Formula Collection for Math 447 Midterm 3 – Not all items are relevant!

(1) **(a)** • power set $2^{\Omega} = \{$ all subsets of $\Omega\}$ • $\forall\, x \dots$: For all $x \dots$ ⊡ $\exists\, x$ s.t. $\dots$ There is an x such that ...
⊡ $\exists!\, x$ s.t. $\dots$ There is a unique x s.t. $\dots$ ⊡ $p \Rightarrow q$ If p is true then q is true ⊡ $p \Leftrightarrow q$ p iff q, i.e., p true if and only if q true
• Intervals: $]a, b[= \{x \in \mathbb{R} : a < x < b$, $]a, b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \leq b$, $[a, b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \leq x \leq b$, etc.
• countable set A: can be sequenced: ⊡ $A = \{a_1, a_2, \dots, a_n\}$ (finite set) ⊡ $A = \{_1, a_2, \dots\}$ ("countably infinite" set)
⊡ $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is uncountable • family $\left(x_i\right)_{i\in I}$: index set I may be uncountable • $\bigcup_{i\in J} A_i = \{x : \exists\, i_0 \in J$ s.t. $x \in A_{i_0}\}$ • $\bigcap_{i\in J} A_i = \{x : \forall\, i \in J\, x \in A_i\}$. • Can use $A \uplus B$ for $A \cup B$ if disjoint sets • **De Morgan:** ⊡ $\left(\bigcup_k A_k\right)^{\complement} = \bigcap_k A_k^{\complement}$ ⊡ $\left(\bigcap_k A_k\right)^{\complement} = \bigcup_k A_k^{\complement}$ • **Distributivity:** ⊡ $\bigcup_j \left(B \cap A_j\right) = B \cap \bigcup_j A_j$ ⊡ $\bigcap_{j\in I} \left(B \cup A_j\right) = B \cup \bigcap_j A_j$
• Cartesian products: $\left|\, X_1 \times \dots \times X_n \,\right| = \left|X_1\right| \cdots \left|X_n\right|$ • Formulas f. preimages of $f : X \to Y$:
Arbitrary index set J and $B, B_j \subseteq Y$: ⊡ $f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j)$ ⊡ $f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$
⊡ $f^{-1}(B^{\complement}) = \left(f^{-1}(B)\right)^{\complement}$ ⊡ $B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$ • $A \subseteq \Omega \Rightarrow \mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 else
• partition B_j ($j \in \mathbb{N}$) of Ω, $A \in \Omega \Rightarrow A = \biguplus_j (A \cap B_j) \Rightarrow P(A) = \dots$; Used to compute $P(B_{j_0} \mid A)$ from $P(A \mid B_j)$ and $P(B_j)$ (must know for all j) • Indicator function: $\mathbf{1}_A(y) =$

(b) Sums and Riemann integrals (Riem-$\int$) and Lebesgue integrals (Leb-$\int$):
• $x_n \geq 0$ or $\sum_n x_n$ abs conv $\Rightarrow \sum_n x_n$ satisfies WHAT? • Leb-$\int$: positive, monotone, linear, mon. + domin. conv.
• step function h: $\int h(\vec{y}) d\vec{y} =$? • simple function g: $\int g d\lambda^d =$? • If both $\int_A f(\vec{y}) d\vec{y}$, $\int_A f d\lambda^d$ exist, they are equal
• Use Fubini for both $\int_A f(\vec{y}) d\vec{y}$ and $\int f d\lambda^d$ to compute multidim $\int$. • $\mathbf{1}_A$ Riem-$\int$-ble $\Rightarrow \lambda^d(A)$ defined how?
• $\mathfrak{B}^d = \sigma\{$ d–dim rectangles $\}$ • $\left[f \geq 0 \text{ or } \int |f| d\lambda^d < \infty\right] \Rightarrow \left[A \mapsto \int_A f\, d\lambda^d \text{ is } \sigma\text{–additive}\right]$
• For what functions φ, ψ is $A \mapsto \sum_{\omega\in A} \varphi(\omega)$, $A \mapsto \int_A \psi(\vec{y}) d\vec{y}$ $\left(= \int_A \psi d\lambda^d\right)$ a probab measure?

(c) • Probability space = sample space (Ω, P) • σ–algebra $\mathfrak{F} \subseteq 2^{\Omega}$: ⊡ $A \in \mathfrak{F} \Rightarrow A^{\complement} \in \mathfrak{F}$ ⊡ $A_n \in \mathfrak{F} \Rightarrow \bigcup_{j=1}^{\infty} A_j \in \mathfrak{F}$ ⊡ $\emptyset \in \mathfrak{F}$ • distribution of random element (rand elem) $X : (\Omega, P) \to \Omega'$: $P_X(B) = P\{X \in B\} = P\left(X^{-1}(B)\right)$ on codomain.
• Conveniences: $P_X(\{x\}) = P\{X = x\}$; $P_X(]a, b]) = P\{a < X \leq b\}$ (if X is random var. (rv), i.e., $\Omega' \subseteq \mathbb{R}$); ...
• discrete probab spaces and random elements and rvs defined how?

