Formula Collection for Math 454 Midterm 3 – Not all items are relevant!

Abbreviations: • rv = r.v. = random variable; r.elem = random element • meas = measure • mble = measurable • fn = function • prob = probability • distr = distribution • conv = convergence • cont = continuous/continuity • Riem-$\int$-ble = Riemann integrable • w.r.t = with respect to • RN = Radon–Nikodým • cond.exp = conditional expectation • BM = Brownian motion • BAM = binom. asset model • BS = Black–Scholes • PF = portfolio • arbPF = arbitrage portfolio • self–fin = self–financing

(a) • power set $2^\Omega = \{$ all subsets of $\Omega\}$ • $\forall\, x \dots$: For all $x \dots$ ⊡ $\exists\, x$ s.t. … There is an x such that …
⊡ $\exists!\, x$ s.t. … There is a unique x s.t. … ⊡ $p \Rightarrow q$ If p is true then q is true ⊡ $p \Leftrightarrow q$ p iff q, i.e., p true if and only if q true
• Intervals: $]a,b[= \{x \in \mathbb{R} : a < x < b$, $]a,b]_{\mathbb{Z}} = \{x \in \mathbb{Z} : a < x \leq b$, $[a,b]_{\mathbb{Q}} = \{x \in \mathbb{Q} : a \leq x \leq b$, etc.
• countable set A: can be sequenced: ⊡ $A = \{a_1, a_2, \dots, a_n\}$ (finite set) ⊡ $A = \{_1, a_2, \dots\}$ ("countably infinite" set)
⊡ $\mathbb{Z}$ and $\mathbb{Q}$ are countable, but $\mathbb{R}$ is uncountable • family $(x_i)_{i\in I}$: index set I may be uncountable • $\bigcup_{i\in J} A_i = \{x : \exists\, i_0 \in J$ s.t. $x \in A_{i_0}\}$ • $\bigcap_{i\in J} A_i = \{x : \forall\, i \in J\, x \in A_i\}$. • Can use $A \uplus B$ for $A \cup B$ if disjoint sets • **De Morgan:** ⊡ $\left(\bigcup_k A_k\right)^{\complement} = \bigcap_k A_k^{\complement}$ ⊡ $\left(\bigcap_k A_k\right)^{\complement} = \bigcup_k A_k^{\complement}$ • **Distributivity:** ⊡ $\bigcup_j (B \cap A_j) = B \cap \bigcup_j A_j$ ⊡ $\bigcap_{j\in I} (B \cup A_j) = B \cup \bigcap_j A_j$
• Cartesian products: $|\, X_1 \times \dots \times X_n\,| = |X_1| \cdots |X_n|$ • Indicator fn $\mathbf{1}_A(\omega) =$? • preimages of $f : X \to Y$: Arbitrary index set J and $B, B_j \subseteq Y$: ⊡ $f^{-1}(\bigcap_{j\in J} B_j) = \bigcap_{j\in J} f^{-1}(B_j)$ ⊡ $f^{-1}(\bigcup_{j\in J} B_j) = \bigcup_{j\in J} f^{-1}(B_j)$
⊡ $f^{-1}(B^{\complement}) = \left(f^{-1}(B)\right)^{\complement}$ ⊡ $B_1 \cap B_2 = \emptyset \Rightarrow f^{-1}(B_1) \cap f^{-1}(B_2) = \emptyset$ • $A \subseteq \Omega \Rightarrow 1_A(\omega) = 1$ if $\omega \in A$ and 0 else

(b) Integrals $\int_A f d\mu$: • monotone & dominated conv. & other laws for $\int f d\mu$ • Jensen ineq valid for what μ?
• simple fn g: $\int g d\mu =$? • $\int h(\vec{x}) d\vec{x}$ (Riemann) vs $\int h d\lambda^d$ • $\mathfrak{B}^d = \sigma\{$ d–dim rectangles $\}$ • For what φ is $A \mapsto \int_A \varphi(\omega)\mu(d\omega)$ a prob meas? • ILMD method used how? for what theorems?
• Use Fubini for both $\int_A f(\vec{y}) d\vec{y}$ and $\int f d\lambda^d$ to compute multidim $\int$. • 1_A Riem-$\int$-ble $\Rightarrow \lambda^d(A)$ defined how?
• mble $f : (\Omega, \mathfrak{F}, \mu) \to (\Omega', \mathfrak{F}')$ img meas $\mu_f(A')$ (= ???) on $\mathfrak{F}'$. Distrib P_X of a r.elem X defined how?
• $f : \Omega \to \mathbb{R}$, μ, ν σ–finite meas on $(\Omega, \mathfrak{F})$ s.t. $\nu(A) = \int_A f d\mu$, $\Rightarrow$ f =(def) RN derivative $\frac{d\nu}{d\mu}$. • meas $\nu \ll \mu \Leftrightarrow [\mu(A) = 0 \Rightarrow \nu(A) = 0]$; $\nu \sim \mu \Leftrightarrow [\mu(A) = 0 \Leftrightarrow \nu(A) = 0]$; • $[\nu \ll \mu$ and μ, ν σ–finite$] \Leftrightarrow [$ RN derivative $\frac{d\nu}{d\mu}$ exists$]$.
• $\int_\Omega g \circ f(\omega)\mu(d\omega) = \int_{\Omega'} g(\omega')\mu_f(d\omega')$ (what are f, g w.r.t. μ?) • $\int f(\omega)\frac{d\nu}{d\mu}(\omega)\mu(d\omega) = \int f(\omega)\nu(d\omega)$ ($f, \mu, \nu, \frac{d\nu}{d\mu}$ =?);
• $f \leqq g$ μ–a.e. $\Leftrightarrow \int_A f d\mu \leqq \int_A g d\mu$ for all $A \in \mathfrak{F}$ • Y functionally dependent on $X \Leftrightarrow \sigma(\_\_\_) \subseteq \sigma(\_\_\_)$ in what sense?

