Formula Collection for Math 447 Midterm 2 – Not all items are relevant!

Some abbreviations: probab = probability; spc = space; rv = r.v. = random variable; re = r.e. = random element; fn = function; lin = linear; $\int$ = integral, integrable; pos = positive; mon = monotone; distrib = distribution; discr = discrete; cont = continuous; cond = conditional; meas = measure, mble = measurable; defs = defines

(1) (a) ch.2: • power set $2^\Omega = \{$ all subsets of $\Omega\}$ • $\forall\, x \dots$: For all $x \dots$ $\boxdot$ $\exists\, x$ s.t. $\dots$ There is an x such that $\dots$ $\boxdot$ $\exists!\, x$ s.t. $\dots$ There is a unique x s.t. $\dots$ $\boxdot$ $p \Rightarrow q$ If p is true then q is true $\boxdot$ $p \Leftrightarrow q$ p iff q, i.e., p true if and only if q true
• Intervals: $]a,b[= \{x \in \mathbb{R} : a < x < b$, $]a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \leq b$, $[a,b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \leq x \leq b$, etc.
• countable set A: can be sequenced: $\boxdot$ $A = \{a_1, a_2, \dots, a_n\}$ (finite set) $\boxdot$ $A = \{_1, a_2, \dots\}$ ("countably infinite" set) $\boxdot$ $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is uncountable • family $(x_i)_{i \in I}$: index set I may be uncountable • $\bigcup_{i \in J} A_i = \{x : \exists\, i_0 \in J$ s.t. $x \in A_{i_0}\}$ • $\bigcap_{i \in J} A_i = \{x : \forall\, i \in J\ x \in A_i\}$. • Can use $A \biguplus B$ for $A \cup B$ if disjoint sets • **De Morgan:** $\boxdot$ $\left(\bigcup_k A_k\right)^{\complement} = \bigcap_k A_k^{\complement}$ $\boxdot$ $\left(\bigcap_k A_k\right)^{\complement} = \bigcup_k A_k^{\complement}$ • **Distributivity:** $\boxdot$ $\bigcup_j (B \cap A_j) = B \cap \bigcup_j A_j$ $\boxdot$ $\bigcap_{j \in I} (B \cup A_j) = B \cup \bigcap_j A_j$
• Cartesian products: $|X_1 \times \dots \times X_n| = |X_1| \cdots |X_n|$ • Formulas f. preimages of $f : X \to Y$: Arbitrary index set J and $B, B_j \subseteq Y$: $\boxdot$ $f^{-1}(\bigcap_{j \in J} B_j) = \bigcap_{j \in J} f^{-1}(B_j)$ $\boxdot$ $f^{-1}(\bigcup_{j \in J} B_j) = \bigcup_{j \in J} f^{-1}(B_j)$ $\boxdot$ $f^{-1}(B^{\complement}) = \left(f^{-1}(B)\right)^{\complement}$ $\boxdot$ $B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$ • $A \subseteq \Omega \Rightarrow$ indicator fn $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 else • partition B_j $(j \in \mathbb{N})$ of Ω, $A \subseteq \Omega \Rightarrow A = \biguplus_j (A \cap B_j)$

(b) ch.3 & 4: Sums and Riemann integrals (Riem-$\int$) and Lebesgue integrals (Leb-$\int$):
• $x_n \geq 0$ or $\sum_n x_n$ abs conv $\Rightarrow$ $\sum_n x_n$ satisfies WHAT? • Leb-$\int$: positive, monotone, linear, mon. + domin. conv.
• step fn h: $\int h(\vec{y}) d\vec{y} = ?$ • simple fn g: $\int g d\lambda^d = ?$ • If both $\int_A f(\vec{y}) d\vec{y}$, $\int_A f d\lambda^d$ exist, they are equal
• Use Fubini for both $\int_A f(\vec{y}) d\vec{y}$ and $\int_A f d\lambda^d$ to compute multidim $\int$. • $\left[\mathbf{1}_A \text{ Riem-}\int\text{-ble}\right] \Rightarrow \lambda^d(A)$ defined how?
• Borel sets $\mathfrak{B}^d = \sigma\{$ d–dim rectangles $\}$ • $\left[f \geq 0 \text{ or } \int |f| d\lambda^d < \infty\right] \Rightarrow \left[A \mapsto \int_A f\, d\lambda^d \text{ is } \sigma\text{–additive}\right]$
• For what functions φ, ψ is $A \mapsto \sum_{\omega \in A} \varphi(\omega)$, $A \mapsto \int_A \psi(\vec{y}) d\vec{y}$ $\left(= \int_A \psi d\lambda^d\right)$ a probab meas?