• independence for $2, n$, arbitr. many events • $P(A \mid B)$ • general addition & multiplication rules, complement rule, total probability, Bayes formula

(d) Combinatorial Analysis • Think: Does order matter in your probability space or doesn't it?
• multiplication rule for several factors • # of permutations P_r^n vs # of combinations $\binom{n}{r}$ vs $\binom{n}{r_1, \dots, r_k}$
⊡ $0! = 1$, $n! = 1 \cdot 2 \cdots n$; $(n \in \mathbb{N})$ ⊡ several interpretations of $\binom{n}{r_1, \dots, r_k}$

• deck of 52 cards: ⊡ 4 suits (clubs, spades, hearts, diamonds) of 13 each: Ace, $2, 3, \dots, 10$, Jack, Queen, King ⊡ so: 4 2's, 4 3's, 4 Aces, 4 Jacks, ... • Roulette: ⊡ slots $0, 00, 1, 2, \dots, 36$ ⊡ 18 black, 18 red; numbers 1 – 36 in 12 rows $\times$ 3 cols

(e) Random variables (rvs) and random elements
• Discrete rand elem $X : (\Omega, P) \to \Omega'$, $p(x) = p_X(x) = P_X\{x\}$: PMF = probab. mass func (WMS: probab. func.) for X.
• Continuous rand vars $Y : (\Omega, P) \to \mathbb{R}$, $p(y) = p_Y(y)$: PDF for Y. • discrete & cont. rvs: CDF $F_Y(y)$; pth quantile ϕ_p
• $E[Y], Var[Y], \sigma_Y$ of rv Y: ⊡ Remember all formulas! ⊡ $E[g(Y)] = \dots$ • m_k' and m_k; MGF $m_Y(t)$ compute how?
• Each distribution: ⊡ application context? ⊡ $m_Y(t) =$? ⊡ Given $m_Y(t)$: $Y \sim$ WHAT?
• iid sequences of random elements ⊡ Bernoulli trials and sequences ⊡ 0–1 encoded Bernoulli trials
• Discrete rvs: ⊡ Bernoulli(p) ⊡ binom(n, p) ⊡ geom(p) ⊡ neg. binom(p, r): $p(y) = \binom{y-1}{r-1} p^r q^{y-r}$, $\mu = \frac{r}{p}$, $\sigma^2 = \frac{r(1-p)}{p^2}$
⊡ hypergeom(N, R, n) ⊡ Poisson(λ) • Contin rvs: ⊡ uniform(θ_1, θ_2): $\sigma^2 = \frac{(\theta_1 - \theta_2)^2}{12}$
⊡ $\mathcal{N}(\mu, \sigma^2)$: empirical rule =? ⊡ gamma(α, β) vs $\chi^2(\text{df} = \nu)$ vs expon(β) • $2\times$ Tchebysheff – know them both!

Continued on p.2!

(f) Multivariate $\vec{Y} = (Y_1, \dots, Y_k)$:

• joint CDF $F_{\vec{Y}}(\vec{y})$, joint PMF $p_{\vec{Y}}(\vec{y})$, joint PDF $f_{\vec{Y}}(\vec{y})$ ⊡ allow you to see whether the rvs are independent. HOW? ⊡ What condition on $\{(y_1, y_2)$: the PDF or PMF is $> 0\}$? Must determine! • $E[g(Y_1, \dots, Y_n)] = \cdots$ • $Cov[X, Y] = 0$ vs. X, Y independent. Relationship? • $\vec{Y} = (Y_1, Y_2)$: ⊡ marginal CDFs F_{Y_j}, PMFs p_{Y_j} and PDFs f_{Y_j} ⊡ conditional PMF $p_{Y_1|Y_2}(y_1|y_2)$ and PDF $f_{Y_1|Y_2}(y_1|y_2)$ define $E[Y_1 \mid Y_2]$, $Var[Y_1 \mid Y_2]$ how? • $E[g(Y_1, \dots, Y_k \mid Y = y] = ??$

• $E\big[\,E[Y_1 \mid Y_2]\,\big] =$ (11.52) • $Var[Y_1] =$ (11.55)

• connection sums of indep rvs and their MGFs ...

• Given a small 2-dim table (say, 3×4 entries) for a joint PMF, be able to compute marginal and conditional distributions and conditional expectations and variances.

• Order stats: ⊡ We only do them for continuous, iid rvs. ⊡ Find CDFs for $Y(1)$ and $Y(n)$ directly; differentiate to get PDFs ⊡ For $1 < j < n$: maybe find a corresponding multinomial sequence ⊡ $f_{(\bullet)}(\vec{y}) =$ (9.40); proof done how?

(g) Sampling: • "sample" sometimes denotes the random vector (sampling action) $\vec{Y}$ and sometimes "the" realization $\vec{y} = \vec{Y}(\omega)$ ⊡ Random sampling actions (RSAs) are iid, SRS actions are not. ⊡ picked at random $\neq$ random sample: Might be SRS when sampling without replacement!

(h) Odds and ends:

• Sample mean $\overline{Y} = \dots$ of (iid) random sample: $E[\overline{Y}] = \dots$; $Var[\overline{Y}] = \dots$

(i) NOT ON EXAM - MIGHT HELP TO REMEMBER FORMULAS $E[Y] = \dots, E[g(\vec{Y})] = \dots$,:

• abstract integrals $\int \dots dP$: their properties determine those of

⊡ $E[Y] = \int_\Omega Y dP = \int_{\mathbb{R}} y dP_Y$ ⊡ $E[g \circ \vec{Y}] = \int_\Omega g \circ \vec{Y} dP = \int_{\mathbb{R}^n} g(\vec{y}) dP_{\vec{Y}}$ ⊡ $Var[Y] = \int_\Omega g(Y) dP$ (with $g(y) = ??$)

⊡ also applies to conditional expectations, defined by conditional PMFs, PDFs