(c) Cond.Exp: • prob space $(\Omega, \mathfrak{F}, P)$; σ–algebras $\mathfrak{H} \subseteq \mathfrak{G} \subseteq \mathfrak{F}$; $\varphi : \mathbb{R} \to \mathbb{R}$ mble. Then ⊡ $E[c_1 X + c_2 Y|\mathfrak{G}] = c_1 E[X|\mathfrak{G}] + c_2 E[Y|\mathfrak{G}]$. ⊡ If X is $\mathfrak{G}$–measurable, then $E[X \cdot Y|\mathfrak{G}] = X \cdot E[Y|\mathfrak{G}]$. ⊡ $E\big[E[X|\mathfrak{G}]\,\big|\,\mathfrak{H}\big] = E[X|\mathfrak{H}]$. ⊡ X independent of $\mathfrak{G} \Rightarrow E[X|\mathfrak{G}] = E[X]$. ⊡ φ convex $\Rightarrow \varphi\big(E[X \mid \mathfrak{G}]\big) \leqq E[\varphi \circ (X) \mid \mathfrak{G}]$. • countable partition $\Omega = \biguplus [G_j : j \in J]$ s.t. $G_j \in \mathfrak{F}$ and $P(G_j) > 0$ for all j; $\mathfrak{G} := \sigma\{G_j : j \in J\}$; r.v. X; ⊡ Then $E[X \mid \mathfrak{G}](\omega) = \dots$

(d1) 1–dim Stoch. calculus: • simple process $Z_t \Rightarrow$ Itô integral $\int_0^t Z_u dW_u = \dots$ • $dt dW_t =$? $dW_t dW_t =$?;
• Itô isometry =? • Itô processes $dX_t = A_t dt + B_t dW_t$, $dY_t = U_t dt + V_t dW_t$; then ⊡ $dX_t dY_t =$? ⊡ $d[X,X]_t =$?
⊡ When is $t \mapsto \int_0^t X_u dY_u$ a martingale? • Lévy: contin. paths martingale M_t; $M_0 = 0$; $[M,M]_t = t \Rightarrow M_t =$?
• 1-dim and multidim Itô formulas • Itô product rule • GBM and general GBM: ⊡ $dX_t =$? ⊡ meaning of α_t, σ_t

(d2) multidim Stoch. calculus: • d–dim BM $\vec{W}_t =$? • $[W^{(i)}, W^{(j)}]_t =$? • $dW_t^{(i)} dW_t^{(j)} =$? • $dX_t = \Theta_t dt + \vec{\Delta}_t$ • $d\vec{W}_t$ means? • d–dim Lévy =?

(e) financial markets: • contingent claim $\mathcal{X}$: ⊡ simple if $\mathcal{X} = \Phi(S_T)$; ⊡ $\Pi_t(\mathcal{X})$ = correct (no arbitrage) price of $\mathcal{X}$
• self–fin PF $\vec{H}_t =$??; arbPF = ?? replicating PF = ?? • pricing principle = ?? • complete market =?

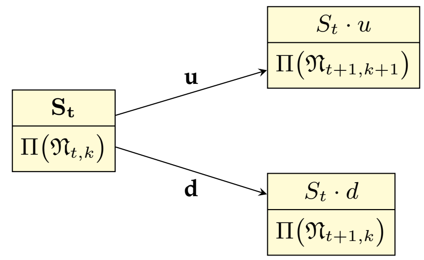
(e1) BAM (binomial asset model):
• Notation: $\widetilde{P}$ = risk–neutral probab (P = real world probab) on $(\Omega, \mathfrak{F}, \mathfrak{F}_t)$
• def. BAM: ⊡ trading times $t = 0, 1, 2, \dots, T$; multiperiod BAM if $T > 1$ ⊡ $B_0 = 1$; $B_t = (1+r)^t$ = money market acct price; $D_t = 1/B_t$ = discount ⊡ $S_0 = s =$ const; ⊡ $u, d, p_u, p_d, \tilde{u}, \tilde{d}$ def'd HOW? ⊡ compute S_{t+1} from S_t HOW?
• PF $\vec{H}_t = (H_t^B, H_t^S)$: ⊡ $\vec{H}_t$ bought at $t-1$, sold at t ⊡ $V_t = V_t^{\vec{H}} = H_t^B B_t + H_t^S S_t$ = sales value of $\vec{H}_t$ ⊡ $x_t = H_t^B B_{t-1}$

= money in the bank at t_1; $\boxdot$ $y_t = H_t^S$ = stock shares bought at $t-1$; $\bullet$ no arb $\Leftrightarrow$ WHAT? $\bullet$ martingales (P? $\widetilde{P}$?) are . . .

