Math 454 - Financial Mathematics - Spring 2025

Selected Solutions for Spring 2025 Midterms

0.1 Midterm 1

Ver A #1; Ver B #4 (means problem 1 of Ver A which is problem 4 of Ver B):
1(a, b): See lecture notes **1(c)**: Example: $f_n(x) = 1/n$ (= const)

Ver A #2; Ver B #5: **2(a)**: $\boxed{\textbf{False}}$ e.g., $\{20\} \in \mathfrak{F}'$, but $\{f = 20\} \notin \mathfrak{F}$ **2(b)**: $\boxed{\textbf{True}}$

Ver A #3; Ver B #9: See lecture notes, Example 4.14

Ver A #4; Ver B #6: $P(G_1) = \frac{2}{6}$, $P(G_2) = \frac{4}{6}$, $\int_{]a,b]} X dP = \frac{3}{6}\left(\frac{b^2}{2} - \frac{a^2}{2}\right)$, $(0 < a \leq b \leq 6)$. So,

- $0 < \omega \leqq 2 \Rightarrow E[X \mid \mathfrak{G}](\omega) = \frac{6}{2} \cdot \frac{3}{6}\left(\frac{2^2}{2} - \frac{0^2}{2}\right) = 3,$
- $2 < \omega \leqq 6 \Rightarrow E[X \mid \mathfrak{G}](\omega) = \frac{6}{4} \cdot \frac{3}{6}\left(\frac{6^2}{2} - \frac{2^2}{2}\right) = 12.$

Thus, $E[X \mid \mathfrak{G}] = 3 \cdot \mathbf{1}_{]0,2]} + 12 \cdot \mathbf{1}_{]2,6]}$

Ver A #5; Ver B #1: $\{\tau_2 \leq t\}$ only needs knowledge of S_u for $u \leq t$: ⊡ $\{\tau_2 \leq t\} \in \mathfrak{F}_t^S$ is $\boxed{\textbf{True}}$
$\{\tau_1 \leq t\}$ and $\{\tau_3 \leq t\}$ need knowledge of S_u for $u \leq 2t$. Thus,
⊡ $\{\tau_1 \leq t\} \in \mathfrak{F}_t^S$ is $\boxed{\textbf{False}}$ ⊡ $\{\tau_3 \leq t\} \in \mathfrak{F}_t^S$ is $\boxed{\textbf{False}}$
The last time of something happening is not known before any time t. Thus,
⊡ $\{\tau_3 \leq t\} \in \mathfrak{F}_t^S$ is $\boxed{\textbf{False}}$ (again!) ⊡ $\{\tau_4 \leq t\} \in \mathfrak{F}_t^S$ is $\boxed{\textbf{False}}$

Ver A #6; Ver B #2: $f^{-1}(\cap \dots) = \cap\big(f^{-1}(\dots)\big)$ and $]-2,0[\,\cap\,[-1,3[\,\cap\,[1,4] = \emptyset \Rightarrow$
the answer is $f^{(-1)}(\emptyset) = \boxed{\emptyset}$

Ver A #7; Ver B #7: The smart way: $Y \sim$ geom$(p = 1/4) \Rightarrow \int Y dP = E[Y] = 1/p = \boxed{\mathbf{4}}$
Alternatively, use $\sum\limits_{j=0}^{\infty} q^j = \frac{1}{1-q}$, where $q = \frac{3}{4} \dots$

Ver A #8; Ver B #8: ⊡ $P\{Y = 1\} = \boxed{\textbf{DNE}}$ ⊡ $P_Y\{2\} = \boxed{\textbf{1/3}}$ ⊡ $P\big\{Y \in \{3,4\}\big\} = \boxed{\textbf{DNE}}$
Here, DNE follows from $\{Y = 1\} \notin \mathfrak{F}$, and $\big\{Y \in \{3,4\}\big\} \notin \mathfrak{F}$. Observe that this implies that Y is not $(\mathfrak{F}, \mathfrak{F}')$–measurable. Thus, Y is $\boxed{\textbf{neither}}$ a random element $\boxed{\textbf{nor}}$ a random variable.

Ver A #9; Ver B #3: See lecture notes, ch.4.

0.2 Midterm 2

Ver A #1; Ver B #7 (means problem 1 of Ver A which is problem 7 of Ver B):

1(a–c) $\boxed{\text{see lecture notes}}$ **1(d)**: We use **($\star$)** $x \mapsto g(x)$ is convex and part **1(c)**:.

$$E[\,U_{t+h} \mid \mathfrak{F}_t\,] \;=\; E[\,g(W_{t+h}) \mid \mathfrak{F}_t\,] \overset{(\star)}{\geq} g\big(E[W_{t+h} \mid \mathfrak{F}_t\,]\big) \overset{\textbf{(1c)}}{=} g(W_t) \;=\; U_t\,. \;\blacksquare$$