(c) ch.7: Combinatorial Analysis
• Think: Does order matter in your probability space or doesn't it?
• multiplication rule for several factors • # of permutations P^n_r vs # of combinations $\binom{n}{r}$ vs $\binom{n}{r_1, \dots, r_k}$
$\boxdot$ $0! = 1$, $n! = 1 \cdot 2 \cdots n$; $(n \in \mathbb{N})$ $\boxdot$ several interpretations of $\binom{n}{r_1, \dots, r_k}$
• deck of 52 cards: $\boxdot$ 4 suits (clubs, spades, hearts, diamonds) of 13 each: Ace, $2, 3, \dots, 10$, Jack, Queen, King $\boxdot$ so: 4 2's, 4 3's, 4 Aces, 4 Jacks, $\dots$ • Roulette: $\boxdot$ slots $0, 00, 1, 2, \dots, 36$ $\boxdot$ 18 black, 18 red; numbers 1 – 36 in 12 rows $\times$ 3 cols

(d) ch.1, 5, 8: generalities about probability spaces:
• probab spc = sample spc (Ω, P) • σ–algebra $\mathfrak{F} \subseteq 2^\Omega$: $\boxdot$ $A \in \mathfrak{F} \Rightarrow A^{\complement} \in \mathfrak{F}$ $\boxdot$ $A_n \in \mathfrak{F} \Rightarrow \bigcup_{j=1}^\infty A_j \in \mathfrak{F}$ $\boxdot$ $\emptyset \in \mathfrak{F}$
• distrib of r.e. $X : (\Omega, P) \to \Omega'$: $P_X(B) = P\{X \in B\} = P\big(X^{-1}(B)\big)$ on codomain.
• Conveniences: $P_X(\{x\}) = P\{X = x\}$; $P_X(]a,b]) = P\{a < X \leq b\}$ (if X is rv, i.e., $\Omega' \subseteq \mathbb{R}$); ...
• discr probab spaces and discr r.e.s and r.v.s defined how?
• independence for $2, n$, arbitr. many events • $P(A \mid B)$ • general addition & multiplication rules, complement rule
• partition B_j $(j \in \mathbb{N})$ of Ω, $A \subseteq \Omega$: $\Rightarrow$ total probability formula = $\dots$, Bayes formula = $\dots$
• Sampling: $\boxdot$ action vs realization $\boxdot$ random sample vs SRS $\boxdot$ urn models with/without replacement

(e) ch.6: Advanced Topics - Measure and Probability $\boxed{\star}$ (may help to recall some formulas)
• meas μ: like probab meas, but $\mu(\Omega) \neq 1$ possible • measures defined by: $\boxdot$ $\mu(A) := P(A)$ $\boxdot$ $\mu(A') := P_X(A') = P\{X \in A'\}$ (for r.e. $X : (\Omega, \mathfrak{F}, P) \to (\Omega', \mathfrak{F}')$) $\boxdot$ $\mu(\{\omega\}) := f(\omega) \cdot p(\omega)$ (for PMF p and $f \geq 0$) $\boxdot$ $\mu(A) := |A|$ (counts the elements of A) $\boxdot$ $\mu(]a,b]) := \int_a^b f(y) dy$ defs meas μ on $(\mathbb{R}, \mathfrak{B}^1)$; $\boxdot$ $\mu(B) := \int_B f(\vec{y}) d\vec{y}$ ($B = d$–dim rectangle and $f \geq 0$) defs meas μ on $(\mathbb{R}^d, \mathfrak{B}^d)$; • $\int f d\mu$ (and thus, $\int Y dP$) defined like for Lebesgue meas

$\mu = \lambda^d$: first simple f, then $f \geq 0$, then general f. • $\int f d\mu$ is pos, mon, lin; mon. + domin. conv.

• $f : (\Omega, \mathfrak{F}, \mu) \to (\Omega', \mathfrak{F}')$ is mble if $\big[A' \in \mathfrak{F}' \Rightarrow f^{-1}(A') \in \mathfrak{F}\big]$ Then $\mu_f(A') = \mu\big(f^{-1}(A')\big)$ defs image meas μ_f on $\mathfrak{F}'$

• f, g real–valued: $\int_\Omega g \circ f d\mu = \int_{\mathbb{R}} g d\mu_f$

• For μ =probab meas P and r.v. Y: $E[Y] = \int Y dP$ has properties of $\int$(!), e.g., for $g : \mathbb{R} \to \mathbb{R}$: $\int_\Omega g \circ Y dP = \int_{\mathbb{R}} g(y) dP_Y$ ⊡ so $E[g \circ Y] = \dots$

(f) ch.9, 10: Discrete r.e.s and continuous r.v.s

• discr r.e. $X : (\Omega, P) \to \Omega'$, PMF $p(x) = p_X(x) = \dots$ • cont r.v.s $Y : (\Omega, P) \to \mathbb{R}$, PDF $p(y) = p_Y(y) = \dots$

• discr & cont rvs: CDF $F_Y(y)$; pth quantile ϕ_p • $E[Y], Var[Y], \sigma_Y$ of rv Y: ⊡ Remember all formulas! ⊡ $E[g(Y)] = \dots$
• m'_k and m_k; MGF $m_Y(t)$ • Each distrib: ⊡ application context? ⊡ $m_Y(t)$ =? ⊡ Given $m_Y(t)$: $Y \sim$ WHAT?

• iid sequences of random elements ⊡ Bernoulli trials and sequences ⊡ 0–1 encoded Bernoulli trials

• discr rvs: ⊡ Bernoulli(p) ⊡ binom(n, p) ⊡ geom(p) ⊡ neg. binom(p, r): $p(y) = \binom{y-1}{r-1} p^r q^{y-r}$, $\mu = \frac{r}{p}$, $\sigma^2 = \frac{r(1-p)}{p^2}$

⊡ hypergeom(N, R, n): $\mu = \frac{nR}{N}$, $\sigma^2 = \frac{nR}{N} \cdot \frac{N-R}{N} \cdot \frac{N-n}{N-1}$ ⊡ poisson(λ): $m_Y(t) = e^{\lambda(e^t - 1)}$ ⊡ approximate binom(n, p) with Poisson(λ), HOW?