$\bullet$ hedge PF for simple claim $\mathcal{X} = \Phi(S_T)$ at $t+1$ is $H_{t+1}^B = (1+r)^{-t}x_{t+1}$ and $H_{t+1}^S = y_{t+1}$, where x_{t+1}, y_{t+1} for the node $\mathfrak{N}_{t,k}$ (remember: $\vec{H}_t$ = purchased at time $t-1$!) in the tree excerpt shown below are, if $\Pi\big(\mathfrak{N}_{t,k}\big) =$ option price $\Pi_t(\mathcal{X})(\omega)$ $\Leftrightarrow$ $S_t(\omega) = su^k d^{t-k}$ $\Leftrightarrow$ exactly k upward moves and $t-k$ downward moves

$$x_{t+1} = \frac{1}{1+R} \cdot \frac{u\Pi\big(\mathfrak{N}_{t+1,k}\big) - d\Pi\big(\mathfrak{N}_{t+1,k+1}\big)}{u-d}\,,$$

$$y_{t+1} = \frac{1}{s} \cdot \frac{\Pi\big(\mathfrak{N}_{t+1,k+1}\big) - \Pi\big(\mathfrak{N}_{t+1,k}\big)}{u-d}\,.$$



$\boxdot$ Is one period BAM complete? $\boxdot$ Is multiperiod BAM complete?

(e2) contin time financial markets with only 1 riskless asset and self–fin PF $\vec{H}_t$: $\bullet$ PF Value $V_t = V_t^{\vec{H}} = H_t^B B_t + \sum_{j>0} H_t^{(j)} S_t^{(j)}$ $\bullet$ $X_t := H_t^B B_t = V_t - \sum_{j>0} H_t^{(j)} S_t^{(j)}$ $\bullet$ budget eqn: $dV_t = H_t^B dB_t + \sum_{j>0} H_t^{(j)} dS_t^{(j)}$

$\bullet$ If only 1 stock price S_t: $\boxdot$ can write $Y_t := H_t^S$ $\boxdot$ $V_t = X_t + H_t^S S_t = X_t + Y_t S_t$ $\boxdot$ budget eqn: $dV_t = H_t^B dB_t + Y_t dS_t$

(e3) Black–Scholes market $\bullet$ model of (classical) BS market: $\boxdot$ constant $r, \alpha, \sigma \in \mathbb{R}$ meaning =?. $\boxdot$ $dB_t =$? $\boxdot$ $dD_t =$? $\boxdot$ $dS_t =$? $\boxdot$ $(t,x) \mapsto \pi(t,x)$ is C^2 means ? $\boxdot$ $\Phi(S_T) = \pi$(WHAT?, WHAT?) $\bullet$ hedge $\vec{H}_t$ for simple claim $\Phi(S_T)$ $\Rightarrow$ $\boxdot$ $d\big(e^{-rt}V_t^{\vec{H}}\big) = (\alpha - r)H_t^S e^{-rt} S_t dt + \sigma H_t^S e^{-rt} S_t dW_t = H_t^S d\big(e^{-rt}S_t\big)$ $\boxdot$ delta–hedging rule: $H_t^S =$? $\boxdot$ BS PDE: $\pi_t(t,x) + rx\,\pi_x\big(t,x\big) + \frac{1}{2}\sigma^2 x^2 \pi_{xx}(t,x) = r\,\pi\big(t,x\big)$; π must satisfy $\pi(T,x) = \Phi$(WHAT?) $\bullet$ put–call parity means ...

$\bullet$ Europ. call: $\boxdot$ $\Phi(x) =$? $\boxdot$ $c(t,x) = \text{BSM}(\tau, x; K, r, \sigma) = xN\big(d_+(\tau,x)\big) - Ke^{-r\tau}N\big(d_-(\tau,x)\big)$, where $0 \le t < T$, $\tau = T - t$, $x > 0$, $d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right]$ $\bullet$ fwd contract: $\boxdot$ $\Phi(x) =$? $\boxdot$ $f(t,x) =$?

$\bullet$ Europ. put: $\boxdot$ $\Phi(x) =$? $\boxdot$ $p(t,x) =$? $\bullet$ fwd price For_t defined how? $\bullet$ Greeks: Delta, Gamma, Rho, Theta, Vega are??

$\bullet$ Amer. puts and calls: which one should never be exercised before T?