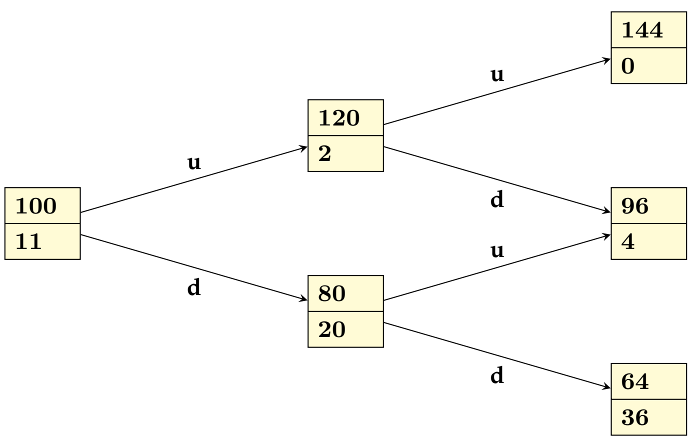
Ver A #2; Ver B #8: **($\star$)** Contract function: $\Phi(su) = 35$ and $\Phi(sd) = 0$.

($\star\star$)) Risk–neutral probabs: $\tilde{p}_u = \tilde{p}_d = \frac{1}{2}$

- From **($\star$)**) and **($\star\star$)**): $\Pi_0(\mathcal{X}) \;=\; \frac{1}{2}\cdot 35 + \frac{1}{2}\cdot 0 \;=\; \boxed{\frac{35}{2}} \;=\; 17.5.$
- From **($\star$)**): $H_1^B \;=\; \frac{1}{1+r}\cdot\frac{u\Phi(sd)-d\Phi(su)}{u-d} \;=\; \boxed{-52.5}$ and $H_1^S \;=\; \frac{1}{S_0}\cdot\frac{\Phi(su)-\Phi(sd)}{u-d} \;=\; \boxed{0.7}$

Ver A #3; Ver B #6: $\boxed{\text{see lecture notes}}$

Ver A #4; Ver B #3: **4(a)**: From diagram below, $\Pi_0(\mathcal{X}) \;=\; \boxed{11}$



4(b): $H_1^B \;=\; \frac{1}{1+r}\cdot\frac{u\Phi(sd)-d\Phi(su)}{u-d} \;=\; \boxed{56}$ and $H_1^S \;=\; \frac{1}{S_0}\cdot\frac{\Phi(su)-\Phi(sd)}{u-d} \;=\; \boxed{-0.45}$

Ver A #5; Ver B #1: **5(a)**: Both cases $x < K, x \geq K$, yield $\boxed{\Phi(x) = x - K}$.

5(b): $\boxed{\text{Forward contract}}$ at T with strike K

Ver A #6; Ver B #2: **6(a)**: $\tilde{p}_u \;=\; \frac{(1+r)-d}{u-d} \;=\; \boxed{\frac{1}{3}}$ and $\tilde{p}_d \;=\; \frac{u-(1+r)}{u-d} \;=\; \boxed{\frac{2}{3}}$

6(b.1): Under P: $\boxed{\text{None!}}$ **6(b.2)**: Under $\widetilde{P}$: $\boxed{(21/20)^{-t}S_t,\; (21/20)^{-t}V_t,\; (21/20)^{-t}\Pi(\mathcal{X})_t}$

Ver A #7; Ver B #4: **7(a)**: $\boxed{(x-75)^+}$ **7(b)**: $\boxed{(75-x)^+}$ **7(c)**: $\boxed{x-75}$

Ver A #8; Ver B #5: **8(a)**: $V_0 = H_1^B(1+r)^0 + H_1^S\cdot S_0 \;=\; \boxed{80}$ **8(b)**: $\boxed{\textbf{No}}$ (e.g. $V_0 \neq 0$)

8(c): $1 + r \;=\; u\tilde{p}_u + d\tilde{p}_d = \frac{3}{2} \;\Rightarrow\; r \;=\; \boxed{1/2}$

8(d): $\big[D_tV_t = \text{martingale};\; D_t = 1/\frac{3}{2}\;\big] \;\Rightarrow\; \big[\,E[V_2] = (D_t)^{-2}E[V_0] = 180;$

$E[V_3] = (D_t)^{-3}E[V_0] = 270\;\big]$ Thus, $E\left[\frac{1}{3}V_2 + \frac{1}{9}V_3\right] \;=\; 60 + 30 \;=\; \boxed{90}$.

0.3 Midterm 3

Ver A #1; Ver B #3 (means problem 1 of Ver A which is problem 3 of Ver B):
$f_t(t,x) - \frac{1}{2} f_{xx}(t,x) = 0$, so $dY_t \;=\; 4Y_t \, dW_t$. Now, integrate.

Ver A #2; Ver B #4: $dW_t^{(1)} dW_t^{(2)} = 0$. Itô's product rule: $d(X_tY_t) \;=\; \left(\frac{-4X_t}{t^2+1}\right) \, dW_t^{(2)} \;+\; 3t \, Y_t \, dW_t^{(1)}$.
Integrals w.r.t. Brownian motion only (no dt term) are martingales. ■

Ver A #3; Ver B #5: See lecture notes. **3(b)** only is true.

Ver A #4; Ver B #1: See lecture notes. **Ver A #5; Ver B #2**: See lecture notes, ch.11.3.

Ver A #6; Ver B #6: Mechanical application of Itô's formula to $f(t,x,y) = \frac{7x}{5y}$.