)^{\complement}$ is closed According to $(\star)$ that closed set equals $f^{-1}(F)$. This proves **(a)**.

(b) PROOF of f^{-1} (closed) = closed $\Rightarrow$ f is continuous:

It suffices to show that $f^{-1}(V)$ is open for any open $V \subseteq Y$ because this implies the continuity of f. So let $V \subseteq Y$ be open. Then $V^{\complement}$ is closed, hence $f^{-1}(V^{\complement})$ is closed by our assumption, hence its complement $\left(f^{-1}(V^{\complement})\right)^{\complement}$ is open According to $(\star)$ that open set equals $f^{-1}(V)$. This proves **(b)**. ■

Problem 21.22. (**Continuity**). Let $X := \mathbb{R}$, equipped with the standard Euclidean metric $d(x, x') = |x - x'|$. Let $f_n : \mathbb{R} \to \mathbb{R}$; $\quad x \mapsto \frac{\sin(n^2 x)}{n}$.

- **(a)** Prove that $f_n(\cdot) \overset{uc}{\to} 0$ on $\mathbb{R}$.
- **(b)** Prove that $f_n(x)$ does not converge pointwise on $\mathbb{R}$. Hint: What exactly must you show?

Solution to #21.22:

PROOF of **(a)**: Let $\delta > 0$. It follows from $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ that there exists $N \in \mathbb{N}$ such that $\frac{1}{n} = |\frac{1}{n} - 0| < \delta$ for all $n \geq N$. We conclude from $|\sin t| \leq 1$ for all $t \in \mathbb{R}$ that $|f_n(x) - 0| = \frac{\sin(n^2 x)}{n} \leq \frac{1}{n} < \delta$ for all $n \geq N$ and also for all $x \in X$. This proves that $f_n(\cdot) \overset{uc}{\to} 0$ on X.

PROOF of **(b)**: Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then $f'(x) = \frac{n^2 \cos(n^2 x)}{n} = n \cos(n^2 x)$. Thus $f_n'(0) = n \cos(0) = n$. It follows that $\lim\limits_{n\to\infty} f_n'(0) = \infty$, thus $f_n'(x)$ does converge at $x = 0$. ■

Problem 21.23. (**Compactness**).

Let $(X, \mathfrak{U})$ be a topological space. Prove that X is compact if and only if every family $(F_i)_{i \in I}$ of closed sets has the **finite intersection property** (short: **fip**), i.e., $(F_i)_{i \in I}$ satisfies the following: If every finite selection $F_{i_1}, F_{i_2}, \dots, F_{i_k}$ of members of $(F_i)_i$ has nonempty intersection then $\bigcap\limits_{i \in I} F_i \neq \emptyset$.

Note that an equivalent formulation is the contrapositive: If $\bigcap\limits_{i \in I} F_i = \emptyset$ then there must be finitely many members $F_{i_1}, F_{i_2}, \dots, F_{i_n}$ such that $F_{i_1} \cap \dots \cap F_{i_n} = \emptyset$.

Solution to #21.23:

PROOF: We recall the definition of compactness: Every family $(U_i)_{i \in I}$ of open sets which covers X possesses a finite selection $U_{i_1}, U_{i_2}, \dots, U_{i_k}$ which covers X.

(a) PROOF of X is compact $\Rightarrow$ every family $(F_i)_{i \in I}$ of closed sets has the fip:

We work with the contrapositive: Assume that $\bigcap\limits_{i \in I} F_i = \emptyset$ For each $i \in I$ let $U_i := F_i^{\complement}$. Then each U_i is open as the complement of a closed set. It follows from de Morgan that $\bigcup\limits_{i \in I} U_i = \emptyset^{\complement} = X$, i.e., the family $(U_i)_i$ is an open covering of X. Since X is compact we can extract a finite subcover $U_{i_1}, \dots, U_{i_k}$: $U_{i_1} \cup \dots \cup U_{i_k} = X$. We obtain from de Morgan that $F_{i_1} \cup \dots \cup F_{i_k} = X^{\complement} = \emptyset$. It follows that $(F_i)_{i \in I}$ has the fip, and this proves **(a)**. Since $\bigcup\limits_{i \in I} U_i = X$ de Morgan yields $\bigcup\limits_{i \in I} U_i = X$

(b) PROOF that if every family $(F_i)_{i \in I}$ of closed sets has the fip then X is compact:

Let $(U_i)_{i \in I}$ be a family of open sets which covers X. We must extract a finite covering. For each $i \in I$ let $F_i := U_i^{\complement}$. Then F_i is closed as the complement of an open set. Since $\bigcup\limits_{i \in I} U_i = X$ de Morgan yields $\bigcap\limits_{i \in I} F_i = X^{\complement} = \emptyset$. But the family of closed sets $(F_i)_{i \in I}$ possesses the fip, thus there exist finitely many members $F_{i_1}, \dots, F_{i_k}$ such that $F_{i_1} \cap \dots \cap F_{i_k} = \emptyset$. From de Morgan we obtain $U_{i_1} \cup \dots \cup U_{i_k} = \emptyset^{\complement} = X$. We have extracted a finite covering from the family $(U_i)_{i \in I}$, thus X is compact. This proves **(b)**. ■

21.5 Non–essential Definitions

A non–essential Definition is one which the student need not remember in the sense that it will not occur in a quiz or exam. It is possible though that such a definition will be referenced in later parts of the document.

For example, the non–essential term "abelian group", defined in Definition **??** on p.**??**, is referenced in example **??** which can be found on p.**??**.

This chapter contains the beginnings of a list of non–essential definitions. It is broken down into several lists on a chapter by chapter basis.

> Generally speaking, any definition that is given in an optional (starred) chapter or in a construct other than a proper definition, e.g., in a footnote or a remark, is considered non–essential, and it very likely will not be included in those lists

Ch.??: The Axiomatic Method:

binary operation (not essential until its formal definition in ch.**??**) • semigroup, monoid • abelian group (but remember commutative group) • linear function, additivity, homogeneity • commutative ring with unit

Ch.??: Functions and Relations:
linear/total ordering • linearly/totally ordered set • inverse relation • maps to operator (but remember assignment operator)

22 Other Appendices

22.1 Greek Letters

The following section lists all greek letters that are commonly used in mathematical texts. You do not see the entire alphabet here because there are some letters (especially upper case) which look just like our latin alphabet letters. For example: $A =$ Alpha $B =$ Beta. On the other hand there are some lower case letters, namely epsilon, theta, sigma and phi which come in two separate forms. This is not a mistake in the following tables!

α	alpha	θ	theta	ξ	xi	ϕ	phi
β	beta	ϑ	theta	π	pi	φ	phi
γ	gamma	ι	iota	ρ	rho	χ	chi
δ	delta	κ	kappa	ϱ	rho	ψ	psi
ϵ	epsilon	$\varkappa$	kappa	σ	sigma	ω	omega
ε	epsilon	λ	lambda	ς	sigma		
ζ	zeta	μ	mu	τ	tau		
η	eta	ν	nu	υ	upsilon		

Γ	Gamma	Λ	Lambda	Σ	Sigma	Ψ	Psi
Δ	Delta	Ξ	Xi	Υ	Upsilon	Ω	Omega
Θ	Theta	Π	Pi	Φ	Phi		

22.2 Notation

This appendix on notation has been provided because future additions to this document may use notation which has not been covered in class. It only covers a small portion but provides brief explanations for what is covered.

For a complete list check the list of symbols and the index at the end of this document.

Notation 22.1. a) If two subsets A and B of a space Ω are disjoint, i.e., $A \cap B = \emptyset$, then we often write $A \uplus B$ rather than $A \cup B$ or $A + B$. Both $A^{\complement}$ and, occasionally, $\complement A$ denote the complement $\Omega \setminus A$ of A.

b) $\mathbb{R}_{>0}$ or $\mathbb{R}^+$ denotes the interval $]0, +\infty[$, $\mathbb{R}_{\geq 0}$ or $\mathbb{R}_+$ denotes the interval $[0, +\infty[$,

c) The set $\mathbb{N} = \{1, 2, 3, \cdots\}$ of all natural numbers excludes the number zero. We write $\mathbb{N}_0$ or $\mathbb{Z}_+$ or $\mathbb{Z}_{\geq 0}$ for $\mathbb{N} \uplus \{0\}$. $\mathbb{Z}_{\geq 0}$ is the B/G notation. It is very unusual but also very intuitive. □

Definition 22.1.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of real numbers. We call that sequence **nondecreasing** or **increasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.

We call it **strictly increasing** if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$.

We call it **nonincreasing** or **decreasing** if $x_n \geq x_{n+1}$ for all n.
We call it **strictly decreasing** if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$. $\square$

References

[1] Heinz Bauer. Approximation and abstract boundaries. American Mathematics Monthly Vol.85 No.8 p.632-647, 1978.

[2] Matthias Beck and Ross Geoghegan. The Art of Proof. Springer, 1st edition, 2010.

[3] Norman B. Haaser and Joseph A. Sullivan. Real Analysis. publisher = Van Nostrand, year = 1971, 1st edition.

[4] Walter Rudin. Principles of Mathematical Analysis. McGraw-Hill, New York, San Francisco, Toronto, London, 2nd edition, 1964.

List of Symbols

$r^\star$ – rational cut , 722

$\complement A$ – complement , 752
$\lambda^1, \lambda^2, \ldots, \lambda^n,$ – Lebesgue measure , 731
$\mathbb{N}, \mathbb{N}_0$, 752
$\mathbb{R}^+, \mathbb{R}_{>0}$, 752
$\mathbb{R}_+, \mathbb{R}_{\geq 0}$, 752
$\mathbb{R}_{>0}, \mathbb{R}^+$, 752
$\mathbb{R}_{\geq 0}, \mathbb{R}_+$, 752
$\mathbb{Z}_+, \mathbb{Z}_{\geq 0}$, 752
$A \uplus B$ – disjoint union , 752
$A^\complement$ – complement , 752
$(\Omega, \mathfrak{F})$ – measurable space , 729
$(\Omega, \mathfrak{F}, \mu)$ – measure space , 730
$\liminf\limits_{n\to\infty} A_n$ – limit inferior for sets , 733
$\limsup\limits_{n\to\infty} A_n$ – limit superior for sets , 733
$\mu(\cdot)$ – finite measure , 730
$\underline{\mu}(\cdot)$ – measure , 730
$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ – extended real numbers , 728
$\overline{\mathbb{R}}_+$ – nonnegative extended , 728
$\mathfrak{F}$ – σ–algebra , 729

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